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CAYLEY DIGRAPHS AND $(1, j, n)$-SEQUENCINGS OF THE ALTERNATING GROUPS A_n

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The problem of finding a sequencing $\Pi_1, \Pi_2, \ldots, \Pi_{|A_n|}$ for the elements of the alternating group A_n which minimizes the cost function

$$C = \sum_{i=1}^{|A_n|-1} c(\Pi_i^{-1} \circ \Pi_{i+1}),$$

where $c(\Pi) = |\{j\colon \Pi(j) \neq j\}|$, is solved. For $n \geq 3$ the sequencing is constructed by finding a directed Hamiltonian path in the Cayley digraph D_n of A_n, determined by the generating set $B_n = \{(1, j, n)\colon j \in \{2, 3, \ldots, n-1\}\}$. We further consider the question of finding a directed Hamiltonian cycle in D_n.

1. Introduction

Recently interest has arisen (see [10, 11, 8 and 9]) in generating a sequencing of the elements of a permutation group subject to various constraints. Of special interest is the problem of generating a sequencing $\Pi_1, \Pi_2, \ldots, \Pi_{|G|}$ of the elements of a permutation group G so that the total cost

$$C = \sum_{i=1}^{|G|-1} c(\Pi_i^{-1} \circ \Pi_{i+1})$$

is minimized, where $c : G \to \mathbb{R}^+$ is a cost function. Of course $\Pi_i \circ (\Pi_i^{-1} \circ \Pi_{i+1}) = \Pi_{i+1}$, so that $c(\Pi_i^{-1} \circ \Pi_{i+1})$ is the cost of "proceeding by multiplication" from Π_i to Π_{i+1} in the sequencing.

In particular, Tannenbaum in [10] raised the question of finding such a sequencing when $G = A_n$ $(n \geq 3)$ with its natural action on $\{1, 2, \ldots, n\}$, $c(\Pi) = |\{j\colon \Pi(j) \neq j\}|$ so that $c(\Pi) \geq 3$ for each non-identity element $\Pi \in A_n$, and the set of allowable multipliers for use in the sequencing is

$$B_n = \{(1, j, n)\colon j \in \{2, 3, \ldots, n-1\}\},$$

which is a minimal generating set of A_n. In this paper we construct many such sequencings for each $n \geq 3$. Terms not defined in this article can be found in [1] or [7].

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2. Notation and definitions

In writing permutations explicitly we use the standard cycle notation and the composition, $\Pi_1 \circ \Pi_2$, of Π_1 and Π_2 will always mean Π_1 followed by Π_2. The identity permutation will be denoted by ι. We write $\Pi(h)$ for the image of h under Π and we often abbreviate $\Pi(h)=k$ by writing "Π sends $h \mapsto k$". We define

$$\Sigma_n(h_1, h_2, \ldots, h_m; k_1, k_2, \ldots, k_m) = \{\Pi \in A_n: \Pi(h_r) = k_r,\ r \in \{1, 2, \ldots, m\}\}.$$

Of course $\Sigma_n(k;k)$ is the stabilizer of k in A_n. We also let $\tau_j = (1, j, n)$ with n being unambiguously determined by the context in which we write τ_j. Given a digraph D and $\Sigma \subseteq V(D)$, we let $\langle \Sigma \rangle$ denote the subgraph of D induced by Σ.

For $n \geq 3$, we let D_n denote the Cayley digraph of A_n with respect to B_n. That is, D_n is the digraph with $V(D_n) = A_n$ and $E(D_n) = \bigcup_{j=2}^{n-1} E_j$, where $E_j = \{(\Pi, \Psi): \Pi, \Psi \in A_n \text{ and } \Pi \circ \tau_j = \Psi\}$. The elements of E_j will be called j-edges. We let $\mathrm{Aut}(D_n)$ denote the automorphism group of D_n.

We note that the existence of a sequencing of A_n in which all multipliers are permutations in B_n is equivalent to the existence of a directed Hamiltonian path in D_n. We shall also consider the existence of a directed Hamiltonian cycle in D_n. (The Hamiltonian problem for various Cayley digraphs has been studied in [5, 2, 4, and 12].)

We shall henceforth refer to directed paths and directed cycles as dipaths and dicycles, respectively. We call a dipath with initial vertex x and terminal vertex y an x–y dipath; and we write $I(P)$ and $T(P)$ to denote the initial and terminal vertices, respectively, of a dipath P. When P and Q are dipaths with $T(P) = I(Q)$ we write PQ to denote the concatenation of P and Q.

For each $\sigma \in A_n$, we let the mapping $m_\sigma : A_n \to A_n$ (pre-multiplication by σ) be defined by $m_\sigma(\Pi) = \sigma \circ \Pi$. We let $\mathcal{M} = \{m_\sigma : \sigma \in A_n\}$. For each $\sigma \in S_{n-2}$, where S_{n-2} acts on $\{2, 3, \ldots, n-1\}$, we let the mapping $c_\sigma : A_n \to A_n$ (conjugation by σ) be defined by $c_\sigma(\Pi) = \sigma^{-1} \circ \Pi \circ \sigma$. (Recall that the cycle types of Π and $c_\sigma(\Pi)$ are the same.) We let $\mathcal{C} = \{c_\sigma : \sigma \in S_{n-2}\}$.

3. Preliminary lemmas

The first two lemmas are standard results and we omit their proofs.

Lemma 1. *If $n \geq 3$, then $\mathcal{M} \leq \mathrm{Aut}(D_n)$ and m_σ preserves each E_j set-wise. Furthermore, the mapping $m : A_n \to \mathcal{M}$ defined by $\sigma \mapsto m_\sigma$ is an anti-isomorphism.*

Lemma 2. *If $n \geq 3$, then D_n is vertex transitive.*

Lemma 3. *If $n \geq 3$, then $\mathcal{C} \leq \mathrm{Aut}(D_n)$ and the mapping $c : S_{n-2} \to \mathcal{C}$ given by*

$\sigma \mapsto c_\sigma$ is an isomorphism. If $n \geq 4$, if $j_1, j_2 \in \{2, 3, \ldots, n-1\}$ are distinct, and if $\alpha = (j_1, j_2)$, then c_α preserves E_j set-wise for $j \notin \{j_1, j_2\}$ and interchanges E_{j_1} and E_{j_2}.

Proof. The proof of the lemma is immediate for $n = 3$; so we assume that $n \geq 4$ and that $j_1, j_2 \in \{2, 3, \ldots, n-1\}$ are distinct.

Since $c_{\gamma \circ \beta} = c_\gamma \circ c_\beta$, $\mathscr{C}$ is a group of permutations of $V(D_n)$ and c is an epimorphism. If $j \notin \{j_1, j_2\}$, then α fixes j (and 1 and n); hence, if $\Pi \circ \tau_j = \Psi$, then $c_\alpha(\Pi) \circ \tau_j = c_\alpha(\Psi)$. Thus, E_j is preserved set-wise by c_α.

Similarly, if $\Pi \circ \tau_{j_1} = \Psi$, then $c_\alpha(\Pi) \circ \tau_{j_2} = c_\alpha(\Psi)$ and the same is true when the roles of j_1 and j_2 are reversed. Hence, c_α interchanges E_{j_1} and E_{j_2}.

So we see that $c_\alpha \in \mathrm{Aut}(D_n)$ and since S_{n-2} is generated by the set of its transpositions, $\mathscr{C} \leq \mathrm{Aut}(D_n)$.

All that remains to be shown is that c is injective. We note that $c_{\sigma_1} = c_{\sigma_2}$ if and only if $\sigma_1 \circ \sigma_2^{-1} \in Z(A_n)$, the center of A_n. An easy exercise in group theory shows that $Z(A_n)$ is trivial for $n \geq 4$. Hence, $c_{\sigma_1} = c_{\sigma_2}$ implies that $\sigma_1 = \sigma_2$ so that c is indeed an isomorphism. □

We remark that since c_α fixes ι, c_α interchanges the j_1-edge and the j_2-edge leaving ι.

Lemma 4. *If $n \geq 3$, then D_n is edge-transitive.*

Proof. Given $(\Pi_1, \Psi_1) \in E_{j_1}$ and $(\Pi_2, \Psi_2) \in E_{j_2}$, let $\alpha = (j_1, j_2)$ and let $a = m_{\Pi_1^{-1}} \circ c_\alpha \circ m_{\Pi_2}$. A routine computation shows that $a(\Pi_1) = \Pi_2$ and $a(\Psi_1) = \Psi_2$. Since $a \in \mathrm{Aut}(D_n)$ by Lemma 1 and Lemma 3, the proof is complete. □

Lemma 5. *If $n \geq 4$ and $k \in \{1, 2, \ldots, n\}$, then the left cosets of $\Sigma_n(k; k)$ in A_n are precisely the sets $\Sigma_n(1; k), \Sigma_n(2; k), \ldots, \Sigma_n(n; k)$.*

If $k \in \{2, 3, \ldots, n-1\}$, then

(i) *each of the n subgraphs $\langle \Sigma_n(1; k) \rangle, \ldots, \langle \Sigma_n(n; k) \rangle$ is isomorphic to D_{n-1},*
(ii) *if $j \neq k$, then every j-edge of D_n is an edge in exactly one of these n subgraphs, and*
(iii) *if $(\Pi, \Psi) \in E_k$ and $\Pi \in \Sigma_n(h_1, h_2, h_3;\ k, 1, n)$, then $\Psi \in \Sigma_n(h_2, h_3, h_1;\ k, 1, n)$.*

Proof. If $\omega \in A_n$ and ω sends $h \mapsto k$, then for $\sigma \in \Sigma_n(k; k)$, $\omega \circ \sigma$ sends $h \mapsto k$ so that the left coset $\omega \Sigma_n(k; k) = \Sigma_n(h; k)$.

We now assume that $k \in \{2, 3, \ldots, n-1\}$. Consider the action of A_{n-1} on the set $\{1, 2, \ldots, n\} \setminus \{k\}$, and let $\wedge : A_{n-1} \to \Sigma_n(k; k)$ be the mapping given by $\sigma \mapsto \hat{\sigma}$, where $\hat{\sigma}(h) = \sigma(h)$ for $h \neq k$ and $\hat{\sigma}(k) = k$. It is immediate that $\wedge$ is a group isomorphism and since $\tau_j = (1, j, n) \in A_{n-1}$ and $\hat{\tau}_j = (1, j, n) \in A_n$ for $j \neq k$, we see that $\wedge$ is a digraph isomorphism from D_{n-1} to $\langle \Sigma_n(k; k) \rangle$. For $h \neq k$, let

$\sigma \in A_n$ be any permutation which sends $h \mapsto k$ and (recalling Lemma 1) consider the automorphism m_σ of D_n. The restriction of m_σ to $\Sigma_n(k;k)$ is a bijection from $\Sigma_n(k;k)$ to $\Sigma_n(h;k)$ and hence is a digraph isomorphism from $\langle \Sigma_n(k;k) \rangle$ to $\langle \Sigma_n(h;k) \rangle$. This completes the proof of (i).

If $j \neq k$, if Π sends $h \mapsto k$, and if $\Pi \circ \tau_j = \Psi$, then since $\tau_j(k) = k$, Ψ sends $h \mapsto k$ so that the j-edge (Π, Ψ) is an edge of the digraph $\langle \Sigma_n(h;k) \rangle$.

If $\Pi \in \Sigma_n(h_1, h_2, h_3;\ k, 1, n)$ and $\Pi \circ \tau_k = \Psi$, then it is immediate that $\Psi \in \Sigma_n(h_2, h_3, h_1;\ k, 1, n)$ and so the k-edge (Π, Ψ) leaves the subgraph $\langle \Sigma_n(h_1;k) \rangle$ and enters the subgraph $\langle \Sigma_n(h_2;k) \rangle$. □

A standard exercise in group theory (see [7]) shows that A_n is generated by its set of 3-cycles for $n \geqslant 3$. Since (j_1, j_2, j_3), $(1, j_1, j_2)$, and (j_1, j_2, n) can be written as products of elements of B_n, B_n is indeed a generating set for A_n. The following lemma is a graph theoretic restatement of this fact.

Lemma 6. *If $n \geqslant 3$, then D_n is strongly connected.*

The proof that B_n generates A_n shows that if $n \geqslant 3$, then the diameter of D_{n+1} is at most 3 more than the diameter of D_n. We state here without proof (see [3]) that the diameters of D_4, D_5, D_6, D_7, and D_8 are 4, 5, 6, 8 and 10 respectively. Determination of the diameter of D_n in the general case seems to be an interesting problem.

4. The main results

A directed graph D is homogeneously traceable if for each $v \in V(D)$ there exists a Hamiltonian dipath, P, in D with $I(P) = v$. We now investigate the existence of Hamiltonian dipaths in the directed graphs D_n ($n \geqslant 3$). The digraphs D_3 and D_4 are investigated by hand and D_5 is investigated by computer search. For $n \geqslant 6$ we use induction and automorphisms to insure the existence of many Hamiltonian dipaths in the digraphs $\langle \Sigma_n(h;k) \rangle$, $h \in \{1, 2, \ldots, n\}$, for a particular $k \in \{2, 3, \ldots, n-1\}$ depending on a vertex ω which sends $1 \mapsto$ n. We then piece these together to form the desired ι–ω Hamiltonian dipath in D_n. The method required to effect this piecing is less straightforward than one might expect.

In fact, for $n \neq 4$ we are able to choose ω so that the ι–ω Hamiltonian dipath can be extended to a Hamiltonian dicycle.

Theorem 7. *For $n = 3$ and $n \geqslant 5$, given any $\omega \in A_n = V(D_n)$ such that ω sends $1 \mapsto n$, there exists an ι–ω Hamiltonian dipath. For $n = 4$, there exists an ι–$(1, 4)(2, 3)$ Hamiltonian dipath.*

Proof. For $n=3$, $\omega=(1,3,2)$ is the only permutation which sends $1\mapsto 3$ and since D_3 is the directed 3-cycle $[\iota, (1,2,3), (1,3,2), \iota]$, there is an ι–ω Hamiltonian dipath in D_3.

For $n=4$, we use the numbering of the elements of A_4 shown in Appendix A. Here Π_5, Π_7, and Π_{12} all send $1\mapsto 4$, but a routine hand computation shows that there are exactly 4 Hamiltonian dipaths in D_4 having ι as initial vertex and each of these has Π_{12} as terminal vertex. These four paths are P_1 and P_2 (listed in Appendix C) and the images, $c_{(2,3)}(P_1)$ and $c_{(2,3)}(P_2)$, of P_1 and P_2 under the automorphism $c_{(2,3)}$. Thus since D_4 is vertex transitive, D_4 is homogeneously traceable but not Hamiltonian.

For $n=5$, we use the numbering of the elements of A_5 shown in Appendix B. By implementing a "foresightful" backtracking algorithm on a computer we find the three Hamiltonian dipaths in D_5 listed as R_1, R_2, and R_3 in Appendix D.

By Lemma 3, if R is a Hamiltonian dipath in D_5 and $\sigma\in S_3$ (acting on the set $\{2,3,4\}$) then the image $c_\sigma(R)$ is also a Hamiltonian dipath in D_5. The Hamiltonian dipaths $c_{(2,4)}(R_1)$, $c_{(3,4)}(R_1)$, $c_{(2,4)}(R_2)$, $c_{(3,4)}(R_2)$, $c_{(2,3)}(R_2)$, $c_{(2,3,4)}(R_2)$, $c_{(2,4,3)}(R_2)$, $c_{(2,4)}(R_3)$, and $c_{(2,3)}(R_3)$ all have $\Pi_1=\iota$ as initial vertex; and their terminal vertices are Π_{17}, Π_{13}, Π_{45}, Π_{27}, Π_{35}, Π_{39}, Π_{43}, Π_{57}, and Π_{54} respectively. Thus, for each of the twelve vertices, ω, of D_5 which send $1\mapsto 5$ we have exhibited an ι–ω Hamiltonian dipath.

Having proved the theorem for $n=5$, we now assume that $n\geqslant 6$ and that the theorem is true for $n-1$. We are given $\omega\in A_n=V(D_n)$ such that ω sends $1\mapsto n$ and we shall construct an ι–ω Hamiltonian dipath P in D_n. Since ω is an even permutation, there exists $k\in\{2,3,\ldots,n-1\}$ such that $\omega(k)\neq k$. We fix this k for the remainder of the proof and we let $h^*=\omega^{-1}(k)$ (of course $h^*\neq k$ and $\omega\in\Sigma_n(h^*;k)$). We shall be concerned with the induced subgraphs $\langle\Sigma_n(1;k)\rangle$, $\langle\Sigma_n(2;k)\rangle,\ldots,\langle\Sigma_n(n;k)\rangle$ and we begin by proving the following claim involving the homogeneous traceability of these subgraphs.

Claim. If $h\in\{1,2,3,\ldots,n\}$, if $\Pi\in\Sigma_n(h;k)=V(\langle\Sigma_n(h;k)\rangle)$, and if $\mathscr{E}(\Pi)=\{\Psi\in\Sigma_n(h;k)$: there exists a Π–Ψ Hamiltonian dipath in $\langle\Sigma_n(h;k)\rangle\}$, then $\mathscr{E}(\Pi)\supseteq\Sigma_n(h,\Pi^{-1}(1);k,n)$.

Proof. Note first that since the theorem is true for $n-1$ and since $\langle\Sigma_n(k;k)\rangle\cong D_{n-1}$ by Lemma 5, $\mathscr{E}(\iota)\supseteq\Sigma_n(k,1;k,n)=\Sigma_n(k,\iota^{-1}(1);k,n)$. That is, the claim is true for $\Pi=\iota$.

Next we observe that $\Sigma_n(h,\Pi^{-1}(1);k,n)=m_\Pi(\Sigma_n(k,1;k,n))$. This follows immediately from the following facts: m_Π is injective, $|\Sigma_n(h,\Pi^{-1}(1);k,n)|=|\Sigma_n(k,1;k,n)|$, and $\Sigma_n(h,\Pi^{-1}(1);k,n)\supseteq m_\Pi(\Sigma_n(k,1;k,n))$.

Now given $\Psi\in\Sigma_n(h,\Pi^{-1}(1);k,n)$ we know from the previous observation that there exists $\sigma\in\Sigma_n(k,1;k,n)$ such that $\Pi\circ\sigma=\Psi$. Since the theorem is true for $n-1$ and since $\langle\Sigma_n(k;k)\rangle\cong D_{n-1}$ by Lemma 5, $\sigma\in\Sigma_n(k,1;k,n)\subseteq\mathscr{E}(\iota)$ so that there exists an ι–σ Hamiltonian dipath, P, in $\langle\Sigma_n(k;k)\rangle$. By Lemma 1,

$m_\Pi \in \mathrm{Aut}(D_n)$ and hence $m_\Pi(P)$ is a simple dipath in D_n. In fact, if $\alpha \in \Sigma_n(k;k)$ then $m_\Pi(\alpha) \in \Sigma_n(h;k)$; so since all the vertices of P are in $\Sigma_n(k;k)$, all the vertices of $m_\Pi(P)$ are in $\Sigma_n(h;k)$ and hence $m_\Pi(P)$ is a Hamiltonian dipath in $\langle \Sigma_n(h;k)\rangle$. Since $I(m_\Pi(P)) = \Pi \circ \iota = \Pi$ and $T(m_\Pi(P)) = \Pi \circ \sigma = \Psi$, $\Psi \in \mathscr{E}(\Pi)$. Therefore, $\mathscr{E}(\Pi) \supseteq \Sigma_n(h,\ \Pi^{-1}(1);\ k, n)$ and the proof of the claim is complete. □

We shall now construct the desired dipath P. A schematic diagram of the construction is presented in Fig. 1. We let $\Pi = (1, k, n)$ and we note that

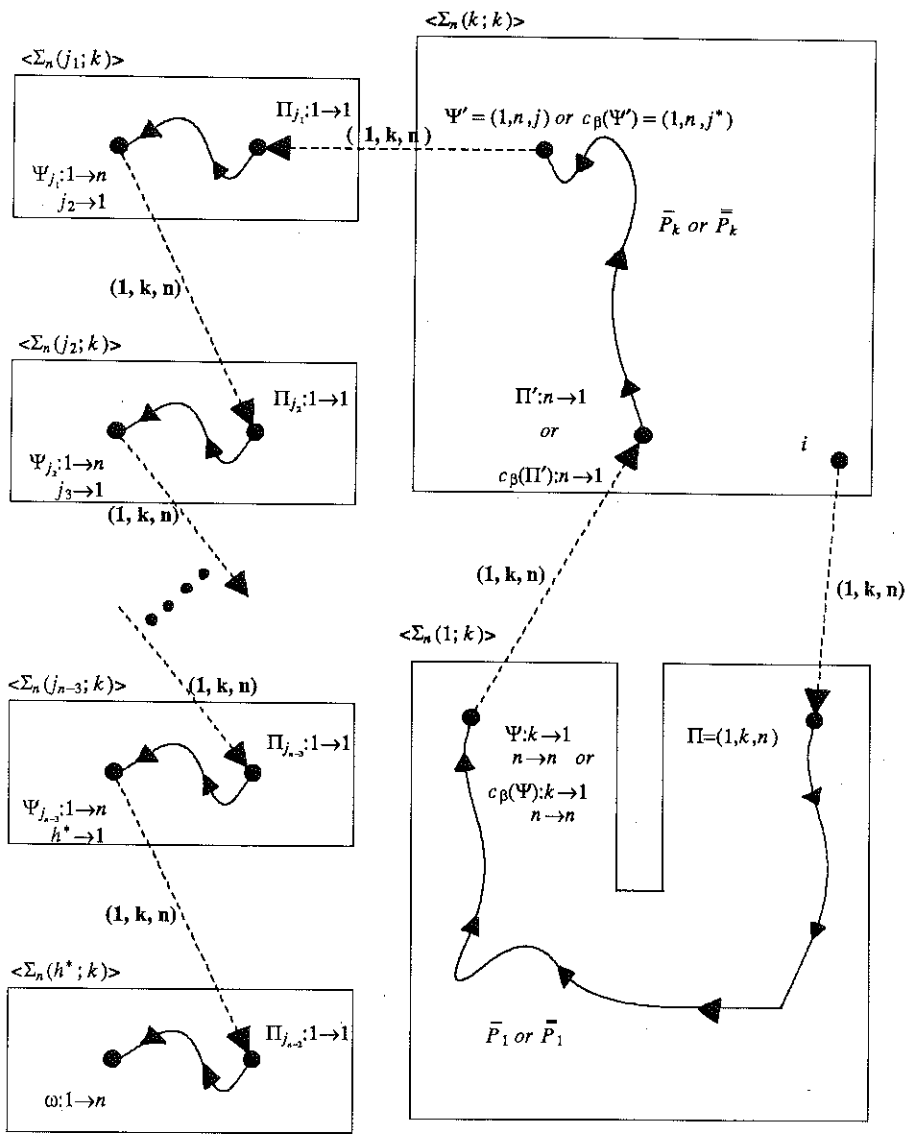


Fig. 1.

$\Pi \in \Sigma_n(1;k)$. By the claim, $\mathscr{E}(\Pi) \supseteq \Sigma_n(1, n; k, n)$ and hence there exists $\Psi \in \Sigma_n(1, k, n; k, 1, n)$ such that there is a Π–Ψ Hamiltonian dipath $\bar{P}_1$ in $\langle \Sigma_n(1;k) \rangle$. We let $\Pi' = \Psi \circ (1, k, n)$ and we note that $\Pi' \in \Sigma_n(k, n; k, 1)$. By the claim, $\mathscr{E}(\Pi') \supseteq \Sigma_n(k, n; k, n)$ and hence there exists a Π'–ι Hamiltonian dipath $\bar{Q}_k$ in $\langle \Sigma_n(k;k) \rangle$ since $\iota \in \Sigma_n(k, n; k, n)$. We let Ψ' be the penultimate vertex in $\bar{Q}_k$ and we let $\bar{P}_k$ be the initial segment of $\bar{Q}_k$ which satisfies $T(\bar{P}_k) = \Psi$ (that is, deleting the last edge, (Ψ', ι), of $\bar{Q}_k$ results in $\bar{P}_k$). Now, for some $j \in \{2, 3, \ldots, n-1\} \setminus \{k\}$ we have $\Psi' \circ (1, j, n) = \iota$ and hence $\Psi' = (1, n, j)$. If $j = h^*$ (recall that $h^* = \omega^{-1}(k)$) we fix $j^* \in \{2, 3, \ldots, n-1\} \setminus \{k, j\}$ and we let $\beta = (j, j^*)$. We now consider the mapping c_β: we note that if $\alpha \in \Sigma_n(1;k)$, then $c_\beta(\alpha) \in \Sigma_n(1;k)$ and if $\alpha \in \Sigma_n(k;k)$, then $c_\beta(\alpha) \in \Sigma_n(k;k)$. So by letting $\bar{\bar{P}}_1 = c_\beta(\bar{P}_1)$ and $\bar{\bar{P}}_k = c_\beta(\bar{P}_k)$ we conclude (since $c_\beta \in \mathrm{Aut}(D_n)$ by Lemma 3 and since $c_\beta(\Pi) = \Pi$) that $\bar{\bar{P}}_1$ is a Π–$c_\beta(\Psi)$ Hamiltonian dipath in $\langle \Sigma_n(1;k) \rangle$ and (since $c_\beta(\Psi') = (1, n, j^*)$) that $\bar{\bar{P}}_k$ is a $c_\beta(\Pi') - (1, n, j^*)$ dipath in $\langle \Sigma_n(k;k) \rangle$ that includes every vertex of $\langle \Sigma_n(k;k) \rangle$ except $c_\beta(\iota) = \iota$. Furthermore, since (Ψ, Π') is a k-edge of D_n we know by Lemma 3 that $(c_\beta(\Psi), c_\beta(\Pi'))$ is also a k-edge of D_n.

We are now prepared to define the initial segment of our path P. If $j \neq h^*$ we let $P_{1,k} = [\iota, \Pi]\bar{P}_1[\Psi, \Pi']\bar{P}_k$. If $j = h^*$ we let $P_{1,k} = [\iota, \Pi]\bar{\bar{P}}_1[c_\beta(\Psi), c_\beta(\Pi')]\bar{\bar{P}}_k$. In both cases $P_{1,k}$ is a Hamiltonian dipath in the subgraph $\langle \Sigma_n(k;k) \cup \Sigma_n(1;k) \rangle$ of D_n. $I(P_{1,k}) = \iota$ and $T(P_{1,k}) = (1, n, j_1)$, where $j_1 \in \{2, 3, \ldots, n-1\} \setminus \{k, h^*\}$ (in fact $j_1 = j$ if $j \neq h^*$ and $j_1 = j^*$ if $j = h^*$).

We now define the other segments of our dipath P. We let $(j_1, j_2, \ldots, j_{n-2})$ be a fixed ordering of the elements of $\{2, 3, \ldots, n-1\} \setminus \{k\}$, where $j_{n-2} = h^*$ and of course j_1 is determined by the terminal vertex, $(1, n, j_1)$, of $P_{1,k}$. We let $\Pi_{j_1} = (1, n, j_1) \circ (1, k, n)$ and we let Ψ_{j_1} be a fixed element of $\Sigma_n(j_1, j_2, 1; k, 1, n)$. Next we proceed inductively as follows: for $i \in \{2, 3, \ldots, n-3\}$, having defined $\Pi_{j_{i-1}}$ and $\Psi_{j_{i-1}}$ we let $\Pi_{j_i} = \Psi_{j_{i-1}} \circ (1, k, n)$ and we let Ψ_{j_i} be a fixed element of $\Sigma_n(j_i, j_{i+1}, 1; k, 1, n)$. Finally we let $\Pi_{j_{n-2}} = \Psi_{j_{n-3}} \circ (1, k, n)$. We note that for each $i \in \{1, 2, \ldots, n-3\}$, $\Pi_{j_i} \in \Sigma_n(j_i, 1; k, 1)$. By the claim, $\mathscr{E}(\Pi_{j_i}) \supseteq \Sigma_n(j_i, 1; k, n)$ and since $\Psi_{j_i} \in \Sigma_n(j_i, 1; k, n)$ we conclude that there exists a Π_{j_i}–Ψ_{j_i} Hamiltonian dipath $\bar{P}_{j_i}$ in $\langle \Sigma_n(j_i; k) \rangle$. Also, we note that $\Pi_{j_{n-2}} \in \Sigma_n(h^*, 1; k, 1)$ (since we required that $j_{n-2} = h^*$); by the claim, $\mathscr{E}(\Pi_{j_{n-2}}) \supseteq \Sigma_n(h^*, 1; k, n)$ and since $\omega \in \Sigma_n(h^*, 1; k, n)$ we conclude that there exists a $\Pi_{j_{n-2}}$–ω Hamiltonian dipath $\bar{P}_{j_{n-2}}$ in $\langle \Sigma_n(h^*; k) \rangle$. We let $P_{j_1} = [(1, n, j_1), \Pi_{j_1}]\bar{P}_{j_1}$ and for $i \in \{2, 3, \ldots, n-2\}$, we let $P_{j_i} = [\Psi_{j_{i-1}}, \Pi_{j_i}]\bar{P}_{j_i}$.

Finally we let $P = P_{1,k}P_{j_1}P_{j_2} \ldots P_{j_{n-3}}P_{j_{n-2}}$. Thus, P is an ι–ω Hamiltonian dipath in D_n and the proof is complete. □

Corollary 8. *The digraph D_4 is homogeneously traceable but not Hamiltonian; and for $n = 3$ and $n \geq 5$, D_n is Hamiltonian.*

Proof. The statement concerning D_4 was verified in the proof of Theorem 7. Thus we assume that $n = 3$ or $n \geq 5$ and we let $\omega = (1, n, j)$ where $j \in \{2, 3, \ldots, n-1\}$. Since ω sends $1 \mapsto n$, we conclude by Theorem 7 that there is an ι–ω Hamiltonian dipath P in D_n. The dipath $P[\omega, \iota]$ is a Hamiltonian dicycle in D_n. □

Theorem 7 made verification of Corollary 8 easy since we knew the final vertex of P would be of the form $(1, n, j)$. It seems natural to investigate possible restrictions on terminal vertices of Hamiltonian dipaths in D_n. Thus for $\Pi \in A_n$ we let $\mathscr{F}(\Pi) = \{\Psi \in A_n$: there exists a Π–Ψ Hamiltonian dipath in $D_n\}$.

Corollary 9. *If* $\Pi \in A_n$, *then*

(a) *if* $n = 3$, $\mathscr{F}(\Pi) = \{\Pi \circ (1, 3, 2)\}$,
(b) *if* $n = 4$, $\mathscr{F}(\Pi) = \{\Pi \circ (1, 4)(2, 3)\}$,
(c) *if* $n \geqslant 5$, $\mathscr{F}(\Pi) \supseteq \{\Psi \in A_n : \Psi^{-1}(n) = \Pi^{-1}(1)\}$.

Proof. We first observe that for any $n \geqslant 3$, $\{\Psi \in A_n : \Psi^{-1}(n) = \Pi^{-1}(1)\} = m_\Pi(\Sigma_n(1; n))$. This follows immediately from the following facts: m_Π is injective, $|\{\Psi \in A_n : \Psi^{-1}(n) = \Pi^{-1}(1)\}| = |\Sigma_n(1; n)|$, and $\{\Psi \in A_n : \Psi^{-1}(n) = \Pi^{-1}(1)\} \supseteq m_\Pi(\Sigma_n(1; n))$.

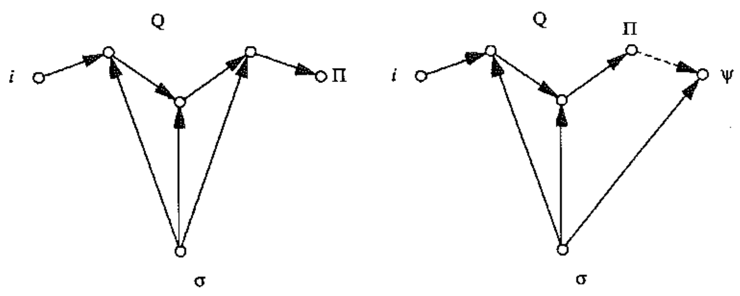
We now assume $n = 3$ or $n \geqslant 5$. Given $\Psi \in A_n$ such that $\Psi^{-1}(n) = \Pi^{-1}(1)$ we know that there exists σ_Ψ such that $\Pi \circ \sigma_\Psi = \Psi$ and $\sigma_\Psi(1) = n$. By Theorem 7 we know that there exists an ι–σ_Ψ Hamiltonian dipath, P, in D_n. By Lemma 1, $m_\Pi \in \mathrm{Aut}(D_n)$ so that $m_\Pi(P)$ is a Hamiltonian dipath with $I(m_\Pi(P)) = \Pi$ and $T(m_\Pi(P)) = \Pi \circ \sigma_\Psi = \Psi$. Hence $\Psi \in \mathscr{F}(\Pi)$ and thus $\mathscr{F}(\Pi) \supseteq \{\Psi \in A_n : \Psi^{-1}(n) = \Pi^{-1}(1)\}$.

For $n = 4$, the above argument shows that for $\Pi \in A_4$, $\mathscr{F}(\Pi) \supseteq \{\Pi \circ (1, 4)(2, 3)\}$. Since $\mathscr{F}(\iota) = \{(1, 4)(2, 3)\}$ and since D_4 is vertex transitive (Lemma 2), we conclude that $\mathscr{F}(\Pi) = \{\Pi \circ (1, 4)(2, 3)\}$. For $n = 3$, equality is immediately observed. □

5. Queries and remarks

In view of Corollary 9, a natural question to ask is, "Is it true that for $n \geqslant 5$, $\mathscr{F}(\Pi) = \{\Psi \in A_n : \Psi^{-1}(n) = \Pi^{-1}(1)\}$?" Since D_n is vertex transitive, it suffices to consider the question of equality between $\mathscr{F}(\iota)$ and $\{\Psi \in A_n : n = \Psi(1)\}$. The question appears to be difficult and of considerable interest. We were unable to construct Hamiltonian dipaths in D_n by any direct construction having a "theoretical" basis. Our inductive method of proving Thorem 7 does not work for D_5 using D_4 because of the small size of the sets $\mathscr{F}(\Pi)$ in D_4.

Our backtracking algorithm for D_5 was "foresightful" in the following way (see Fig. 2). In extending a dipath Q with $T(Q) = \Pi$, a vertex σ not in Q is called bad if the edges leaving σ are (σ, α_1), (σ, α_2), and (σ, α_3) and all of α_1, α_2, and α_3 are already in Q. If the edges leaving Π are (Π, Ψ_1), (Π, Ψ_2), and (Π, Ψ_3), then



σ is a bad vertex   Ψ would create the bad vertex σ

Fig. 2.

in trying to extend Q by adding the edge (Π, Ψ_i) we of course check that Ψ_i is not already in Q, but we also check to make sure that the dipath consisting of Q followed by the edge (Π, Ψ_i) would have no more than one bad vertex (since if this dipath is to be extended to a Hamiltonian dipath, P, a bad vertex must be $T(P)$). This additional heuristic check speeds up the program and enables us to generate approximately 24 000 Hamiltonian dipaths in D_5 using approximately 144 minutes of CPU time on a VAX 750 computer. Each of these dipaths has ι as its initial vertex and one of the twelve vertices of D_5 which sends $1 \mapsto 5$ as its terminal vertex. This suggests that $\mathcal{F}(\iota) = \{\Psi \in A_5 : 5 = \Psi(1)\}$, but certainly is not conclusive since we estimate that we produced approximately 1/15 000 of all Hamiltonian dipaths in D_5 with initial vertex ι. We also mention that for each $j \in \{2, 3, 4\}$, every $\Psi \in A_5$ which sends $1 \mapsto 5$ is the terminal vertex of at least one Hamiltonian dipath in D_5 having $(\iota, (1, j, 5))$ as its initial edge.

We have seen that D_4 is a homogeneously traceable non-Hamiltonian directed graph of order 12 which has the following properties:

P_1: asymmetry, P_3: vertex transitivity,

P_2: regularity, P_4: edge transitivity.

Let P be a subset of $\{P_1, P_2, P_3, P_4\}$. We ask the following questions. Does D_4 have the smallest order among all homogeneously traceable non-Hamiltonian directed graphs having the properties in P? If so, is D_4 the unique such directed graph of order 12?

The Cayley digraph, $D(G, S)$, of any group, G, with respect to a generating set, S, of G is vertex transitive (see the proofs of Lemma 1 and Lemma 2). Although the Lovász conjective (every connected vertex transitive graph has a Hamiltonian path) remains unsettled, Nijenhuis and Wilf [6] showed that for a particular S consisting of a 5-cycle and a transposition, $D(S_5, S)$ does not have a Hamiltonian dipath. Nevertheless, it is of great interest to determine which groups, G, have the property that $D(G, S)$ has a Hamiltonian dipath (dicycle) for every generating set, S, of G. An excellent result in this area, due to Witte (see [12]), is that each $D(G, S)$ has a Hamiltonian dicycle when G is a p-group. In particular, determining those S for which $D(A_n, S)$ is traceable (Hamiltonian) seems to be an interesting problem.

The authors would like to express their thanks to the referee for his helpful suggestions for improving this paper.

Appendix A

A numbering of the elements of A_4.

$\Pi_1 = \iota$,	$\Pi_5 = (1, 4, 3)$,	$\Pi_9 = (1, 3, 2)$,
$\Pi_2 = (2, 3, 4)$,	$\Pi_6 = (1, 2, 4)$,	$\Pi_{10} = (1, 2)(3, 4)$,
$\Pi_3 = (2, 4, 3)$,	$\Pi_7 = (1, 4, 2)$,	$\Pi_{11} = (1, 3)(2, 4)$,
$\Pi_4 = (1, 3, 4)$,	$\Pi_8 = (1, 2, 3)$,	$\Pi_{12} = (1, 4)(2, 3)$.

Appendix B

A numbering of the elements of A_5.

$\Pi_1 = \iota$,	$\Pi_{21} = (1, 3, 2)$,	$\Pi_{41} = (1, 4, 2, 5, 3)$,
$\Pi_2 = (3, 4, 5)$,	$\Pi_{22} = (1, 2, 3, 4, 5)$,	$\Pi_{42} = (1, 4, 2, 3, 5)$,
$\Pi_3 = (3, 5, 4)$,	$\Pi_{23} = (1, 5, 4, 3, 2)$,	$\Pi_{43} = (1, 5, 3, 2, 4)$,
$\Pi_4 = (2, 4, 5)$,	$\Pi_{24} = (1, 2, 3, 5, 4)$,	$\Pi_{44} = (1, 4, 3, 2, 5)$,
$\Pi_5 = (2, 5, 4)$,	$\Pi_{25} = (1, 4, 5, 3, 2)$,	$\Pi_{45} = (1, 5, 2, 3, 4)$,
$\Pi_6 = (2, 3, 5)$,	$\Pi_{26} = (1, 2, 4, 3, 5)$,	$\Pi_{46} = (2, 3)(4, 5)$,
$\Pi_7 = (2, 5, 3)$,	$\Pi_{27} = (1, 5, 3, 4, 2)$,	$\Pi_{47} = (2, 4)(3, 5)$,
$\Pi_8 = (2, 3, 4)$,	$\Pi_{28} = (1, 2, 4, 5, 3)$,	$\Pi_{48} = (2, 5)(3, 4)$,
$\Pi_9 = (2, 4, 3)$,	$\Pi_{29} = (1, 3, 5, 4, 2)$,	$\Pi_{49} = (1, 3)(4, 5)$,
$\Pi_{10} = (1, 4, 5)$,	$\Pi_{30} = (1, 2, 5, 3, 4)$,	$\Pi_{50} = (1, 4)(3, 5)$,
$\Pi_{11} = (1, 5, 4)$,	$\Pi_{31} = (1, 4, 3, 5, 2)$,	$\Pi_{51} = (1, 5)(3, 4)$,
$\Pi_{12} = (1, 3, 5)$,	$\Pi_{32} = (1, 2, 5, 4, 3)$,	$\Pi_{52} = (1, 2)(4, 5)$,
$\Pi_{13} = (1, 5, 3)$,	$\Pi_{33} = (1, 3, 4, 5, 2)$,	$\Pi_{53} = (1, 4)(2, 5)$,
$\Pi_{14} = (1, 3, 4)$,	$\Pi_{34} = (1, 3, 2, 4, 5)$,	$\Pi_{54} = (1, 5)(2, 4)$,
$\Pi_{15} = (1, 4, 3)$,	$\Pi_{35} = (1, 5, 4, 2, 3)$,	$\Pi_{55} = (1, 2)(3, 5)$,
$\Pi_{16} = (1, 2, 5)$,	$\Pi_{36} = (1, 3, 2, 5, 4)$,	$\Pi_{56} = (1, 3)(2, 5)$,
$\Pi_{17} = (1, 5, 2)$,	$\Pi_{37} = (1, 4, 5, 2, 3)$,	$\Pi_{57} = (1, 5)(2, 3)$,
$\Pi_{18} = (1, 2, 4)$,	$\Pi_{38} = (1, 3, 4, 2, 5)$,	$\Pi_{58} = (1, 2)(3, 4)$,

$\Pi_{19} = (1, 4, 2)$, $\Pi_{39} = (1, 5, 2, 4, 3)$, $\Pi_{59} = (1, 3)(2, 4)$,
$\Pi_{20} = (1, 2, 3)$, $\Pi_{40} = (1, 3, 5, 2, 4)$, $\Pi_{60} = (1, 4)(2, 3)$,

Appendix C

The Hamiltonian dipaths P_1 and P_2 in D_4.

P_1: $\Pi_1\ \Pi_6\ \Pi_7\ \Pi_2\ \Pi_{11}\ \Pi_9\ \Pi_4\ \Pi_5\ \Pi_3\ \Pi_{10}\ \Pi_8\ \Pi_{12}$
edges of P_1: $\tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2$

P_2: $\Pi_1\ \Pi_6\ \Pi_{10}\ \Pi_5\ \Pi_3\ \Pi_9\ \Pi_4\ \Pi_{11}\ \Pi_7\ \Pi_2\ \Pi_8\ \Pi_{12}$
edges of P_2: $\tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_2$

Appendix D

Hamiltonian dipaths in A_5. For $m \in \{1, 2, 3\}$, $R_m = R^*Q_m$.

R^*: $\Pi_1\ \Pi_{16}\ \Pi_{17}\ \Pi_6\ \Pi_{20}\ \Pi_{57}\ \Pi_7\ \Pi_{55}\ \Pi_{13}\ \Pi_2\ \Pi_{30}\ \Pi_{27}\ \Pi_8\ \Pi_{22}\ \Pi_{45}\ \Pi_{48}\ \Pi_{58}$
edges of R^*: $\tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_4\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2$

Q_1: $\Pi_{58}\ \Pi_{51}\ \Pi_3\ \Pi_{32}\ \Pi_{52}\ \Pi_{24}\ \Pi_{35}\ \Pi_{46}\ \Pi_{36}\ \Pi_{29}\ \Pi_{49}\ \Pi_{14}\ \Pi_{12}\ \Pi_{56}\ \Pi_{21}\ \Pi_{34}\ \Pi_{43}$
edges of Q_1: $\tau_2\ \tau_4\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_4\ \tau_4\ \tau_2\ \tau_2\ \tau_4\ \tau_3\ \tau_2$

$\Pi_{47}\ \Pi_{59}\ \Pi_{38}\ \Pi_{33}\ \Pi_{40}\ \Pi_{39}\ \Pi_4\ \Pi_{53}\ \Pi_{31}\ \Pi_{37}\ \Pi_{60}\ \Pi_{42}\ \Pi_{41}$
$\tau_3\ \tau_4\ \tau_2\ \tau_4\ \tau_3\ \tau_3\ \tau_4\ \tau_3\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2$

$\Pi_{25}\ \Pi_{50}\ \Pi_{15}\ \Pi_{44}\ \Pi_{23}\ \Pi_9\ \Pi_{26}\ \Pi_{28}\ \Pi_{18}\ \Pi_{54}\ \Pi_5\ \Pi_{19}\ \Pi_{10}\ \Pi_{11}$
$\tau_2\ \tau_3\ \tau_2\ \tau_4\ \tau_4\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_4\ \tau_4\ \tau_2\ \tau_4$

Q_2: $\Pi_{58}\ \Pi_{51}\ \Pi_3\ \Pi_{49}\ \Pi_{36}\ \Pi_{29}\ \Pi_{59}\ \Pi_{38}\ \Pi_{33}\ \Pi_{14}\ \Pi_{12}\ \Pi_{56}\ \Pi_{21}\ \Pi_{34}$
edges of Q_2: $\tau_2\ \tau_4\ \tau_3\ \tau_2\ \tau_2\ \tau_4\ \tau_4\ \tau_2\ \tau_2\ \tau_4\ \tau_2\ \tau_2\ \tau_4\ \tau_2$

$\Pi_{40}\ \Pi_{39}\ \Pi_9\ \Pi_{26}\ \Pi_{28}\ \Pi_{43}\ \Pi_{47}\ \Pi_{41}\ \Pi_{25}\ \Pi_{50}\ \Pi_{15}\ \Pi_{10}\ \Pi_{11}\ \Pi_5$
$\tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_4\ \tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_4\ \tau_2\ \tau_4$

$\Pi_{19}\ \Pi_{42}\ \Pi_{35}\ \Pi_{46}\ \Pi_{24}\ \Pi_{32}\ \Pi_{52}\ \Pi_{18}\ \Pi_{54}\ \Pi_4\ \Pi_{53}\ \Pi_{31}\ \Pi_{37}\ \Pi_{60}\ \Pi_{44}\ \Pi_{23}$
$\tau_3\ \tau_4\ \tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_4\ \tau_2\ \tau_2\ \tau_4\ \tau_3\ \tau_3\ \tau_2\ \tau_3\ \tau_4$

Q_3: $\Pi_{58}\ \Pi_{26}\ \Pi_{39}\ \Pi_9\ \Pi_{34}\ \Pi_{43}\ \Pi_{47}\ \Pi_{28}\ \Pi_{18}\ \Pi_{54}\ \Pi_4\ \Pi_{40}\ \Pi_{59}\ \Pi_{38}\ \Pi_{29}$
edges of Q_3: $\tau_4\ \tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_4\ \tau_4\ \tau_2$

$\Pi_{49}\ \Pi_{11}\ \Pi_5\ \Pi_{52}\ \Pi_{24}\ \Pi_{32}\ \Pi_{23}\ \Pi_3\ \Pi_{15}\ \Pi_{44}\ \Pi_{31}\ \Pi_{37}\ \Pi_{53}\ \Pi_{19}$
$\tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_2\ \tau_4\ \tau_2\ \tau_2\ \tau_3\ \tau_3\ \tau_2\ \tau_2$

$\Pi_{10}\ \Pi_{50}\ \Pi_{41}\ \Pi_{25}\ \Pi_{60}\ \Pi_{42}\ \Pi_{35}\ \Pi_{46}\ \Pi_{36}\ \Pi_{21}\ \Pi_{12}\ \Pi_{56}\ \Pi_{33}\ \Pi_{14}\ \Pi_{51}$
$\tau_3\ \tau_2\ \tau_2\ \tau_3\ \tau_2\ \tau_4\ \tau_2\ \tau_3\ \tau_4\ \tau_2\ \tau_2\ \tau_4\ \tau_2\ \tau_3$

References

[1] M. Behzad, G. Chartrand, and L. Lesniak–Foster, Graphs and Digraphs, Wadsworth International Mathematics Series (1979).

[2] E. Durnberger, Connected Cayley graphs of semi-direct products of cyclic groups of prime order by Abelian groups are Hamiltonian, Discrete Math. 46 (1983) 55–68.

[3] R.J. Gould and R. Roth, On the diameter of the (i, j, n) Cayley Graph of A_n, in preparation.

[4] J.B. Klerlein, Hamiltonian cycles in Cayley color graphs, J. Graph Theory 2 (1978) 65–68.

[5] D. Marušič, Hamiltonian circuits in Cayley graphs, Discrete Math. 46 (1983) 49–54.

[6] A. Nijenhuis and H. Wilf, Combinatorial Algorithms, second edition, NEXPER Chap. (Academic Press, New York, 1978).

[7] J.J. Rotman, The Theory of Groups, second edition, Allyn and Bacon Ser. in Adv. Math. (1973).

[8] R. Sedgewick, Permutation Generation Methods, Computer Surveys 9 (1977) 137–164.

[9] P.J. Slater, Generating all Permutations by Graphical Transpositions, Ars Combin. 5 (1978) 219–225.

[10] P. Tannenbaum, Minimal Cost Permutation Generating Algorithms, Proc. Fourteenth SE Conf. on Combinatorics, Graph Theory, and Computing, to appear.

[11] M. Tchuente, Generation of permutations by graphical exchanges, Ars Combin. 14 (1982) 115–122.

[12] D. Witte, Cayley digraphs of prime power order are Hamiltonian, to appear in Discrete Math.