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Forbidden Subgraphs and Hamiltonian Properties in the Square of a Connected Graph

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ABSTRACT

Various Hamiltonian-like properties are investigated in the squares of connected graphs free of some set of forbidden subgraphs. The star $K_{1,4}$, the subdivision graph of $K_{1,3}$, and the subdivision graph of $K_{1,3}$ minus an endvertex play central roles. In particular, we show that connected graphs free of the subdivision graph of $K_{1,3}$ minus an endvertex have vertex pancyclic squares.

In this article, all graphs are finite, undirected, without loops or multiple edges. Terms not defined here can be found in [1]. If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G induced by U is the graph with vertex set U and whose edge set consists of those edges of G incident with two elements of U. A graph is *Hamiltonian* if it contains a cycle through all its vertices. A graph is *vertex pancyclic* if each of its vertices lies on a cycle of length ℓ, for each ℓ, $3 \leq \ell \leq |V(G)|$. The *square* of a graph G, denoted G^2, is that graph obtained from G by inserting an edge between any two vertices at distance 2 apart in

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G. A graph G is $(H_1, H_2, \ldots, H_k)$-*free* $(k \geq 1)$, if G contains no induced subgraph isomorphic to H_i, for any $i = 1, 2, \ldots, k$. If $k = 1$, we simply say G is H_1-free.

The investigation of Hamiltonian properties in the square of a graph was spurred by the classical result of Fleischner [2].

Theorem A [2]. If G is a 2-connected graph, then G^2 is Hamiltonian.

Harary and Schwenk [5] were able to characterize when the square of a tree is Hamiltonian based on the subdivision graph of $K_{1,3}$ (see Fig. 1).

Theorem B (Harary and Schwenk [5]). For any tree T, T^2 is Hamiltonian if and only if T is $S(K_{1,3})$-free.

Until recently, few results had been obtained on the large class of connected graphs not covered by Theorems A and B. Then Matthews obtained the following.

Theorem C (Matthews [6]). If G is a connected $K_{1,3}$-free graph, then G^2 is vertex pancyclic.

The purpose of this paper is to extend the result of Matthews and obtain other Hamiltonian-like results on the square of a connected graph. We begin with a useful lemma.

Lemma 1. Let G be a $K_{1,4}$-free graph. For each vertex v of G, its neighborhood $N(v)$ can be partitioned into at most three sets so that the graph induced by each set contains a spanning path.

Proof. If $\deg v \leq 3$ the result is immediate. So suppose $\deg v \geq 4$ and that $N(v) = \{v_1, v_2, \ldots, v_k\}$ $(k \geq 4)$. Let $P_1 : v_1v_2 \cdots v_i$ be a path of maximum length in $\langle N(v)\rangle$. If P_1 contains all of $N(v)$ we are done, so assume $v_{i+1} \notin P_1$. Let $P_2 : v_{i+1}v_{i+2} \cdots v_j$ be a path of maximum length starting with v_{i+1} in $\langle N(v) - V(P_1)\rangle$. If $N(v) = V(P_1) \cup V(P_2)$ then we are done, so let S_3 denote the vertices that remain; that is, $S_3 = N(v)$

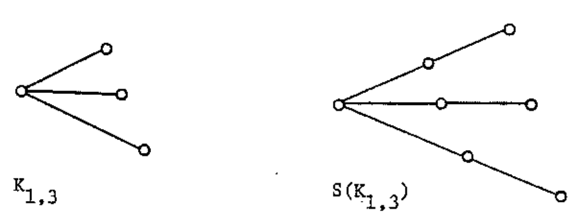

FIGURE 1. The star $K_{1,3}$ and its subdivision graph $S(K_{1,3})$.

$- V(P_1) - V(P_2)$. Clearly $V(P_1) \cup V(P_2) \cup S_3$ partitions $N(v)$, and $\langle V(P_1)\rangle$ and $\langle V(P_2)\rangle$ each contain spanning paths. If $|S_3| = 1$ we are again done, so we suppose that $w_1, w_2 \in S_3$.

Now consider the graph $H = \langle v, v_i, v_j, w_1, w_2\rangle$. The graph H is isomorphic to $K_{1,4}$ unless w_1w_2 is an edge of G. This follows since v_i (and similarly v_j) cannot be adjacent to any vertex off the path P_1 (and similarly P_2). This implies that $\langle S_3\rangle$ is complete and hence contains a spanning path. ∎

We note that this technique could be used to extend Lemma 1 to $K_{1,n}$-free graphs; however, this would not add to the results to follow.

Theorem D (Fleischner [3]). If G is a graph, then G^2 is Hamiltonian if and only if G^2 is vertex pancyclic.

In attempting to generalize Theorem C, and in view of Theorem B, stars and the graph $S(K_{1,3})$ naturally come to mind. Further, using the characterization of square traceable graphs [4], one realizes that even $K_{1,4}$-free trees are not necessarily square traceable. We now present a generalization of Theorem C, based on the graph Y of Figure 2.

Theorem 2. If G is a connected Y-free graph, then G^2 is vertex pancyclic.

Proof. From Theorem D, it suffices to show that G^2 is Hamiltonian. Thus we choose a longest cycle C in G^2. If C contains all vertices of G^2 we are done, so we assume there exists $x \in V(G^2)$ such that x is not on C. Since G is connected, we may choose x so that it is adjacent in G to some vertex of C. Further, without loss of generality we may assume x is adjacent in G to x_1 and that the cycle C is

$$C : x_1x_2x_3 \cdots x_nx_1.$$

Since C is a longest cycle in G^2, $x_1x_2 \in E(G^2) - E(G)$, for otherwise $xx_2 \in E(G^2)$ and a cycle longer than C would result. We note that for every $uv \in E(G^2) - E(G)$ which lies on C, there exists a vertex w on C adjacent in G to both u and v. If w was not on C, then a longer cycle would be immediate.

We proceed by showing that this finite cycle C contains an infinite sequence of distinct vertices $x_{i_1}, x_{i_2}, \ldots$, with the following properties

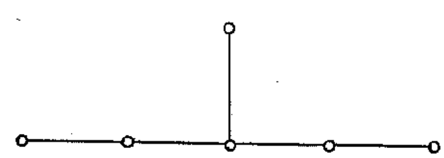

FIGURE 2. The graph $Y = S(K_{1,3})$ minus an endvertex.

holding for each x_{i_j}:

(a) $x_{i_k} \neq x_{i_j}$ and $x_{i_k+1} \neq x_{i_j}$ for each $k < j$,

(b) $x_1 x_{i_j} \in E(G)$,

(c) $x_{i_j} x_{i_j+1} \in E(G^2) - E(G)$,

(d) $x_{i_{j+1}} x_{i_j}$ and $x_{i_{j+1}} x_{i_j+1} \in E(G)$.

By the previous observations, let x_{i_1} be a vertex on C with $x_{i_1}x_1$ and $x_{i_1}x_2$ edges in G. If $x_{i_1}x_{i_1+1} \in E(G)$, then $x_2 x_{i_1+1} \in E(G^2)$ and since $xx_{i_1} \in E(G^2)$ then

$$x_1 x x_{i_1} x_{i_1-1} \cdots x_2 x_{i_1+1} x_{i_1+2} \cdots x_n x_1$$

would be a cycle in G^2 longer than C. Thus, $x_{i_1}x_{i_1+1} \in E(G^2) - E(G)$, and we see that x_{i_1} meets conditions (a)–(d) [meeting (d) vacuously]. Since $x_{i_1}x_{i_1+1} \in E(G^2) - E(G)$, there exists x_{i_2} on C with $x_{i_1}x_{i_2}$ and $x_{i_1+1}x_{i_2}$ edges of G [thus x_{i_2} meets property (d)]. We note that by the maximality of C, $x_{i_2} \neq x_1$ and $x_{i_2} \neq x_2$. Consider in G the graph $H \cong \langle\{x, x_1, x_2, x_{i_1}, x_{i_1+1}, x_{i_2}\}\rangle$. Since $H \cong Y$, further edges must be present in G between vertices of H. We already know $xx_2 \notin E(G)$. If $xx_{i_1} \in E(G)$, then $xx_2 \in E(G^2)$ and again a longer cycle results. If $xx_{i_1+1} \in E(G)$, then since $xx_{i_1} \in E(G^2)$, a longer cycle is again immediate. If $xx_{i_2} \in E(G)$, then both xx_{i_1} and xx_{i_1+1} are edges of G^2 and once more a longer cycle is produced. Thus no further edge of H involves x.

Since x_{i_2} is not x_1 or x_2, the edges $x_1x_{i_1+1}$ and $x_2x_{i_1+1}$ are not in G. If $x_2x_{i_2} \in E(G)$, we obtain a cycle longer than C since $x_2x_{i_1+1}$ would be an edge of G^2. Thus, since this induced subgraph is not isomorphic to Y, we must have that $x_1x_{i_2} \in E(G)$, [and hence $xx_{i_2} \in E(G^2)$]. Therefore, x_{i_2} meets properties (a) and (b).

Now suppose $x_{i_2}x_{i_2+1}$ is an edge of G. This implies that $x_{i_2+1}x_{i_1+1} \in E(G^2)$. Then

$$x_{i_1} x x_{i_2} x_{i_2-1} \cdots x_{i_1+1} x_{i_2+1} x_{i_2+2} \cdots x_{i_1}$$

is a cycle in G^2 longer than C. Thus $x_{i_2}x_{i_2+1} \in E(G^2) - E(G)$ and condition (c) is met; hence x_{i_2} meets all properties (a)–(d).

Now suppose we have chosen $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ satisfying conditions (a)–(d). We now produce a vertex $x_{i_{k+1}}$ also satisfying these conditions.

Let $x_{i_{k+1}}$ be a vertex on C such that $x_{i_{k+1}}x_{i_k}$ and $x_{i_{k+1}}x_{i_k+1}$ are edges of G [that is, property (d) holds]. Further suppose that $x_{i_{k+1}} = x_{i_j}$ for some $j < k$. The vertex x_{i_j} was chosen on C with property (d); that is

$x_{i_{j-1}+1}x_{i_j} \in E(G)$. Hence $x_{i_{j-1}+1}x_{i_k+1} \in E(G^2)$. Then

$$x_{i_{j-1}}xx_{i_k}x_{i_k-1} \cdots x_{i_{j-1}+1}x_{i_{j-1}+2} \cdots x_{i_j-1}$$

is a cycle longer than C.

A similar argument shows that $x_{i_{k+1}} \neq x_{i_j+1}$, and so property (a) holds. Since $\langle x, x_1, x_{i_{k-1}+1}, x_{i_k}, x_{i_k+1}, x_{i_{k+1}} \rangle$ cannot be isomorphic to Y in G, a case analysis similar to that performed on H earlier shows that $x_1x_{i_{k+1}} \in E(G)$, and thus property (b) holds. If $x_{i_{k+1}}x_{i_{k+1}+1} \in E(G)$, then $x_{i_k+1}x_{i_{k+1}+1} \in E(G^2)$ and, as above, a cycle longer than C results; thus property (c) must hold as well. Hence $x_{i_{k+1}}$ exists and meets the stated conditions. But this implies that there are infinitely many vertices in this finite graph, a contradiction. Hence G^2 must be Hamiltonian and therefore vertex pancyclic. ∎

To further generalize Theorem C, we must include $S(K_{1,3})$ in our set of forbidden subgraphs. The graph $S(K_{1,3})$ itself shows that either it or one of its subgraphs must be in any set of forbidden subgraphs. Our next result includes the graphs of Figure 3 in the set of forbidden subgraphs.

Theorem 3. If G is a connected $(K_{1,4}, S(K_{1,3}), F, W)$-free graph of order $p \geq 3$, then G^2 is vertex pancyclic.

Proof. Again from Theorem D, it suffices to show G^2 is Hamiltonian. We proceed by induction on the order of the graph. If G has three or four vertices the result follows easily. Hence we assume G has order at least 5. If G has maximum degree 2, then G is a cycle or a path and the result is again obvious.

Thus assume there exists a vertex v of degree at least three in G. Let $N(v) = \{v_1, v_2, \ldots, v_k\}(k \geq 3)$. By Lemma 1, $N(v)$ can be partitioned into at most three sets S_i $(i = 1, 2, 3)$ such that $\langle S_i \rangle$ contains a spanning path $(i = 1, 2, 3)$. Let such a spanning path of $\langle S_i \rangle$ be P_i. Without loss of generality say

$$P_1 : v_1v_2 \cdots v_j, \quad P_2 : v_{j+1}v_{j+2} \cdots v_r, \quad P_3 : v_{r+1}v_{r+2} \cdots v_k$$

[renumber the vertices of $N(v)$ if necessary].

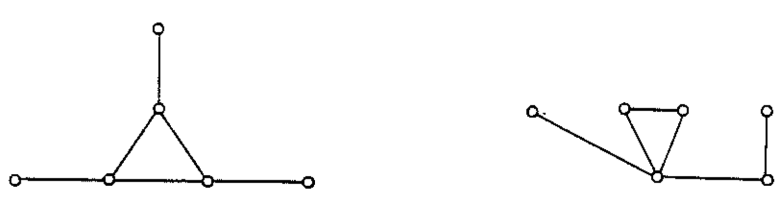

FIGURE 3. The graphs F and W.

Now consider a spanning tree T of G containing all edges from v to vertices of $N(v)$. Let G_i be the subgraph of G induced by v and the vertices of the branch of T containing v_i ($i = 1, 2, \ldots, k$). Since $k \geq 3$, $|V(G_i)| < |V(G)|$. Further, G_i clearly contains none of the graphs $K_{1,4}$, $S(K_{1,3})$, F, or W as induced subgraphs. Thus, by the induction hypothesis, if $|V(G_i)| \geq 3$, then G_i^2 contains a Hamiltonian cycle C_i ($i = 1, 2, \ldots, k$). We note that if $|V(G_i)| = 2$ then G_i^2 is merely traceable.

We now claim that the graph $(G_i - v)^2$ contains a Hamiltonian path from the vertex v_i to a vertex w_i, where $d_G(v, w_i) = 2$; that is, $w_i \in N(v_i)$ ($i = 1, 2, \ldots, k$). [If $|V(G_i)| = 2$, then the path is merely v_i itself.]

To verify this claim, first suppose that C_i contains the edge vv_i. The other edge of C_i incident with v must be of the form vw, where $w \in N(v_i)$. But then, deleting v and its incident edges from C_i leaves a Hamiltonian $(v_i - w)$-path in $(G_i - v)^2$. Hence $w_i = w$ suffices.

Next suppose that the edge vv_i is not on C_i. Say instead that vw and vx are the edges of C_i incident with v, where $w, x \in N(v_i)$. If the edge xv_i (or similarly wv_i) is on C_i, then remove v and its incident edges and consider the path from v_i to w followed by the edge from w to x. This is a Hamiltonian $(v_i - x)$-path in $(G_i - v)^2$, so in this case $w_i = x$ (or similarly w). Next suppose that neither wv_i nor xv_i is on C_i. If v_i is adjacent on C_i to any other $s \in N(v_i)$, then, assuming x is on the segment of C_i between v and s containing v_i, proceed from v_i to x, then to s along the edge xs, then along the cycle C_i from s to w. This is a Hamiltonian $(v_i - w)$-path in $(G_i - v)^2$, so we may let $w_i = w$. We note a similar argument applies to w.

Finally, suppose that no neighbor of v_i is adjacent to v_i on C_i. Thus v_i must be adjacent with two vertices, say b_1 and b_2; where $d_G(v_i, b_j) = 2$, ($j = 1, 2$). Further, suppose the vertex adjacent to both v_i and b_j is a_j ($j = 1, 2$). Without loss of generality we assume that C_i appears as

$$C_i : v, w, L_1, b_1\ v_i, b_2, L_2, x, v$$

where L_1 and L_2 are paths joining w and b_1 and b_2 and x, respectively.

Now, $\langle\{v, a_1, a_2, x, v_i\}\rangle \cong K_{1,4}$ unless, in G, there exists at least one additional edge joining two of these vertices. Since such an edge cannot involve v, three subcases exist.

Subcase 1. If $xa_1 \in E(G)$, then $xb_1 \in E(G^2)$ and the path $v_i, b_2, L_2, x, b_1, L_1, w$ suffices.

Subcase 2. If $xa_2 \in E(G)$, then $\langle\{v, v_i, x, a_2, a_1, b_1\}\rangle \cong W$ unless at least one additional edge exists (and such an edge cannot involve v). If any of the edges xa_1, xb_1, or b_1a_2 is in G, the path of Subcase 1 is again obtained since $xb_1 \in E(G^2)$. If $a_1a_2 \in E(G)$, then $\langle\{v, v_i, a_1, a_2, b_1, b_2\}\rangle$

$\cong F$ unless one of a_1b_2, b_1a_2, or b_1b_2 is in G. In any case, $b_1b_2 \in E(G^2)$ and we obtain the path

$$v_i, x, L_2, b_2, b_1, L_1, w.$$

Subcase 3. If $a_1a_2 \in E(G)$, the argument that ends Subcase 2 suffices.

Thus, in all cases we have found the desired path and the claim is verified.

We now construct a spanning path in $(\cup_{i=1}^{j} G_i - v)^2$ that begins at v_1 [in $N(v)$] and ends with w_j (at distance 2 from v). We call this path P_1^* and it contains the paths $P_1, P_2, \ldots, P_j$ traversed in that order. This is possible since $d_G(w_i, v_{i+1}) \leq 2$ $(i = 1, 2, \ldots, j - 1)$. We also note that similar paths P_2^* in $(\cup_{i=j+1}^{r} G_i - v)^2$ and P_3^* in $(\cup_{i=r+1}^{k} G_i - v)^2$ also exist.

Our final goal is to link the paths P_1^* P_2^*, and P_3^* and the vertex v to obtain a Hamiltonian cycle of G^2. (We note that two or fewer paths make these arguments simpler.) To do this, recall that these paths end with w_j, w_r, and w_k, respectively. The graph $\langle\{v, v_j, w_j, v_r, w_r, v_k, w_k\}\rangle \supset S(K_{1,3})$ and so further edges must be present in G. Also, no other edge may involve v (as we have identified and used all its neighbors). No other edge may involve two of v_j, v_r, and v_k as this contradicts the fact that maximal paths in $N(v)$ were chosen (using the proof of Lemma 1). Thus, either an edge involving two of w_j, w_r, w_k or an edge involving one of v_j, v_r, v_k and one of (of a different subscript) w_j, w_r w_k exists. In either case, this implies that in G^2 the edge between the corresponding w's must exist. Without loss of generality suppose that $w_jw_r \in E(G^2)$. The Hamiltonian cycle of G^2 is then (letting $\overline{P}$ be the reverse of the path P) $v, P_1^*, \overline{P_2^*}, P_3^*, v$.

Thus G^2 is Hamiltonian and hence vertex pancyclic. ∎

We conclude by noting that Theorem 2 and Theorem B lead us to the following conjecture.

Conjecture 1. If G is a connected $S(K_{1,3})$-free graph, then G^2 is vertex pancyclic.

A somewhat lesser result that still generalizes Theorem C and improves upon Theorem 3 would also be of interest.

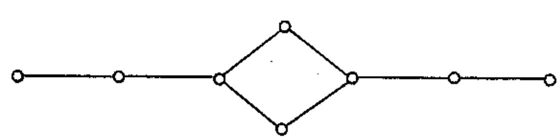

FIGURE 4. A nontraceable graph without $K_{1,4}$, $S(K_{1,3})$, F, or W with vertex pancyclic square.

Conjecture 2. If G is a connected $(S(K_{1,3}), K_{1,4})$-free graph, then G^2 is vertex pancyclic.

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