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Extremal Problems Involving Neighborhood Unions

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ABSTRACT

We examine several extremal problems for graphs satisfying the property $|N(x) \cup N(y)| \geq s$ for every pair of nonadjacent vertices $x, y \in V(G)$. In particular, values for s are found that ensure that the graph contains an s-matching, a 1-factor, a path of a specific length, or a cycle of a specific length.

Graph theory literature abounds with results relating the degrees of the vertices of a graph to various graph properties and parameters. In particular, Ore [7] introduced the idea of bounding the degree sum of pairs of nonadjacent vertices while studying hamiltonian properties in graphs. This concept generalized naturally to hamiltonian properties in digraphs (see [8] and [11]). Win [10] used degree sums while studying disjoint 1-factors, and Bondy and Chvátal [5] solidified this approach as a tool for studying many properties. Meanwhile, Anderson [2] used the idea of bounding the cardinality of the neighborhood $N(X)$, of a set $X \subseteq V(G)$, while studying 1-factors.

We examine a natural hybrid of these two ideas. That is, we consider graphs with the property that for some positive integer s,

$$|N(x) \cup N(y)| \geq s \qquad \text{for every pair of nonadjacent vertices } x, y. \qquad (1)$$

For simplicity, we define NC to be the $\min|N(x) \cup N(y)|$, where the minimum is taken over all pairs of nonadjacent vertices x, y in the graph. In [6], bounds

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on NC were determined that ensured various hamiltonian properties. In particular, the following was shown:

Theorem A ([6]). Let G be a 2-connected graph of order p.

(i) If $\text{NC} \geq (p - 1)/2$, then G contains a spanning path.
(ii) If $\text{NC} \geq (2p - 1)/3$, then G is hamiltonian.

In [1], property (1) was examined in relation to cliques. We now consider matchings, 1-factors, path lengths, and cycle lengths. Notation will follow [3].

1. MATCHINGS

In this section we consider the following question: If a graph satisfies the condition $\text{NC} \geq s$, how large a matching must it contain, and how is this effected by imposing connectivity conditions? We summarize our results in the following theorem. Here, $\beta_1(G)$ denotes the edge-independence number of the graph G.

Theorem 1. Let G be a graph with p vertices satisfying $\text{NC} \geq s$.

(a) If $s \leq \frac{1}{2}(p - 2)$ then $\beta_1(G) \geq s$.
(b) If $s \geq \frac{1}{2}(p - 1)$ and p is odd, then $\beta_1(G) \geq \frac{1}{2}(p - 3)$.
(c) If $s > 2\lfloor\frac{1}{3}p\rfloor - 2$ and p is odd, then $\beta_1(G) = \frac{1}{2}(p - 1)$, unless $p = 5$ and $s = 1$.
(d) If $s \geq \frac{1}{2}(p - 1)$ and p is odd and G is connected, then $\beta_1(G) = \frac{1}{2}(p - 1)$.
(e) If $s \geq \frac{1}{2}p$ and p is even, then $\beta_1(G) \geq \frac{1}{2}(p - 2)$.
(f) If $s > \frac{2}{3}(p - 1) - 1$ and p is even and G connected, then $\beta_1(G) = \frac{1}{2}p$.
(g) If $s \geq \frac{1}{2}p$ and p is even and G is 2-connected, then $\beta_1(G) = \frac{1}{2}p$.

The remainder of this section will be dedicated to verifying these results, and where possible, showing their sharpness. Before proving Theorem 1, we present examples demonstrating the sharpness of these assertions.

Example 1. On the sharpness of Theorem 1.

(a) The complete bipartite graph $K_{s,p-s}$ shows that the conclusion in (a) is best possible, even for graphs with connectivity as high as s.
(b) Consider graphs of the form $K_a \cup K_b \cup K_c$, where a, b, and c are all odd. Then these graphs show that the conclusion in (b) is best possible, even for graphs with s as high as $2\lfloor p/3\rfloor - 2$.
(c) The graph $K_{1,4}$ shows the excluded case. Otherwise, the conclusion is obviously best possible since $\beta_1(G) \leq p/2$.
(d) Again this is clearly best possible.
(e) Consider graphs of the form $K_a \cup K_b$, where a and b are odd. Clearly such graphs show that the conclusion in (e) is best possible, even for

graphs with s as high as $p - 2$ (its maximum value for noncomplete graphs). The graphs K_a, K_b, and K_c with a single vertex identified (with an odd number of a, b, and c being even) show that the conclusion is best possible, even if s is as high as $\frac{2}{3}(p - 1) - 1$.

Both (f) and (g) are clearly best possible.

Before proving Theorem 1, we need the following lemma:

Lemma 2. Let s be a fixed positive integer and let G be a graph of order p with $\beta_1(G) = t$ such that NC $\geq s$.

(i) If $\frac{3s}{4} \leq t < s$, then $2t + 3 \geq p$.
(ii) If $\frac{s}{2} \leq t < 3s/4$, then $2t + 2 \geq p$.

Proof. (i) Consider a maximal independent set A of edges in G. Let $|A| = t$ and let $B = V(G) - V(A)$. We show that $|B| \leq 3$. Suppose this is not the case, that is, suppose there exists $x, y, z, w \in B$. Partition A into the following sets: for each $a \in \{x, y, z, w\} = T$, let

$$A_a = \{uv \in A : u \in N(a), \text{ but } u, v \notin N(T - \{a\})\} \quad \text{and} \quad C = A - \cup A_a.$$

Assume without loss of generality that $|A_x| \geq |A_y| \geq |A_z| \geq |A_w|$, and that $|C| = c$.

From the maximality of A, it is clear that $N(x) \subseteq V(\langle A_x \cup C \rangle)$, and that similar statements hold for $N(y)$, $N(z)$, and $N(w)$. It is also clear that $|A_x \cup A_y| \geq (t - c)/2$ so that $|A_z \cup A_w| \leq (t - c)/2$. But by the maximality of t, the vertices z and w must be nonadjacent. Also, from the maximality of t and the definition of C, $|N_c(z) \cup N_c(w)| \leq c$. Hence $s \leq |N(z) \cup N(w)| \leq 2(t - c)/2 + c = t$, contradicting the fact that $t < s$.

(ii) An analogous argument may be used.

Proof of Theorem 1. Parts (a) and (b) follow from Lemma 2. To prove (c), suppose that the result fails to hold. Now recall Berge's defect form of Tutte's condition for the existence of a 1-factor (see [4] and [9]), which says that if p is odd, then $\beta_1(G) = \frac{1}{2}(p - 1)$ unless there is some set of r vertices whose removal leaves a graph with at least $r + 3$ odd components. But this implies that

$$\begin{aligned} s &\leq \frac{2}{r + 3}(p - r) + r - 2 \\ &\leq \frac{1}{2}(p - 1) - 1, \end{aligned}$$

since the hypothesis that $s > 2\lfloor \frac{1}{3}p \rfloor - 2$ ensures that G cannot have three odd components, and we have that $r \geq 1$. Further, we clearly have that $r \leq \frac{1}{2}(p - 3)$. Thus, we have an immediate contradiction, except when $p = 5$ and $s = 1$.

To prove (d), note that the connectivity of G implies $r \geq 1$, so we may apply the last argument.

Part (e) also follows from Lemma 2.

To prove (f), suppose the results fails to hold. By Tutte's condition [9], since p is even and there is not a 1-factor, then there must exist some set of r vertices whose removal leaves a graph with at least $r + 2$ odd components. But this implies that

$$\begin{aligned} s &\leq \frac{2}{r+2}(p - r) + r - 2 \\ &\leq \frac{2}{3}(p - 1) - 1, \end{aligned}$$

since G is connected, $r \geq 1$ and clearly $r \leq \frac{1}{2}(p - 2)$. But this contradicts the fact

$$s > \frac{2}{3}(p - 1) - 1.$$

If G is 2-connected, then $r \geq 2$ and so $s \leq (p - 2)$; thus, applying the previous argument, (g) follows as well. ∎

We conclude this section by noting that the conditions of Theorem A(ii), immediately imply that the graph contains two disjoint 1-factors.

2. PATHS AND CYCLES

In this section we consider the question, If a graph satisfies the condition $\mathrm{NC} \geq s$, how long a path must it contain, how long a cycle must it contain, and how much are these parameters increased by imposing connectivity conditions? We summarize the results of this section in the following theorem:

Theorem 2. Let G be a graph with p vertices satisfying $\mathrm{NC} \geq s$.

(a) If G is 2-connected, then G contains a path of order at least $\frac{3}{2}s + 2$ or (if $p < \frac{3}{2}s + 2$) G is traceable.

(b) If G is 2-connected, then G contains a cycle of order at least $s + 2$ or (if $p < s + 2$) G is complete. If in addition, s is odd and $p > s + 2$, then G contains a cycle of order at least $s + 3$.

(c) If G is connected, then G contains a path of order at least $s + 2$ or (if $p < s + 2$) G is complete. If, in addition, s is even and $p > s + 2$, then G contains a path of order at least $s + 3$.

(d) If G is connected and $s \geq 3$, then G contains a cycle of order at least $\frac{1}{2}(s + 2)$ or (if $p < s + 2$) G is complete.

(e) If G is not connected, then at most one component of G has fewer than $\frac{1}{2}(s + 2)$ vertices and if there is one, then it is complete. Every other component has at least $\frac{1}{2}(s + 2)$ vertices, and either is complete or contains a path of order at least $s + 2$ and (if $s \geq 3$) a cycle of order at least $\frac{1}{2}(s + 2)$.

Before proving Theorem 2, we consider examples that demonstrate the sharpness of these results.

Example 2. On the sharpness of Theorem 2.

(a) Consider graphs consisting of four or more copies of $K_{(s+4)/2}$ (s even) or $K_{(s+5)/2}$ (s odd) with exactly two vertices in common. These graphs show the results in (a) are best possible.
(b) Use the examples of (a).
(c) Graphs consisting of three or more copies of $K_{(s+3)/2}$ (s odd) or $K_{(s+4)/2}$ (s even) with a single vertex in common show the conclusions of (c) are best possible for arbitrarily large values of p.
(d) Graphs consisting of copies of $K_{(s+2)/2}$ (s even) or $K_{(s+3)/2}$ (s odd) arranged in a row, with a single edge connecting each two consecutive complete graphs, show that the conclusion of (d) is best possible for arbitrarily large values of p. These examples reduce to paths if $s = 1$ or 2 and show the need for $s \geq 3$ to force a cycle.

Proof of Theorem 2. We begin with a proof of (b) because it will make the proof of (a) somewhat easier.

Let G be a 2-connected graph of order $p \geq s + 2$ such that NC $\geq s$. Let $x_1, x_2, \ldots, x_t$ be the vertices in order along a path P of maximum length in G. Let x_i and x_j (possibly $i = j$) be vertices on P that are adjacent to x_1, and suppose that, among all possible longest paths P and vertices x_i, we have chosen ones that maximize i. Then x_1 and x_{j-1} are adjacent only to vertices in $\{x_1, \ldots, x_i\}$ (since clearly x_{j-1} is an end vertex of a path of the same length as P). Suppose $i \leq s + 1$. Then since NC $\geq s$, we see that x_1 and x_{j-1} must be adjacent. Taking j to be $i, i - 1, \ldots$ in turn, we see that x_1 is adjacent to all of $x_2, \ldots, x_i$. All vertices $x_1, \ldots, x_{i-1}$ are thus end vertices of paths of the same length as P, and so are adjacent only to vertices in $\{x_1, \ldots, x_i\}$, which contradicts the 2-connectedness of G. Thus $i \geq s + 2$, and G contains a cycle of length $s + 2$ or more.

To prove the second part of (b), suppose that s is odd and $p > s + 2$, and suppose that there is no cycle of order $s + 3$ or more. Then, with P and x_i as before, $i = s + 2$. Note that $t > s + 2$, since G is connected and contains a cycle of order $s + 2$, and so contains a path whose order is greater than $s + 2$.

Now x_t and all successors along P of neighbors of x_t are end vertices of paths of the same length as P. If none of these vertices are adjacent to any of $x_1, \ldots, x_{i-1}$, then we can repeat the previous argument at the other end of P to deduce that G contains another cycle of order $s + 2$ that has at most one vertex (namely x_i) in common with the first; whence, since G is 2-connected, it certainly contains a cycle of order $s + 3$ or more (and, we note for future reference, a path of order at least $\frac{3}{2}s + 2$).

So we may suppose that at least one of these vertices, without loss of generality, x_t itself, is adjacent to $r \geq 1$ vertices $x_{j_1}, \ldots, x_{j_r}$ $(2 \leq j_1 < \ldots < j_r \leq i - 1)$. For use later, in the proof of (a), we note that we now assume only that G contains no P_{t+1} and no P_t that starts with a cyle of order $s + 3$ or more. Note that

$$x_{j_1+1}, \ldots, x_{j_r+1} \notin N(x_1) \cup N(x_t), \tag{1}$$

otherwise G would contain a C_t (hence, a P_{t+1}) or a P_t with a longer initial cycle. Choose $k > j_r$ minimal such that $x_k \in N(x_1)$. Since G contains a P_t starting with a cycle of order $t - (k - j_r - 1)$ (omitting $x_{j_r+1}, \ldots, x_{k-1}$),

$$t - (k - j_r - 1) \leq s + 2. \tag{2}$$

By (1), (2), and the definition of k, and since $\mathrm{NC} \geq s$ and x_1 and $x_t \notin N(x_1) \cup N(x_t)$,

$$s \leq |N(x_1) \cup N(x_t)| \leq t - r - (k - j_r - 2) \leq s + 1 - r. \tag{3}$$

Thus $r = 1$ and we have equality throughout. In particular (writing j for $j_r = j_1$),

$$\begin{aligned} N(x_t) &\subseteq \{x_j, x_i, x_{i+1}, \ldots, x_{t-1}\}, \\ t - k + j + 1 &= s + 2, \end{aligned} \tag{4}$$

and the path

$$x_1, x_2, \ldots, x_j, x_t, x_{t-1}, \ldots, x_k, x_{k-1}, \ldots, x_{j+1} \tag{5}$$

starts with a cycle of order $s + 2$.

We can now apply the same argument to path (5) to deduce (inter alia) that

$$N(x_{j+1}) \subseteq \{x_j, x_k, x_{k-1}, \ldots, x_{j+2}\}.$$

It follows that

$$N(x_{j+1}) \cup N(x_t) \subseteq \{x_j, x_{j+2}, x_{j+3}, \ldots, x_k, x_i, x_{i+1}, \ldots, x_{t-1}\}$$

so that, since $\mathrm{NC} \geq s$,

$$s \leq t - j - 1 - \max(0, i - 1 - k).$$

Since the cycle $x_j, x_{j+1}, \ldots, x_t, x_j$ has order $t - j + 1 \leq s + 2$, we must have

$t - j + 1 = s + 2$ and $k = i$ or $i - 1$. But if $k = i - 1$, then the cycle

$$x_j, x_{j+1}, \ldots, x_k, x_1, x_i, x_{i+1}, \ldots, x_t, x_j$$

has order $t - j + 2 = s + 3$. Thus $k = i$ and

$$t - j + 1 = i = t - i + j + 1 = s + 2$$

(using (4)), whence (summing)

$$2t + 2 = 3s + 6. \tag{6}$$

This is clearly impossible if s is odd, and so the proof is complete.

Let us now turn to (a). Suppose G is a 2-connected graph of order p satisfying $\mathrm{NC} \geq s$ and containing no path of order greater than $t < \min(p, \frac{3}{2}s + 2)$. Since G is connected, it does not contain a C_t (otherwise it would contain a P_{t+1}). Suppose that G contains a C_{t-1}. Then every component of $G - C_{t-1}$ is a single vertex. Let x and y be two of them (note $p - (t - 1) \geq 2$), and observe that, since $\mathrm{NC} \geq s$, $N(x) \cup N(y)$ contains two consecutive vertices of C_{t-1}, which is impossible since this implies that G contains a P_{t+1} or a C_t. (This does not work if $2s \leq t - 1$, that is, $s = 1$ and $2s = t - 1 = 2$, but then C_{t-1} is meaningless anyway.) Thus G does not contain a C_t or a C_{t-1}.

We now follow an argument very similar to that in (b). Let $x_1, x_2, \ldots, x_t$ be the vertices in order along a path P of maximum length in G, and let P and $x_i \in N(x_1)$ be chosen so that i is as large as possible. As in (b), we may suppose without loss of generality that x_t is adjacent to $r \geq 1$ vertices $x_{j_1}, \ldots, x_{j_r}$ $(2 \leq j_1 < \cdots < j_r \leq i - 1)$. The following four remarks all follow from the fact that G does not contain a C_t, a C_{t-1}, or a P_{t+1}, or a P_t with an initial cycle of more than i vertices:

(i) No x_{j_k+1} is adjacent to any vertex outside P or to any x_m $(i + 1 \leq m \leq t)$ (otherwise we could incorporate $x_m, \ldots, x_t$ into the initial cycle of P). Hence, no two x_{j_k}'s are consecutive in P.

(ii) The vertices $x_1, x_{j_1+1}, x_{j_2+1}, \ldots, x_{j_r+1}, x_t$ are mutually nonadjacent.

(iii) If $x_k \in N(x_1)$ then x_{k-1} and $x_{k-2} \notin N(x_t)$.

(iv) If $x_k \in N(x_1)$ and $k \leq j_r$, then x_{k-1} and $x_{k-2} \notin N(x_{j_r+1})$.
If $x_k \in N(x_1)$ and $k > j_r$, then x_{k+1} and $x_{k+2} \notin N(x_{j_r+1})$.

Now let $S = N(x_1)$, $R = N(x_{j_r+1})$, $T = N(x_t)$, and let

$$A = \{x_{k-1} : x_k \in S \text{ and } 2 \leq k \leq j_r\} \cup \{x_k : x_k \in S \text{ and } j_r + 2 \leq k \leq i\},$$
$$B = \{x_k : x_k \in R \text{ and } 2 \leq k \leq j_r\} \cup \{x_{k-1} : x_k \in R \text{ and } j_r + 2 \leq k \leq i\},$$

Using (i)–(iv) we see that

$$|A| = |S| = d(x_1), \qquad |B| = |R| = d(x_{j_r} + 1), \qquad |T| = d(x_t),$$
$$A \cap B = \varnothing,$$
$$x_{j_1+1}, x_{j_2+1}, \ldots, x_{j_{r-1}+1} \notin A \cup B, \quad \text{and}$$
$$x_{j_1}, x_{j_2}, \ldots, x_{j_r} \in A \cup B \quad \text{if and only if they are in } R \cap T.$$

It follows that

$$|R| + |S| - |R \cap T| = |A \cup B| - |R \cap T| \le i - (r - 1) - r$$
$$= i - 2r + 1. \tag{7}$$

Also, since $x_i \in S$,

$$|T| \le t - i + r \quad \text{and} \quad |T| - |S \cap T| \le t - i + r - 1. \tag{8}$$

Since NC $\ge s$, using (7) and (8), we obtain

$$\begin{aligned} 2s &\le |R \cup T| + |S \cup T| \\ &= |R| + |S| + 2|T| - |R \cap T| - |S \cap T| \\ &\le i - 2r + 1 + 2(t - i + r) - 1 = 2t - i \end{aligned} \tag{9}$$

Now, in view of the previous proof of (b), we may suppose that $i \ge s + 2$, and $i \ge s + 3$ if s is odd. Thus if the result does not hold then we must have $i = s + 2$ (s even) and $s + 3$ (s odd), and we must have equality throughout the argument that gave rise to (9). In particular,

$$x_{j_1-1} \in N(x_{j_r+1}), \tag{10}$$

since otherwise either $x_{j_1} - 1 \notin A \cup B$ or $x_{j_1} \in S \cap T$, and in either case we would have strict inequality in (9).

If s is even, we can now get the result simply by following the proof of the second part of (b) down as far as (6), which is the needed contradiction. If s is odd, we start to modify the proof of the second part of (b). With k as in that proof, the path

$$x_1, x_2, \ldots, x_{j_1-1}, x_{j_r+1}, x_{j_r}, \ldots, x_{j_1}, x_t, x_{t-1}, \ldots, x_k, x_{k-1}, \ldots, x_{j_r+2} \tag{11}$$

starts with a cycle of order $t - (k - j_r - 2)$, so that (2) is replaced by

$$t - (k - j_r - 2) \le s + 3. \tag{12}$$

Equation (3) is thus unchanged, and so, as before, $r = 1$ and we have equality in (12). Thus

$$d(x_t) \leq t - i + r = \frac{1}{2}(3s + 3) - (s + 3) + 1 = \frac{1}{2}(s - 1).$$

But path (11) also starts with an $(s + 3)$-cycle, and so, by the same reasoning,

$$d(x_{j_r+2}) \leq \frac{1}{2}(s - 1).$$

Thus x_t and x_{j_r+2} violate the condition $NC \geq s$. This completes the proof of (a). Note that (c) can be proved directly; however, it follows immediately from (b). Next we consider (d). Suppose (d) fails to hold, and let $P : x_1, x_2, \ldots, x_t$ be a path of maximal length in G. Since G is connected, we know x_1 and x_t are not adjacent, or a longer path would exist. Since each is adjacent only to vertices of P, both $|N(x_1)| \leq (s - 1)/2$, and $|N(x_t)| \leq (s - 1)/2$, or otherwise they would lie on a cycle with at least $(s/2) + 1$ vertices. But then $|N(x_1) \cup N(x_t)| \leq s - 1$, a contradiction. Finally, (e) follows directly from (c) and (d). This completes the proof.

REMARKS

It would be interesting to further explore the effect of higher connectivity and neighborhood unions on path and cycle length. For example, if G is 3-connected (with sufficiently large order) and $NC \geq s$, then does G contain a cycle of order at least $\frac{3}{2}(s + 1)$ and a path of order at least $2s + 1$? Further, if G is 4-connected (with sufficiently large order), then does G contain a $2s$ cycle? This is best possible, even for graphs with connectivity as high as s, as the complete bipartite graph $K_{s,p-s}$ shows. We also conjecture that Theorem 2(a) can be extended in the case s is odd and $p \geq \frac{3}{2}(s + 1) + 2$ to show that G contains a path of order at least $\frac{3}{2}(s + 1) + 2$.

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