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GRAPHS WITH AN ASCENDING SUBGRAPH DECOMPOSITION

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Abstract. It has been conjectured that if a graph G has $\binom{n+1}{2}$ edges, then the edge set of G can be partitioned into n graphs $G_1, G_2, \ldots, G_n$ such that G_i has i edges $(1 \le i \le n)$, and G_i is isomorphic to a subgraph of G_{i+1} $(1 \le i < n)$. Such a graph G is said to have an *ascending subgraph decomposition (ASD)*. If the maximum degree of G is less than $\lfloor(2-\sqrt{2})n\rfloor$, then G has an ASD. Futher, if G is a forest with maximum degree at most $\lfloor(3-\sqrt{3})n/2\rfloor$, then G has an ASD. All of the graphs in the decomposition are unions of paths of length at most 3 (length at most 2 for forests).

1. INTRODUCTION

For graphs G and H, $G \le H$ indicates that G is isomorphic to a subgraph of H. A graph G is decomposed into subgraphs $G_1, G_2, \ldots, G_n$ if the edge set $E(G)$ of G can be partitioned into n sets $E(G_i)$, $(1 \le i \le n)$. A graph G with $\binom{n+1}{2}$ edges has an *ascending subgraph decomposition* (abbreviated ASD) if G can be decomposed into subgraphs G_i $(1 \le i \le n)$ such that G_i has size i and $G_i \le G_{i+1}$ for $i < n$. The following conjecture was stated in [1].

CONJECTURE: If G is a graph with $\binom{n+1}{2}$ edges, then G has as ASD.

The conjecture has been verified for special classes of graphs. In [4], it was shown that any star forest with $\binom{n+1}{2}$ edges has an ASD, and if G has at most $n+2$ vertices and $\binom{n+1}{2}$ edges, then G has an ASD. In both cases, all of the graphs in the decomposition were star forests. Using matchings as the graphs in the decomposition, the following theorem and some related results were verified in [5].

THEOREM 1 (Fu [5]): If a graph G has $\binom{n+1}{2}$ edges and maximum degree $\Delta(G) \le \lfloor(n-1)/2\rfloor$, then G has an ASD.

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We will generalize this result by increasing the upper bound on the maximum degree condition that implies an ASD. To do this we allow the graphs in the decomposition to be the vertex disjoint union of short paths, and not just matchings. In particular, the following will be proved.

THEOREM 2: If G is a graph with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor(2-\sqrt{2})n\rfloor$, then G has an ASD. Also, each of the graphs in the decomposition of G is the disjoint union of paths of length at most 3.

If the graph G is a forest, a stronger result can be proved.

THEOREM 3: If G is a forest with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor(3-\sqrt{3})n/2\rfloor$, then G has an ASD. Also, each of the graphs in the decomposition of G is the disjoint union of paths of length at most 2.

2. NOTATION AND PRELIMINARIES

Notation and definitions not explicitly mentioned will follow [2]. Frequently used notation will be described in this section, and other special notation or definitions will be presented as they are needed.

The general strategy in the proofs of both Theorem 2 and Theorem 3 will be to first use the edge chromatic number of a graph G to decompose the graph G into matchings of approximately the same size. Next, the union of some pairs of these matchings will be decomposed into two graphs, with the larger graph being the vertex disjoint union of paths of some small fixed length k $(1 \le k \le 3)$ and a matching, and with the smaller graph being just a matching. The larger graphs will form the decomposition graphs at the "top end" of an ASD for G. Therefore, for some t and k, the largest graph in the chain of the ASD will have t components that are paths of length k with the remaining edges being a matching. All of the other graphs in the chain will have fewer (possibly 0) paths of length k.

The objective of this section is to develop the results on the decomposition of the union of two matchings that will be needed in the proofs of Theorem 2 and Theorem 3.

By P_k we will mean a path of length k (k edges and $k+1$ vertices). A graph that is the vertex disjoint union of m copies of a graph H will be denoted by mH. Thus, for example, a matching with m edges will be designated by mP_1. We will be dealing with graphs that are the vertex disjoint union of short paths (length at most 3), so it will be convenient to have special notation to describe these graphs. Let r_1, r_2, r_3, and r be non-negative integers with $r_1+2r_2+3r_3=r$. If a graph H has r edges and is the vertex disjoint

union of r_i paths of length i, $(1 \le i \le 3)$, then H will be called an $(r_1;r_2;r_3)$-graph. Thus $H = r_1P_1 \cup r_2P_2 \cup r_3P_3$.

A graph that is the union of two matchings is bipartite and has maximal degree at most 2, and thus is a vertex disjoint union of paths and even cycles. The following well-known but trivial result concerning such graphs will be useful. A short proof of the result is also included, since it uses a technique that will be used frequently in the remaining proofs.

LEMMA 4: If R and B are two edge disjoint matchings, then there is a decomposition of $R \cup B$ into two matchings whose size differ by at most 1.

PROOF: The graph $R \cup B$ is the disjoint union of paths and even cycles. If R has at least 2 more edges than B, then there is at least one odd length alternating path in $R \cup B$ with one more edge in R than in B. Interchanging the edges between R and B in this odd path will decrease the number of edges in R and increase the number of edges in B. Repeated application of this fact yields the result. ■

There are many possible decompositions of a graph that is the edge disjoint union of two matchings. The decomposition given by the next lemma is the basis for several decompositions that will be used in the proofs of the main theorems.

LEMMA 5: If R and B are two edge disjoint matchings with m edges each, then $R \cup B$ is the edge disjoint union of graphs H and M, where the components of H are paths of length 1, 2 and 3, and M is a matching with at most $\lfloor 2m/3 \rfloor$ edges.

PROOF: The graph $R \cup B$ is the disjoint union of paths and even cycles. We will think of the edges of this graph as being 2-colored, with the edges from R being red and the edges from B being blue. We will now alter this coloring to increase the number of blue edges.

If P is an odd length path with both end edges colored red, or just an isolated red edge, then change the color of each edge of P. This altering of the coloring will increase the number of blue edges, but both the red and blue graphs will remain matchings. We can now assume that the first edge of any path is colored blue. For each component of $R \cup B$ mark the red edge (if one exists) adjacent to this first blue edge if the component is a path, and mark an arbitrary red edge if the component is an even cycle. Change the coloring in these components by coloring blue every other red edge starting with the marked red edge; however, do not change the color of the "last" red edge in an even cycle. After this change in coloring, the red edges in any path or cycle will still form a matching, and the blue edges will form paths of length at most 3. In fact, all of the paths of blue edges contained in one of the components of $R \cup B$ will have length 3, except for possibly one path.

We can now define the graphs H and M, by placing the blue edges in H and the red edges in M. All paths of length at most 3 in $R \cup B$ are clearly in H; in fact, a path of length k will have $\lfloor k/4 \rfloor$ edges in M, and a cycle of (even) length k will have $\lceil k/4 \rceil$ edges in M. In particular, a cycle of length 6 will have two edges in M, and $1/3$ is an upper bound on the fraction of edges in M for any component in $R \cup B$. This implies that the number of edges in M is at most $\lfloor 2m/3 \rfloor$, which completes the proof. ■

Note that in the decomposition of $R \cup B$ derived in Lemma 5, the coloring of each path and cycle of length at least two was altered. However, one can choose to alter the coloring on only some paths and cycles. This will result in a decomposition of $R \cup B$ of the same form, but the number of edges in H will be decreased. This may be necessary in some applications, if H has too many edges.

For some non-negative integers r_1, r_2, r_3, and r with $r_1 + 2r_2 + 3r_3 = r$, the graph H of Lemma 5 is an $(r_1;r_2;r_3)$-graph. However, the paths of length 1 and 2 in H are of two types: type one paths come from a path that has length 1 or 2 respectively in $R \cup B$, and type two paths come from the decomposition of a path or even cycle of length at least 5 in $R \cup B$. Let r_i' and r_i'' $(1 \le i \le 2)$ be the number of paths of length i of type one and two respectively in H. Obviously $r_i' + r_i'' = r_i$, and since the decomposition of any longer path in $R \cup B$ will generate at most one path of length 1 or 2 in H, $r_1' + r_2'' \le r_3$. This additional information about H can be described by calling H an $(r_1', r_1''; r_2', r_2''; r_3)$-graph.

Clearly, any $(r_1;r_2;r_3)$-graph has subgraphs isomorphic to each of the following:

$$(r_1 + r_2 + 2r_3; 0; 0) - \textit{graph},$$
$$(r_1 + 2r_3; r_2; 0) - \textit{graph},$$
$$(r_1 + r_2; 0; r_3) - \textit{graph}.$$

However, more can be concluded about subgraphs of the graph H of Lemma 5. Also, we can conclude more about the decompositions of a graph that is the union of two matchings like the $R \cup B$ graph of Lemma 5. The next lemma describes a class of these decompositions.

LEMMA 6: Let R and B be two edge disjoint matchings, each with m edges, and assume that $R \cup B$ is the edge disjoint union of graphs H and M as described in Lemma 5. If H is an $(r_1', r_1''; r_2', r_2''; r_3)$-graph with $r_1 = r_1' + r_1''$, $r_2 = r_2' + r_2''$, and $r = r_1 + 2r_2 + 3r_3$, then M is a matching with at most $\lfloor (2m - r_1' - 2r_2')/3 \rfloor$ edges, and, in addition, $R \cup B$ can be decomposed into a matching and a graph isomorphic to each of the following:

a $(i;0;0)$ – graph

with $r_1'' + r_2 + 2r_3 \le i \le r_1 + r_2 + 2r_3$;

a $(i;j;0)$ – graph

with $0 \le j \le r_2$, and $r_1'' + r_2 - j + 2r_3 \le i \le r_1 + r_2 - j + 2r_3$;

a $(i;0;k)$ – graph

with $0 \le k \le r_3$, and $r_1'' + r_2 + 2(r_3 - k) \le i \le r_1 + r_2 + 2(r_3 - k)$.

PROOF: The restriction on the number of edges in M is a direct consequence of the observation made in the proof of Lemma 5, but applied only to the paths and cycles of $R \cup B$ with at least 4 edges.

Because of the way H and M of Lemma 5 were selected, the following edges can be removed from H and added to M without destroying the fact that M is a matching;

(1) the "middle" edge of any path of length 3 in H,
(2) one of the edges of any path of length 2 in H, and
(3) the edge of a type one path of length 1 in H.

The appropriate moving of edges from H to M gives the above decompositions of $R \cup B$, and completes the proof of Lemma 6. ■

In the case when G is a forest, Theorem 2 can be improved, and a weaker condition on the maximum degree of G that insures an ASD is given in Theorem 3. The basis for this improvement is that in a forest, the union of two edge disjoint matchings will contain only paths. Obviously, any path can be decomposed into a matching and a $(i;j;0)$-graph; thus, no paths of length 3 will be needed in the next two lemmas. We state these lemmas without proof, since the techniques used in the proofs of Lemmas 5 and 6 will clearly suffice.

LEMMA 7: Let R and B be two edge disjoint matchings of a forest having m edges each. Then, $R \cup B$ is the edge disjoint union of graphs H and M, where H is the disjoint union of paths of length 1 and 2, and M is a matching with at most $\lfloor 2m/3 \rfloor$ edges.

LEMMA 8: Let R and B be two edge disjoint matchings of a forest having m edges each, and assume that $R \cup B$ is the edge disjoint union of graphs H and M as described in Lemma 7. If H is an $(r_1', r_1''; r_2; 0)$-graph with $r_1 = r_1' + r_1''$, and $r = r_1 + 2r_2$, then, M is a matching with at most $\lfloor (2m - r_1')/3 \rfloor$ edges, and, in addition, $R \cup B$ can be decomposed into a matching and a graph isomorphic to each of the following:

a $(i;0;0)$ – graph

with $\lfloor m - r_1'/2 \rfloor \le i \le \lceil m + r_1'/2 \rceil$;

a $(i;j;0)$ – graph

with $0 \le j \le r_2$, and $r_1'' + r_2 - j \le i \le r_1 + r_2 - j$.

3. MAIN RESULTS

With the notation and preliminary results of the previous section, we are prepared to prove the main two results, Theorem 2 and Theorem 3.

THEOREM 2: If G is a graph with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (2-\sqrt{2})n \rfloor$, then G has an ASD. Futhermore, each of the graphs in the decomposition of G is the vertex disjoint union of paths of length at most 3.

PROOF: Let $t = \lfloor (2-\sqrt{2})n \rfloor$, $s = \lfloor \binom{n+1}{2}/t \rfloor$ and $s' = \binom{n+1}{2} - st$. By Vizing's theorem [6], the edge chromatic number of G is at most t. Therefore, G can be decomposed into t matchings, and by Lemma 4, we can assume that the number of edges in any pair of these matchings differ by at most 1. Thus, each matching has at least s edges. Note that $s \ge \lfloor (n+1)/(4-2\sqrt{2}) \rfloor$. It follows immediately that G can be decomposed into $t+1$ matchings, say $M_1, M_2, \ldots, M_{t+1}$, each with s edges, except for M_{t+1} which has s' (possibly 0) edges. Using these matchings, we will construct graphs G_i $(1 \le i \le n)$ that form an ASD for G. Note that if $n \le 5$, then $\Delta(G) = 1$ and G is a matching, so Theorem 2 follows trivially. Thus, we can assume $n > 5$, which implies that $s \le n$.

For $1 \le i < j \le t$, consider the graph $G_{ij} = M_i \cup M_j$. By Lemma 5, G_{ij} has a decomposition into a matching M_{ij} and a $(r_1', r_1''; r_2', r_2''; r_3)$-graph H_{ij} with $r_1 = r_1' + r_1''$, $r_2 = r_2' + r_2''$, and $r = r_1 + 2r_2 + 2r_3$. (Of course, the parameters r_1, r_2, r_3, and r depend on i and j.) Also, by Lemma 6, we have that $r_1'' + r_2'' + r_3 \ge \lceil 2(2s - r_1' - 2r_2')/3 \rceil$.

The graphs G_{ij} will be partitioned into three categories that depend on the parameters r_1', r_1'', r_2', r_2'' and r_3. It will be shown that if G_{ij} is in category k $(k = 1, 2,$ or $3)$, then H_{ij} has a subgraph with n edges that is the vertex disjoint union of a matching and paths of length k.

(1) If $r_1' \ge 2(n-s)$, then place G_{ij} in category 1. Because of the nature of the subgraph H_{ij}, at least one half of the edges of G_{ij} that do not contribute to the parameter r_1' contribute to the expression $r_1'' + r_2 + 2r_3$. Therefore, $r_1'' + r_2 + 2r_3 \ge (2s - r_1')/2$, and so H_{ij} has a subgraph that is a matching with at least

$$r_1 + r_2 + 2r_3 \ge r_1' + (2s - r_1')/2 \ge n \text{ edges.}$$

(2) If $r_2' \ge (n-s)$, and G_{ij} is not in category 1, then place G_{ij} in category 2. Again, because of the nature of the subgraph H_{ij}, at least one half of the edges of G_{ij} that do not contribute to the parameter r_2', contribute to the expression $r_1 + r_2'' + 2r_3$. Therefore, $r_1 + r_2'' + 2r_3 \ge (2s - 2r_2')/2$, and so H_{ij} has a subgraph that is the union of paths of length 2 and a matching, and this subgraph has at least

$$r_1 + 2r_2 + 2r_3 \ge 2r_2' + (2s - 2r_2')/2 \ge n \text{ edges.}$$

(3) The remaining G_{ij} will be placed in category 3. In this case, H_{ij} has a subgraph that is the union of paths of length 3 and a matching, and the subgraph has at least

$$r_1 + r_2 + 3r_3 \ge r_1' + r_2' + 2(2s - r_1' - 2r_2')/3 \ge 4s/3 - r_2'/3 \ge n \text{ edges.}$$

The above choices of categories for the G_{ij} confirm our claim.

Consider this as defining a 3-coloring of the edges of the complete graph K_t whose vertices are the t matchings $M_1, M_2, \ldots, M_t$. Thus, the edge between M_i and M_j is assigned color k if the graph G_{ij} is in category k $(1 \le k \le 3)$. Since, the 3-color generalized Ramsey number for matchings $r(pP_1, pP_1, pP_1)$ is bounded above by $4p-2$, (see [3]), it follows that there is a monochromatic matching in this coloring of K_t with at least $\lfloor (t+2)/4 \rfloor$ edges. Direct calculation verifies that $t + 2 \ge 4(n-s)$, so it follows that there are $n - s$ disjoint pairs of matchings that are in the same category k, for some $1 \le k \le 3$.

The argument for each of the categories $k = 1$, 2 and 3 is essentially the same, so we will describe in detail only the representative case $k = 2$. With no loss of generality, we can assume that the first $2(n-s)$ matchings give the $(n-s)$ pairs in the same category 2. For $1 \le i \le n-s$, Lemma 6 (and the note made after the proof of Lemma 5) implies that the union of the i^{th} pair of matchings can be decomposed into a $(2s-n+i-1; n-s-i+1; 0)$-graph G_{n-i+1} with $n-i+1$ edges, and a graph $G_{2s-n+i-1}$ that is a matching with $2s-n+i-1$ edges. Thus, in particular, G_n has $n-s$ paths of length 2 with the remaining edges forming a matching, and G_{s+1} has 1 path of length 2 and the remaining edges form a matching. If we let $G_s = M_{2(n-s)+1}$, a matching with s edges, then the G_j $(2s-n \le j \le n)$ form the top part of an ascending chain of graphs with the subgraph property.

The $t + 2s - 2n$ matchings $M_{2(n-s)+2}, M_{2(n-s)+3}, \ldots, M_{t+1}$ induce a graph L with $\binom{2s-n}{2}$ edges. If $2s-n$ is even, then L can be decomposed into $(2s-n)/2$ matchings with each matching having $2s-n-1$ edges. One of these matchings gives G_{2s-n-1}, and any one of the remaining matchings can be decomposed into two matchings with j and $2s-n-j-1$ edges respectively, for $(1 \le j < (2s-n)/2)$. For $2s-n$ odd, a decompostion of L into $(2s-n-1)/2$ matchings with each matching having $2s-n$ edges will give a similar decomposition. In either case, the G_i for $i < 2s-n$ can be determined. Therefore, for $k=2$, the graph G has an ASD.

The arguments for the cases $k=1$ and 3 are similar to the case $k=2$. Of course, the graphs G_j for $j > s$ will involve paths of length 1 and 3 respectively, instead of paths of length 2. This completes the proof of Theorem 2. ■

Since $(2-\sqrt{2})n > 7n/12 > n/2$, Theorem 2 improves by more than $n/12$ the result in [5] on the maximum degree in a graph G that implies as ASD. However, as noted earlier, the decomposition graphs given in [5] involved only matchings.

We now give an outline of the proof of Theorem 3, which is the analogue of Theorem 2 for forests. The critical difference in the proof of Theorem 3 is that there are only two categories of unions of pairs of disjoint matchings. Thus, the Ramsey result needed in Theorem 3 is a 2-color result instead of the 3-color Ramsey number needed in Theorem 2.

THEOREM 3: If G is a forest with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor(3-\sqrt{3})n/2\rfloor$, then G has an ASD. Futhermore, each of the graphs in the decomposition of G is a disjoint union of paths of length at most 2.

OUTLINE OF PROOF: Let $t = \lfloor(3-\sqrt{3})n/2\rfloor$, $s = \lfloor\binom{n+1}{2}/t\rfloor$ and $s' = \binom{n+1}{2} - st$. As before, we can assume that G is decomposed into $t+1$ matchings, say $M_1, M_2, \ldots, M_{t+1}$, each with s edges $(s \ge \lfloor n/(3-\sqrt{3})\rfloor)$, except for M_{t+1} which has s' (possibly 0) edges. Using these matchings, we will construct graphs G_i $(1 \le i \le n)$ that form as ASD for G.

For $1 \le i < j \le t$, we consider the graph $G_{ij} = M_i \cup M_j$, which has a decomposition into a matching M_{ij} and an $(r_1', r_1''; r_2; 0)$-graph H_{ij} with $r_1 = r_1' + r_2''$, $r = r_1 + 2r_2$ and $r_1'' + 2r_2 \ge \lceil 2(2s - r_1')/3\rceil$. Similar to what was done in the proof of Theorem 2, but now using Lemma 8, the graphs G_{ij} will be partioned into two categoreis. If $r_1 \ge 2(n-s)$, then G_{ij} is in category 1; otherwise, G_{ij} is in category 2.

Consider this as defining a 2-coloring of the edges of a K_t, whose vertices are the t matchings $M_1, M_2, \ldots, M_t$, with the edge between M_i and M_j colored k if the graph G_{ij} is in category k $(k=1,2)$. Since the 2-color generalized Ramsey number $r(pP_1, pP_1)$ is bounded above by $3p-1$, a monochromatic matching in K_t with at least $\lfloor(t+1)/3\rfloor$ edges results. Direct calculations show that $t+1 \ge 3(n-s)$; so, it follows that there are $n-s$ disjoint pairs of matchings that are in the same category, either 1 or 2.

We again consider the case when $k=2$, and assume that the first $2(n-s)$ matchings give

the $n-s$ pairs in the same category 2. Using Lemma 8 we have, for $1 \le i \le n-s$, that the union of the i^{th} pair of matchings can be decomposed into a $(2s-n+i-1; n-s-i+1; 0)$-graph G_{n-i+1} with $n-i+1$ edges, and a graph $G_{2s-n+i-1}$ that is a matching with $2s-n+i-1$ edges. If we let $G_s = M_{2(n-s)+1}$, then the graphs G_j $(2s-n \le j \le n)$ are part of the ascending chain desired. The $t+2s-2n$ matching $M_{2(n-s)+2}, M_{2(n-x)+3}, \ldots, M_{t+1}$ induce a graph L with $\binom{2s-n}{2}$ edges. Just as in the proof of Theorem 2, the graph L can be decomposed to obtain the remaining terms of the ascending subgraph decomposition. Therefore, for $k=2$, the graph G has an ASD. The $k=1$ case is similar, so this completes the outline of the proof of Theorem 3. ■

4. OPEN QUESTIONS

There are numerous open questions concerning the existence and nature of ASD's for graphs. The principal question is, of course, to determine if every graph with $\binom{n+1}{2}$ edges has an ASD. There are interesting classes of graphs for which the same questions can be considered. For example, do all such forests have an ASD? One can also restrict the nature of the graphs that are allowed in the decompositions of the ASD. In fact, there is some evidence to indicate that any appropriate forest has an ASD with star forests as the decomposition graphs.

There are also questions directly related to the results presented here. Can one improve the condition on the maximal degree of a graph (forest) with $\binom{n+1}{2}$ edges that insures that there is an ASD with the decomposition graphs being the disjoint union of paths of length at most three (two)? To determine the largest such maximal degree is probably very difficult.

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