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# Neighborhood Closures for Graphs

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In [1], Bondy and Chvatal made the following definition: Let $P$ be a property defined for all graphs of order $n$ and let $k$ be an integer. Then $P$ is said to be *k-degree stable* if, for all graphs $G$ of order $n$, whenever $G+uv$ has property $P$ and $\deg_G u + \deg_G v \geq k$ then $G$ itself has property $P$. Among the results established were the following.

(1) $G$ contains an $sK_2(2s \leq n) \Longrightarrow (2s-1)$-degree stable

(2) $G$ is hamiltonian $\Longrightarrow n$-degree stable

(3) $G$ contains a hamiltonian path $\Longrightarrow (n-1)$-degree stable

(4) $G$ contains a $C_s(5 \leq s \leq n) \Longrightarrow (2n-s)$-degree stable

These results motivated the definition of the $k$-degree closure of a graph. The *k-degree closure* $D_k(G)$ of a graph $G$ of order $n$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $k$ (in the resulting graph at each stage) until no such pair remains. Equivalently, $D_k(G)$ is the smallest graph $H$ of order $n$ such that $G \subset H$ and

$$\deg_H(u) + \deg_H(v) < k$$

for all $uv \notin E(H)$.

Since, for $n$ sufficiently large, the complete graph of order $n$ has all properties (1)–(4), it follows that if $D_k(G)$ is complete, $k = 2s-1, n, n-1$, or $2n-s$,

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respectively, then $G$ contains an $sK_2$, is hamiltonian, contains a hamiltonian path, or contains a $C_s$ respectively.

In [2] and [3], the authors considered several problems for graphs satisfying the condition

$$|N_G(u) \cup N_G(v)| \geq s$$

for every pair $u$, $v$ of nonadjacent vertices of a graph $G$, where the *neighborhood* $N_G(u)$ of a vertex $u$ is the set of all vertices adjacent to $u$. In this paper we examine the analogs of degree stability and degree closure for neighborhood unions.

Let $P$ be a property defined for all graphs $G$ in a class $\mathcal{G}$ and let $k$ be an integer. Then $P$ is $k$-neighborhood stable in $\mathcal{G}$ if whenever $G + uv$ has property $P$, where $G \in \mathcal{G}$ and $|N_G(u) \cup N_G(v)| \geq k$, then $G$ itself has property $P$. The *$k$-neighborhood closure* $N_k(G)$ of a graph $G$ is then defined to be the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices $u$, $v$ for which the cardinality of the union of the neighborhoods of $u$ and $v$ is at least $k$ (in the resulting graph at each stage), until no such pair remains. As with degree closures, it is of interest to know when $N_k(G)$ is complete.

Our first results involve the neighborhood stability of the property of containing an $s$-matching $sK_2$.

**Theorem 1.** *Let $n$, $s$ and $\delta$ be integers satisfying $n \geq 2s + 1 \geq 3$. If $\mathcal{G}$ denotes the class of graphs of order $n$ with minimum degree $\delta(G) \geq \delta$ then the property of containing $sK_2$ is $(2s - 1 - \delta)$-neighborhood stable in $\mathcal{G}$.*

**Proof.** Let $G \in \mathcal{G}$ and $u$, $v$ be nonadjacent vertices of $G$ such that $|N_G(u) \cup N_G(v)| \geq 2s - 1 - \delta$ and $sK_2 \subset G + uv$. If $uv$ is not an edge in the $s$-matching in $G + uv$, then $sK_2 \subset G$. Thus we may assume that

$$F = \{uv, u_1v_1, u_2v_2, \ldots, u_{s-1}v_{s-1}\}$$

is an $s$-matching in $G + uv$. Let

$$W = V(G) - \{u, u_1, \ldots, u_{s-1}, v, v_1, \ldots, v_{s-1}\}.$$

Since $n \geq 2s + 1$, $W \neq \emptyset$. If $\langle W \cup \{u, v\}\rangle_G$ is nonempty, then again $sK_2 \subset G$. Thus we may assume that $W \cup \{u, v\}$ is an independent set of vertices in $G$. If, for some $i$, $u$ is adjacent to one of the vertices $u_i$ and $v_i$, say $u_i$, and $v_i$ is adjacent to a vertex $w$ of $W$, then

$$\{u_1v_1, \ldots, u_{s-1}v_{s-1}\} - \{u_iv_i\} \cup \{uu_i, v_iw\}$$

is an $s$-matching in $G$. Similarly, if, for some $i$, $v$ is adjacent to one of the vertices $u_i$ and $v_i$, say $v_i$, and $u_i$ is adjacent to a vertex $w$ of $W$, then $sK_2 \subset G$. Now, let

$$\mathrm{N} = N_G(u) \cup N_G(v) \subseteq \{u_1, \ldots, u_{s-1}, v_1, \ldots, v_{s-1}\}$$

and let $\overline{\mathrm{N}} = \{y : xy \in F \text{ and } x \in \mathrm{N}\} \subseteq \{u_1, \ldots, u_{s-1}, v_1, \ldots, v_{s-1}\}$. Let $w \in W$. Then $N_G(w) \subseteq \{u_1, \ldots, u_{s-1}, v_1, \ldots, v_{s-1}\}$. Since $|\overline{\mathrm{N}}| \geq 2s - 1 - \delta$ and $|N_G(w)| \geq \delta$, it follows that $\overline{\mathrm{N}} \cap N_G(w) \neq \emptyset$ and the proof is complete. ■

The result of Theorem 1 extends to graphs of order $n + 2s$ if we restrict our attention to connected graphs. The proof relies on Tutte's theorem [4] which states that a nontrivial graph $G$ has a 1-factor if and only if for every proper subset $S$ of $V(G)$, the number of odd (order) components of $G - S$ does not exceed $|S|$.

**Theorem 2.** *Let $n$, $s$, and $\delta$ be integers satisfying $n = 2s \geq 2$. If $\mathcal{G}$ denotes the class of connected graphs $G$ of order $n$ with minimum degree $\delta(G) \geq \delta$, then the property of containing $sK_2$ is $(2s - 1 - \delta)$-neighborhood stable in $\mathcal{G}$.*

**Proof.** Assume, to the contrary, that there exists $G \in \mathcal{G}$ with nonadjacent vertices $u$ and $v$ such that $sK_2 \subset G + uv$ and $|N_G(u) \cup N_G(v)| \geq 2s - 1 - \delta$, but $sK_2 \not\subset G$. Thus $G$ does not satisfy Tutte's theorem but $G + uv$ does. Necessarily, then, there is a set $S$ of $t \geq 1$ vertices of $G$ such that $G - S$ has $t + 2$ odd components $C_1, C_2, \ldots, C_{t+2}$ with $u \in V(C_{t+1})$ and $v \in V(C_{t+2})$. Since

$$N_G(u) \cup N_G(v) \subseteq V(C_{t+1}) \cup V(C_{t+2}) \cup S,$$

we have

$$|V(C_{t+1}) \cup V(C_{t+2}) \cup S| \geq 2s - \delta + 1.$$

Certainly $|V(C_1) \cup \ldots \cup V(C_t)| \geq t$. Now, suppose $\delta \geq t$. Let $w_i \in V(C_i)$, $1 \leq i \leq t$. Then $\deg_G w_i \geq \delta(G) \geq \delta$. Since $N_G(w_i) \subseteq V(C_i) \cup S$, it follows that $|V(C_i)| \geq \delta - t + 1$. Thus we conclude that

$$|V(C_1) \cup \ldots \cup V(C_t)| \geq \max\{t, t(\delta - t + 1)\},$$

implying

$$|V(G)| = 2s \geq 2s - \delta + 1 + \max\{t, t(\delta - t + 1)\}$$

and so

$$\delta \geq 1 + \max\{t, t(\delta - t + 1)\}. \tag{1}$$

If $\delta \leq t$, then equation (1) yields $\delta \geq 1 + t$, a contradiction. Assume, then, that $\delta > t$. In this case, equation (1) becomes

$$\delta \geq 1 + t(\delta - t + 1),$$

and so

$$-t^2 + (1 + \delta)t + (1 - \delta) \leq 0,$$

where $1 \leq t \leq \delta - 1$. However, it is easily verified that $-t^2 + (1 + \delta)t + (1 - \delta) > 0$ for all $t$ satisfying $1 \leq t \leq \delta - 1$, again producing a contradiction. ■

The results of Theorems 1 and 2 are best possible in the following sense. Given integers $n$ and $s$ satisfying $1 \le s \le \frac{n}{2} - 1$, let $q$ be a nonnegative integer such that $q < \min\{s, n-s-1\}$. Let $A$ be a copy of $K_{2q+1}$, $B$ a copy of $\overline{K}_{s-q}$, and $C$ a copy of $\overline{K}_{n-s-q-1}$. Construct a graph $G$ of order $n$ by adding all possible edges between $A$ and $B$ and between $B$ and $C$. Then $sK_2 \subset G$; an $s$-matching $F$ can be obtained by selecting $q$ independent edges from $A$ and $s - q$ independent edges joining vertices of $B$ and $C$. However, $(s+1)K_2 \not\subset G$. Now, let $\delta = \delta(G) = s - q$. Then $2s-1-\delta = s+q-1$. Let $u$ be the vertex of $A$ incident with no edge in $F$ and let $v$ be a vertex of $C$ incident with no edge in $F$. Then $|N_G(u) \cup N_G(v)| = s+q \ge 2s-1-\delta$ and $(s+1)K_2 \subset G + uv$; but as observed above, $(s+1)K_2 \not\subset G$.

Combining the results of Theorems 1 and 2 we obtain the following sufficient condition for a graph to contain $sK_2$.

**Theorem 3.** *Let $n$, $s$, and $\delta$ be integers satisfying $n \ge 2s \ge 2$. If $G$ is a graph of order $n$ with $\delta(G) \ge \delta$ and $N_{2s-1-\delta}(G)$ is complete, then $G$ contains $sK_2$ provided $n \ge 2s+1$ or $G$ is connected.*

**Theorem 4.** *Let $n$ and $s$ be positive integers satisfying $n \ge 2s$. If $G$ is a graph of order $n$ without isolated vertices for which $N_{2s-2}(G)$ is complete, then $G$ contains $sK_2$ provided $n \ge 2s+1$ or $G$ is connected.*

The corresponding result for degree closure states that if $G$ is a graph of order $n \ge 2s \ge 2$ and $D_{2s-1}(G)$ is complete, then $G$ contains $sK_2$. Consider the graph $G$ obtained from two disjoint copies $A$ and $B$ of $K_s - e$, $s \ge 3$, where two vertices of degree $s-2$, one from $A$ and one from $B$, are joined by an edge and the remaining two vertices of degree $s-2$ are joined by a path of length 2. Then $G$ has order $n = 2s+1$. Furthermore, $N_{2s-2}(G)$ is complete and, consequently, $G$ contains $sK_2$ by Theorem 4. However, since the maximum degree of $G$ is $s-1$, the $(2s-1)$-degree closure $D_{2s-1}(G)$ of $G$ is not complete and so the Bondy-Chvátal result does not offer any new information when applied to $G$. By adding vertices to $G$, maintaining a maximum degree of at most $s-1$, we obtain graphs $G'$ of all orders $n \ge 2s+1$ for which $N_{2s-2}(G')$ is complete but $D_{2s-1}(G')$ is not complete and, in fact, $D_{2s-1}(G') = G'$.

Although Theorem 4 is distinct from existing results for graphs of order at least $2s+1$, in the case of connected graphs $G$ of order $n = 2s$, we have that if $N_{2s-2}(G) = K_{2s}$ then $D_{2s-1}(G) = K_{2s}$. This follows from our next result.

**Theorem 5.** *Let $G$ be a connected graph of order $n \ge 2$ for which $D_{n-1}(G) = G$. If $G \ne K_n$, then $N_{n-2}(G) \ne K_n$.*

**Proof.** If $G$ contains 3 mutually nonadjacent vertices, say $u, v, w$, then $|N_G(u) \cup \cup N_G(v)| \le n-3$ and, consequently, $N_{n-2}(G) \ne K_n$. Thus we may assume that $\overline{K}_3$ is not an induced subgraph of $G$. Let $x$ be a vertex of $G$ of minimum degree and let $H = V(G) - \{x\} - N_G(x)$. Then $|H| = n-1-m$, where $m = |N_G(x)| \ge 1$.

Since $G \neq K_n$, we have that $2n-1-m \geq 1$. Furthermore, since $\overline{K}_3$ is not an induced subgraph of $G$, the graph $\langle H \rangle_G$ is complete. Suppose there exists an edge $uw$, where $u \in N_G(x)$ and $w \in H$. Then

$$\deg_G x + \deg_G w = m + [(n-1-m-1)+1] = n-1,$$

which contradicts the assumption that $D_{n-1}(G) = G$. However, if no edge exists between $N_G(x) \cup \{x\}$ and $H$, then $G$ is disconnected. Thus if $G$ is a connected graph of order $n \geq 2$ for which $D_{n-1}(G) = G$ and $G \neq K_n$, then $N_{n-2}(G) \neq K_n$. ∎

**Corollary 5.** *Let $G$ be a connected graph of order $n \geq 2$. If $N_{n-2}(G) = K_n$ then $D_{n-1}(G) = K_n$.*

For $n$ even, the graph $2K_{\frac{n}{2}}$ shows that a graph $G$ of order $n$ must be connected for the condition $N_{n-2}(G) = K_n$ to imply that $D_{n-1}(G) = K_n$.

In light of Corollary 5, it is reasonable to ask whether, for a connected graph $G$ of order $n$, the condition $N_{n-3}(G) = K_n$ implies $D_{n-2}(G) = K_n$ and, more generally, if $N_{n-l}(G) = K_n$ implies $D_{n-l+1}(G) = K_n$. (Clearly, if $N_{n-l}(G) = K_n$ then $D_{n-l}(G) = K_n$.) This, however, is not the case as our next examples illustrate.

For odd integers $n$ and $l$ satisfying $3 \leq l \leq (n+2)/3$ let $G$ be the connected graph constructed from two disjoint copies $A$ and $B$ of $K_{(n-l+2)/2} - \binom{\frac{l-1}{2}}{} K_2$ in the following manner. Introduce $l-2$ new vertices, each adjacent to one vertex of $A$ and one vertex of $B$ so that the resulting graph has $l-2$ vertices of degree 2, two vertices $u$ and $v$ of degree $\frac{n-l+2}{2} - 2$, and $n-l$ vertices of degree $\frac{n-l+2}{2} - 1$. Finally, add the edge $uv$. Then $G$ has order $n$, $N_{n-l}(G) = K_n$, but $D_{n-l+1}(G) = G$. Similarly, for even integers $n$ and $l$ satisfying $4 \leq l \leq (n+4)/3$, $G$ is constructed from two disjoint copies $A$ and $B$ of $K_{(n-l+2)/2} - \binom{\frac{l-2}{2}}{} K_2$ by adding $l-2$ new vertices, each adjacent to one vertex of $A$ and one vertex of $B$ so that the resulting graph has $l-2$ vertices of degree 2 and all remaining vertices of degree $\frac{n-l+2}{2} - 1$. Then, again, $G$ has order $n$, $N_{n-l}(G) = K_n$, but $D_{n-l+1}(G) = G$.

We next turn our attention to hamiltonian properties. Bondy and Chvatal showed that the property of being hamiltonian is $n$-degree stable. We see next that if $\mathcal{G}$ is the class of 2-connected graphs of order $n$ then the property of being hamiltonian is $(n-2)$-neighborhood stable in $\mathcal{G}$.

**Theorem 6.** *Let $\mathcal{G}$ denote the class of 2-connected graphs of order $n \geq 3$. Then the property of being hamiltonian is $(n-2)$-neighborhood stable in $\mathcal{G}$.*

**Proof.** Let $G \in \mathcal{G}$ and $u$, $v$ be nonadjacent vertices of $G$ such that $|N_G(u) \cup \cup N_G(v)| = n-2$ and $G + uv$ has a hamiltonian cycle $C$. If $uv$ is not an edge of $C$, then $C$ is also a hamiltonian cycle of $G$. Thus we may assume that $G$ contains a hamiltonian path $u = u_1, u_2, \ldots, v = u_n$. Since $|N_G(u) \cup N_G(v)| = n-2$, we have that $N_G(u) \cup N_G(v) = \{u_2, u_3, \ldots, u_{n-1}\}$. If, for some $i$, $uu_i$ and $vu_{i-1}$ are edges of $G$, then $G$ contains the hamiltonian cycle

$$u = u_1, u_2, \ldots, u_{i-1}, v = u_n, u_{n-1}, \ldots, u_i, u = u_1.$$

If this is not the case, then for some $t$, $2 \le t \le n-2$, we have that $uu_i \in E(G)$, $2 \le i \le t$, and $vu_i \in E(G)$, $t+1 \le i \le n-1$.

**Case (i).** *$G$ contains an edge joining a vertex in $\{u = u_1, \ldots, u_{t-1}\}$ with a vertex in $\{u_{t+2}, \ldots, u_n = v\}$, say $u_j u_l$, where $1 \le j \le t-1$ and $t+2 \le l \le n$.*

Then $G$ contains the hamiltonian cycle

$$u = u_1, u_2, \ldots, u_j, u_l, u_{l+1}, \ldots, v = u_n, u_{l-1}, u_{l-2}, \ldots, u_{j+1}, u = u_1.$$

**Case (ii).** *Case (i) does not occur.*

Then, since $G$ is 2-connected, $G$ must contain edges $u_j u_{t+1}$ and $u_l u_t$, where $1 \le j \le t-1$ and $t+2 \le l \le n$. But then $G$ contains the hamiltonian cycle

$$u = u_1, u_2, \ldots, u_j, u_{t+1}, u_{t+2}, \ldots, u_{l-1}, v = u_n, u_{n-1}, \ldots, u_l, u_t, u_{t-1}, \ldots, u_{j+1}, u = u_1. \qquad \blacksquare$$

Since every hamiltonian graph is 2-connected, the connectivity condition in Theorem 5 cannot be weakened; consider, for example, the connected graph $G$ obtained by adding an edge between disjoint copies of $K_1$ and $K_{n-1}$. Then for any two nonadjacent vertices $u$ and $v$ of $G$ we have $|N_G(u) \cup N_G(v)| = n-2$ (the strongest possible neighborhood condition for a graph of order $n$) and $G + uv$ is hamiltonian. However, $G$ is not hamiltonian. For $n \ge 7$, let $G$ be the graph obtained from the path $P_n : u = u_1, u_2, \ldots, v = u_n$ as follows. Let $t$ satisfy $4 \le t \le n-3$ and

$$E(G) = E(P_n) \cup \{uu_i :\ 2 \le i \le t-2\} \cup \{vu_i :\ t+2 \le i \le n-1\} \cup \{uu_{t+1}, vu_{t-1}\}.$$

Then $G$ is a 2-connected graph of order $n$, $|N_G(u) \cup N_G(v)| = n-3$, and $G + uv$ is hamiltonian. However, $G$ is not hamiltonian. Thus Theorem 6 cannot be improved.

It is easy to show that the property of containing a hamiltonian path is $(n-1)$-degree stable by using the corresponding result for hamiltonian cycles. Similarly, we have the following

**Theorem 7.** *Let $\mathcal{G}$ denote the class of connected graphs of order $n \ge 2$. Then the property of containing a hamiltonian path is $(n-2)$-neighborhood stable in $\mathcal{G}$.*

**Proof.** If $n = 2$ then the theorem is vacuously true. Thus we may assume $n \ge 3$. Let $G \in \mathcal{G}$ and $u$, $v$ be nonadjacent vertices of $G$ such that $|N_G(u) \cup N_G(v)| = n-2$

and $G + uv$ has a hamiltonian path $P$. Consider the graph $G'$ obtained from $G$ by adding a vertex $w$ adjacent to every vertex of $G$. Then $G'$ is a 2-connected graph of order $n+1$. Furthermore, $|N_{G'}(u) \cup N_{G'}(v)| = n - 1 = (n+1) - 2$ and $G' + uv$ is hamiltonian. Thus, by Theorem 4, $G'$ is hamiltonian and, consequently, $G$ contains a hamiltonian path. ■

As was the case with the property of being hamiltonian, here we cannot weaken the connectivity condition; consider, for example, $K_1 \cup K_{n-1}$. Furthermore, consider the connected graph $G$ of order $n \geq 4$ obtained from disjoint copies of $P_3 : v = v_1, v_2, v_3$ and $P_{n-3} : u = u_1, u_2, \ldots, u_{n-3}$ as follows. Add all edges of the form $uu_i$, $2 \leq i \leq n-3$, and an edge of the form $v_2u_j$, $1 \leq j \leq n-3$. Then $|N_G(u) \cup N_G(v)| = n - 3$ and $G + uv$ has a hamiltonian path; however, $G$ contains no hamiltonian path.

The next two results are directed applications of Theorem 6 and 7.

**Corollary 6.** *Let $G$ be a 2-connected graph of order $n \geq 3$. If $N_{n-2}(G)$ is complete, then $G$ is hamiltonian.*

**Corollary 7.** *Let $G$ be a connected graph of order $n \geq 2$. If $N_{n-2}(G)$ is complete, then $G$ contains a hamiltonian path.*

The corresponding results for degree closures state that if $G$ is a graph of order $n$ and $D_n(G)$ is complete ($D_{n-1}(G)$ is complete), then $G$ is hamiltonian (respectively, $G$ contains a hamiltonian path). By Corollary 5, for a connected graph $G$ of order $n \geq 2$, the condition $N_{n-2}(G) = K_n$ implies that $D_{n-1}(G)$ is complete. Thus Corollary 7 in fact follows from a known result for degree closures. Similarly, for a 2-connected graph $G$ of order $n \geq 6$, the condition $N_{n-2}(G) = K_n$ implies that $D_n(G)$ is complete, so that Corollary 6 can be obtained from a degree closure result. This follows from Theorem 8.

**Theorem 8.** *Let $G$ be a 2-connected graph of order $n \geq 6$ for which $D_n(G) = G$. If $G \neq K_n$, then $N_{n-2}(G) \neq K_n$.*

**Proof.** If $\overline{K}_3$ is an induced subgraph of $G$, then $N_{n-2}(G) \neq K_n$. Thus we may assume that $\overline{K}_3$ is not an induced subgraph of $G$. We note that if $w \in V(G)$, then $\langle V(G) - \{w\} - N_G(w)\rangle$ is complete since there are no induced $\overline{K}_3$'s. Now, let $v$ be a vertex of $G$ such that $\deg_G v \leq n-4$. (Certainly $\delta(G) < \frac{n}{2}$ and for $n \geq 6$ we can thus find such a vertex $v$.) Let $m = \deg_G v$ and $W = V(G) - \{v\} - N_G(v)$. Thus, $\langle W\rangle$ is complete. Since $G$ is 2-connected, there exist $a, b \in N_G(v)$ and $c, d \in W$ such that $ad, bc \in E(G)$. (Note that $a$, $b$, $c$, $d$ are all distinct.) Since $D_n(G) = G$, we see that $N_G(c) = W - \{c\} \cup \{b\}$; for otherwise,

$$\deg_G v + \deg_G c \geq m + n - m - 2 + 2 = n,$$

a contradiction. Similarly, $N_G(d) = W - \{d\} \cup \{a\}$.

Thus,

$$V(G) - \{c\} - N_G(c) = N_G(v) \cup \{v\} - \{b\}$$

and

$$V(G) - \{d\} - N_G(d) = N_G(v) \cup \{v\} - \{a\}.$$

Consequently, $\langle N_G(v) \cup \{v\} - \{b\}\rangle$ and $\langle N_G(v) \cup \{v\} - \{a\}\rangle$ are complete.

However, if $ab \in E(G)$, then $\deg_G a \geq m+1$. Therefore,

$$\deg_G a + \deg_G c \geq m + 1 + n - m - 1 = n,$$

a contradiction since $ac \notin E(G)$.

Thus, $\langle \{v\} \cup N_G(v)\rangle = K_{m+1} - e$, where $e = ab$, $\langle W\rangle$ is complete, $ad, bc \in E(G)$, and $ac, bd \notin E(G)$.

Now, for every $w \in W - \{c,d\}$, the edge $bw$ is not in $G$ for otherwise, $\deg_G b \geq m+1$ and $\deg_G b + \deg_G d \geq m+1+n-m-1 = n$, again a contradiction since $bd \notin E(G)$. Thus, for each $w \in W - \{c,d\}$ we have that $ab, bw \notin E(G)$ which implies that $aw \in E(G)$ since there are no induced $\overline{K}_3$'s in $G$. But then, $\deg_G a = n-3$ and thus,

$$\begin{aligned}\deg_G a + \deg_G c \geq& n - 3 + n - m - 1\\ =& 2n - m - 4\\ =& n - (n - m - 4)\\ \geq& n, \text{ a contradiction.} \quad \blacksquare\end{aligned}$$

The next result follows directly from Theorem 8.

**Corollary 8.** *Let $G$ be a 2-connected graph of order $n \geq 6$. If $N_{n-2}(G) = K_n$ then $D_n(G) = K_n$.*

For $n$ odd, the graph obtained by identifying a vertex of one copy of $K_{(n+1)/2}$ with a vertex of a second copy of $K_{(n+1)/2}$ shows that a graph $G$ of order $n$ must be 2-connected for the condition $N_{n-2}(G) = K_n$ to imply that $D_n(G) = K_n$.

We next turn our attention to a generalization of Theorem 6, that is, the neighborhood stability of containing the cycle $C_k$, $3 \leq k \leq n$, where new results are obtained.

We first note that for $k = 3$, 4, and 5 there exist 2-connected graphs $G$ of order $n$ with nonadjacent vertices $u$ and $v$ such that $|N_G(u) \cup N_G(v)| = n-2$ and $G + uv$ is pancyclic but $G$ contains cycles of all possible orders except $C_k$. For $k = 3$ and $n \geq 5$ let $G$ be the graph obtained from the complete bipartite graph $K\left(\left\lfloor \frac{n-3}{2} \right\rfloor, \left\lceil \frac{n-3}{2} \right\rceil\right)$ by adding a vertex $u$ adjacent to all vertices in one of the partite sets, a vertex $v$ adjacent to all vertices in the other partite set, and a

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vertex $w$ adjacent only to $u$ and $v$. For $k = 4$ and $n = 4t + 2$, $t \geq 2$, we construct $G$ as follows. Let

$$V(G) = \{u, v, u_1, u_2, \ldots, u_{2t}, v_1, v_2, \ldots, v_{2t}\}$$

and

$$\begin{aligned} E(G) = &\{u_i u_{i+1} : i = 1, 3, \ldots, 2t-1\} \cup \{v_i v_{i+1} : i = 1, 3, \ldots 2t-1\} \\ &\cup \{uu_i : 1 \leq i \leq 2t\} \cup \{vv_i : 1 \leq i \leq 2t\} \\ &\cup \{u_i v_{i+1} : 1 \leq i \leq 2t-1\} \cup \{u_{2t} v_1\}. \end{aligned}$$

Finally, for $k = 5$ and $n \geq 9$, we construct $G$ by beginning with disjoint copies of $P_5 : u = u_1, u_2, \ldots, u_5 = v$ and $P_{n-5} : v_1, v_2, \ldots, v_{n-5}$ and add the edge $uu_3$. The construction is completed by adding the edges

$$uv_1, uv_2, uv_3, vv_4, vv_5, vv_6, uv_7, uv_8, uv_9, vv_{10}, vv_{11}, vv_{12}, \text{ etc.}$$

**Theorem 9.** *Let $\mathcal{G}$ denote the class of 2-connected graphs of order $n \geq 6$. Then the property of containing $C_k$, $6 \leq k \leq n$, is $(n-2)$-neighborhood stable in $\mathcal{G}$.*

**Proof.** Let $G \in \mathcal{G}$ and $u$, $v$ be nonadjacent vertices of $G$ such that $|N_G(u) \cup \cup N_G(v)| = n - 2$ and $C_k \subset G + uv$. As in the proof of Theorem 6, we can assume that $G$ contains a path $u = u_1, u_2, \ldots, u_k = v$, and that for some $t$, $2 \leq t \leq k-2$, we have that $uu_i \in E(G)$, $2 \leq i \leq t$, and $vu_i \in E(G)$, $t+1 \leq i \leq k$. Furthermore, there exists no $i$ such that $uu_i$ and $vu_{i-1}$ are edges of $G$, $2 \leq i \leq k$.

**Case (i).** *There exists a $u_j - u_l$ path $P$ in $G$, where $1 \leq j \leq t-1$ and $t+2 \leq l \leq k$, such that $V(P) \cap \{u = u_1, u_2, \ldots, v = u_k\} = \{u_j, u_l\}$.* Assume that $P$ has been chosen as the shortest such path. If $P$ has length 1, then

$$u = u_1, u_2, \ldots, u_j, u_l, u_{l+1}, \ldots, v = u_k, u_{l-1}, u_{l-2}, \ldots, u_{j+1}, u = u_1$$

is a $k$-cycle of $G$. Suppose, then, that $P$ has length 2, say $P : u_j, w, u_l$. Since $|N_G(u) \cup N_G(v)| = n-2$, it follows that at least one of $u$ and $v$ is adjacent to $w$. Without loss of generality, assume $uw \in E(G)$. If $l \geq t+3$, then

$$u = u_1, w, u_l, u_{l+1}, \ldots, v = u_k, u_{l-2}, u_{l-3}, \ldots, u = u_1$$

is a $k$-cycle of $G$. If $l = t+2$ and $t+2 \neq k-1$, then

$$u = u_1, w, u_l, u_{l+1}, \ldots, u_{k-2}, u_k, u_{t+1}, u_t, \ldots, u = u_1$$

is a $k$-cycle of $G$. Finally, if $l = t+2 = k-1$, then $u = u_1, w, u_{t+2}, u_{t+1}, \ldots, u = u_1$ is a $k$-cycle of $G$. Thus we may assume that $P$ has length 3 or more. Again, since $|N_G(u) \cup N_G(v)| = n-2$, every internal vertex of $P$ is adjacent to at least one of

$u$ and $v$. However, it then follows that since $P$ was chosen to be a shortest path from $u_1, u_2, \ldots, u_{t-1}$ to $u_{t+2}, u_{t+3}, \ldots, u_k$ such that $|V(P) \cap \{u_1, u_2, \ldots, u_k\}| = 2$, there exists a $u - v$ path $P'$ of length 3 such that $|V(P') \cap \{u_1, u_2, \ldots, u_k\}| = 2$. Clearly, then, $G$ contains a $k$-cycle.

**Case (ii).** *Case (i) does not occur.* Then, since $G$ is 2-connected there exists a $u_j - u_{t+1}$ path $P_1$, $1 \leq j \leq t-1$, and a $u_t - u_l$ path $P_2$, $t+2 \leq l \leq k$, such that

$$V(P_1) \cap V(P_2) = \emptyset \text{ and } (V(P_1) \cup V(P_2)) \cap \{u_1, u_2, \ldots, u_k\} = \{u_j, u_l, u_t, u_{t+1}\}.$$

Assume that $P_1$ and $P_2$ have been chosen to minimize their total length. Since $|N_G(u) \cup N_G(v)| = n-2$, every internal vertex of $P_1$ and $P_2$ is adjacent to at least one of $u$ and $v$. However, since Case (i) does not occur, it follows that $u$ is adjacent to every internal vertex of $P_1$ and $v$ is adjacent to every internal vertex of $P_2$. Thus, without loss of generality, we may assume that if $|V(P_1)| \geq 3$ then, in fact, $P_1 : u, w, u_{t+1}$ and, similarly, that if $|V(P_2)| \geq 3$, then $P_2 : v, z, u_t$. If $|V(P_1)| = |V(P_2)| = 2$, then

$$u = u_1, u_2, \ldots, u_j, u_{t+1}, u_{t+2}, \ldots, u_{l-1}, v = u_k, u_{k-1}, \ldots, u_l, u_t, u_{t-1}, \ldots, u_{j+1}, u = u_1$$

is a $k$-cycle of $G$. Suppose then, $|V(P_1)| = |V(P_2)| = 3$. If $t \geq 4$, then

$$u = u_1, w, u_{t+1}, u_{t+2}, \ldots, v = u_k, z, u_t, u_{t-1}, \ldots, u_4, u = u_1$$

is a $k$-cycle of $G$. Similarly, if $k - t \geq 4$, then

$$u = u_1, w, u_{t+1}, u_{t+2}, \ldots, u_{k-3}, v = u_k, z, u_t, u_{t-1}, \ldots, u = u_1$$

is a $k$-cycle of $G$. If $t \leq 3$ and $k - t \leq 3$ then, since $k \geq 6$, we must have $t = 3$ and $k - t = 3$. Thus $k = 6$ and $|V(P_2)| = 3$. If $j \geq 3$, then

$$u = u_1, u_3, \ldots, u_j, u_{t+1}, u_{t+2}, \ldots, v = u_k, z, u_t, u_{t-1}, \ldots, u_{j+1}, u = u_1$$

is a $k$-cycle of $G$. If $j = 2$ then

$$u_2, u_{t+1}, u_{t+2}, \ldots, v = u_k, z, u_t, u_{t-1}, \ldots, u_2$$

is a $k$-cycle of $G$. Lastly, if $j = 1$, then either $t \geq 3$ and

$$u = u_1, u_{t+1}, u_{t+2}, \ldots, v = u_k, w, u_t, u_{t-1}, \ldots, u_3, u = u_1$$

is a $k$-cycle of $G$ or $k - t \geq 3$ and

$$u = u_1, u_{t+1}, u_{t+2}, \ldots, u_{k-2}, v = u_k, z, u_t, u_{t-1}, \ldots, u = u_1$$

is a $k$-cycle of $G$. ■

For $n \geq k \geq 6$, consider the graph $G$ obtained from the path $P_k : u = u_1, u_2, \ldots, v = u_k$ as follows. Let $t$ satisfy $3 \leq t \leq k-2$. Add the edges $uu_i$, $2 \leq i \leq t-1$, the edges $vu_i$, $t+1 \leq i \leq k-1$, and the edges $uu_{t+1}, vu_{t-1}$. Finally, add $n-k$ additional vertices, each adjacent to $u_1$ and $u_3$ if $t \geq 4$ or to $u_{k-2}$ and $u_k$ if $t = 3$. Then $G$ is a 2-connected graph of order $n$, $|N_G(u) \cup N_G(v)| = n-3$, and $C_k \subset G + uv$. However, $C_k \not\subset G$. Thus Theorem 9 cannot be improved by weakening the neighborhood condition.



**Theorem 10.** *Let*
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**Theorem 10.** *Let $G$ be a 2-connected graph of order $n \geq 6$. If $N_{n-2}(G)$ is complete, then $C_k \subset G$ for $6 \leq k \leq n$.*

We close by observing that the corresponding result for degree closure states that for integers $n$ and $s$ satisfying $n \geq s \geq 5$, if $D_{2n-s}(G)$ is complete, then $C_s \subset G$. For even $n \geq 8$, let $G$ be the graph obtained from two disjoint copies $A$ and $B$ of $K_{n/2}$ by adding $n/2$ edges from vertices of $A$ to vertices of $B$ so that the resulting graph is $(n/2)$-regular. Then $N_{n-2}(G) = K_n$ so that it follows from Theorem 9 that $C_2 \subset G$ for all $s$ satisfying $6 \leq s \leq n$. However, $D_{2n-s}(G)$ is complete only if $2n - s \leq n$, that is, only if $s = n$.

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