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A NOTE ON GRAPHS WHOSE NEIGHBORHOODS ARE n-CYCLES

ABSTRACT. Let G be a graph, and let v be a vertex of G. We denote by $N(v)$ the set of vertices of G which are adjacent to v, and by $\langle N(v)\rangle$ the subgraph of G induced by $N(v)$. We call $\langle N(v)\rangle$ the neighborhood of v. In a paper of 1968, Agakishieva has, as one of her main theorems, the statement:

"Graphs in which every neighborhood is an n-cycle exist if and only if $3 \leqslant n \leqslant 6$."

It it is the object of this note to provide a list of counter examples to this statement.

By an *automorphism* of a graph we mean a permutation of its vertices which preserves adjacency. The automorphisms of a graph form a group, which we call *the group of the graph*.

By a *map* we mean a decomposition of an unbounded surface into N_2 non-overlapping simply-connected regions (called *faces*) by N_1 arcs (called *edges*) joining pairs of N_0 points (called *vertices*). By an *automorphism* of a map we mean a permutation of its vertices which sends faces to faces and edges to edges, and which preserves incidence. Thus, for any two corresponding faces or vertices, the edges incident with one correspond to the edges incident with the other in the same (or opposite) cyclic order. The automorphisms of a map form a group, which we call the group of the map.

An automorphism of a map is completely determined by its effect on any one face: If we fix a face, with all its sides and vertices, then each neighboring face is likewise fixed, so that such an automorphism must be the identity. A map is said to be *regular* if its group contains two particular automorphisms: one, say ϱ, which cyclically permutes the successive edges of one face, and another, say σ, which cyclically permutes the successive edges meeting at one vertex of this face. It follows that the group of automorphisms of a regular map is transitive on its N_0 vertices, N_1 edges, and N_2 faces. We declare such a map to be of type $\{p, q\}$ if p edges belong to a face and q to each vertex. The notation $\{p, q\}$ is due to Schläfli.

Each regular map of type $\{p, q\}$ has a polygon P of which any two consecutive edges belong to a face of the map, but no three do so. P is called a *Petrie polygon* of the map. Due to the transitivity of the group, all the Petrie polygons of the map have the same length. The largest map of type $\{p, q\}$ that possesses a Petrie polygon of r sides is denoted by $\{p, q\}_r$. For example, $\{3, 6\}_{2n}$ represents a regular map whose faces are triangles, 6 at each vertex, and whose Petrie polygons all have length $2n$. For a thorough discussion of regular maps see [2].

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The five regular polyhedra have the symbols $\{3, 3\}_4$, $\{3, 4\}_6$, $\{4, 3\}_6$, $\{3, 5\}_{10}$, and $\{5, 3\}_{10}$, denoting the tetrahedron, octahedron, cube, icosahedron, and dodecahedron, respectively. From the last four we can derive simpler 'elliptic' maps by identifying pairs of 'antipodal' vertices.

The vertices and edges of any map M may be regarded as forming a graph G. If M is $\{p, q\}_r$, a convenient symbol for the corresponding graph G is $(p, q)_r$. It is clear that any automorphism of M induces an automorphism of G, and the group of M is a subgroup of the group G. For example, the group of $\{3, 5\}_5$ is A_5, but $(3, 5)_5$ is K_6, whose group is S_6. The icosahedral group A_5 is transitive on the 6 Petrie polygons such as $v_5v_6v_2v_3v_1v_5$ in Figure 1

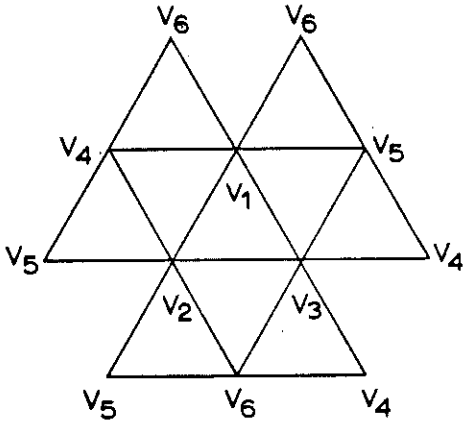


Fig. 1.

(where a specimen of $\{3, 5\}_5$ is shown, unfolded to form a planar 'net'). But the symmetric group is transitive on 72 pentagons such as $v_2v_3v_5v_6v_4v_2$. Each map $\{p, q\}_r$ has a companion $\{r, q\}_p$ in which the roles of faces and Petrie polygons are interchanged; therefore $(p, q)_r = (r, q)_p$.

One requirement for a graph $(p, q)_r$ to have neighborhoods all of which are q-cycles is evidently $p = 3$. This is satisfied by $(3, 3)_4$, $(3, 4)_6$, and $(3, 5)_{10}$, whose neighborhoods are 3-cycles, 4-cycles, and 5-cycles, respectively. When $q \geqslant 5$, there is a second requirement; namely:

THEOREM. *If $q \geqslant 5$, then the graph $(3, q)_r$ has neighborhoods which are all q-cycles only if $r \geqslant 7$.*

Proof. Assume $q \geqslant 5$ and that every neighborhood of $G = (3, q)_r$ is a q-cycle. Let v be a vertex of G and consider the subgraph G_v which consists of v, the edges incident with v, and the q-cycle that forms the neighborhood of

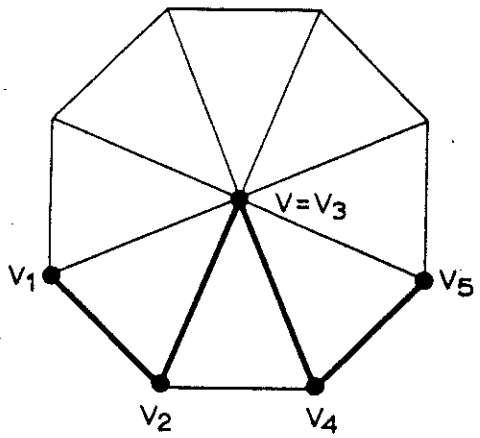
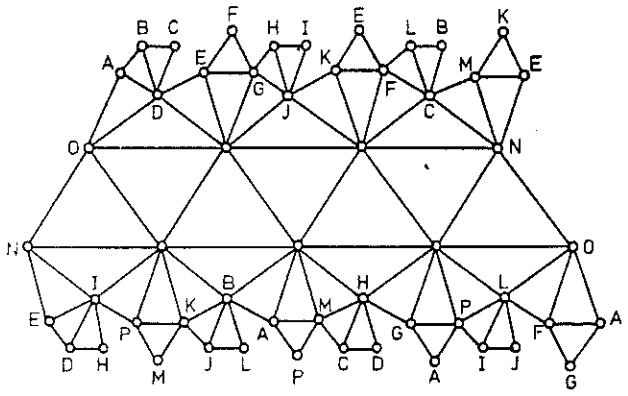


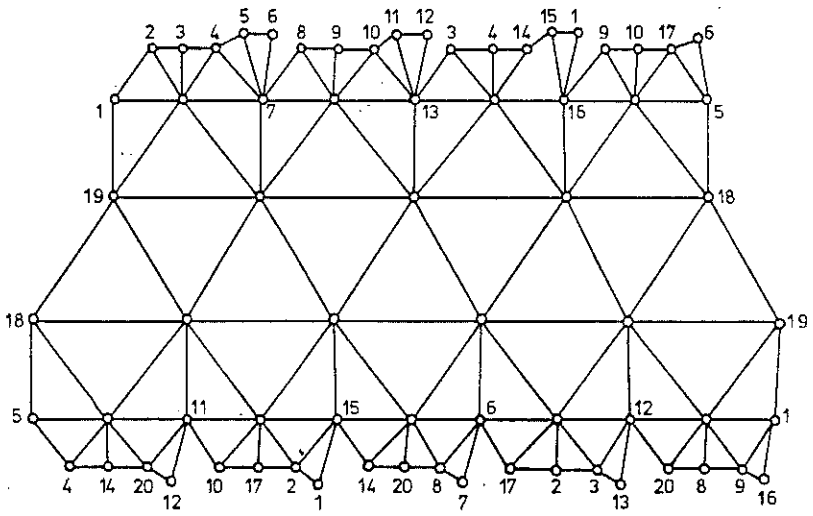
Fig. 2.

v. Let $P = v_1 v_2 \ldots v_r v_1$ be an r-cycle of G which corresponds to a Petrie polygon of $\{3, q\}_r$ and assume that P has an edge in common with G_v. Then P has exactly four edges in common with G_v, as shown in Figure 2. Relabel the vertices of P, if necessary, so that these edges are $v_1 v_2$, $v_2 v_3$, $v_3 v_4$, and $v_4 v_5$. Clearly $r \geqslant 5$. We shall now exclude the cases $r = 5$ and $r = 6$. If $r = 5$, then $v_5 v_1$ is the remaining side of P, and is an extraneous edge of the neighborhood of v. Here $v = v_3$.



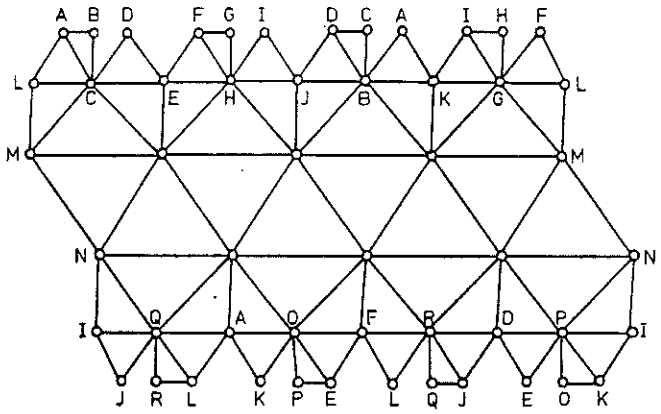
$\{3,8\}_7$: 21 vertices
84 edges
56 faces

Fig. 3.



$\{3,7\}_9$ 36 vertices
126 edges
84 faces

Fig. 4.



$\{3,7\}_8$: 24 vertices
84 edges
56 faces

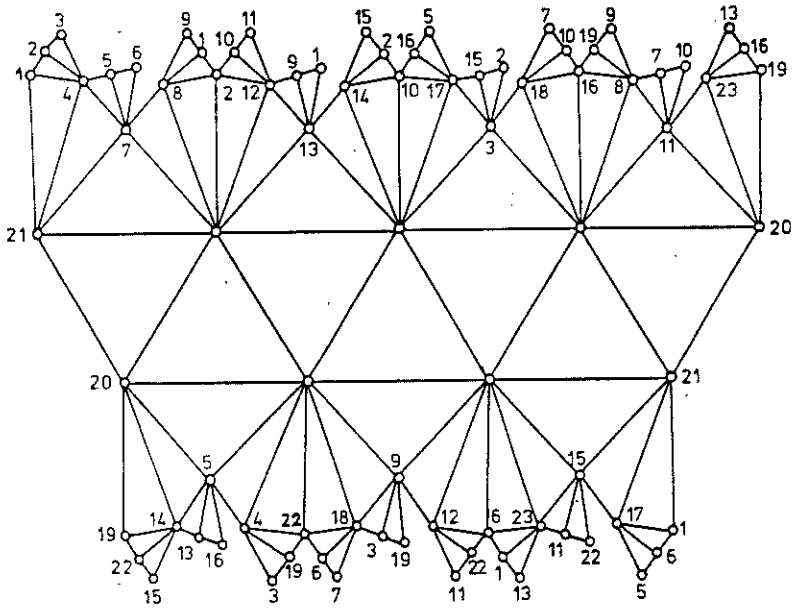
Fig. 5.

If $r=6$, then v_6 does not belong to G_v. However, in this case v_5v_6 and v_6v_1 are two sides of a triangle. The third side is v_5v_1, and is an extraneous edge of $\langle N(v)\rangle$.

The graphs $(3, 6)_{2n}$ exist for each $n\geqslant 4$ and provide an infinite class of graphs, the neighborhoods of which are all 6-cycles. The map $\{3, 6\}_{2n}$ has n^2 vertices, $3n^2$ edges, and $2n^2$ faces. It can be drawn on the surface of a torus.

It is interesting that the neighborhoods of the graph $(3, 5)_5$ are isomorphic to the complete graph K_5, while those of $(3, 6)_6$ are isomorphic to the complete bipartite graph (Thomsen graph) $K_{3,3}$. The map $\{3, 5\}_5$ can be drawn on the real projective plane, and can be obtained by identifying opposite vertices of an icosahedron.

On page 139 of [2] there is a listing of the known regular maps of the form $\{p, q\}_r$. Among the entries on the list (or rather, their duals), we have investigated $\{3, 8\}_7$, $\{3, 7\}_8$, $\{3, 7\}_9$, $\{3, 9\}_7$ and $\{3, 7\}_{12}$, and have found that each of these yields an example of $(3, q)_r$, $r\geqslant 7$, in which every neigh-



$\{3,9\}_7$: 28 vertices
126 edges
84 faces

Fig. 6.

borhood is a q-cycle. It is likely that the other graphs $(3, q)_r$, $r \geqslant 7$, that can be obtained from the list in [2] are also examples.

In Figures 2–5 we have depicted the regular maps $\{3, 8\}_7$, $\{3, 7\}_9$, $\{3, 7\}_8$, and $\{3, 9\}_7$, respectively, each one 'unfolded' to form a planar map. To 're-fold' these maps, one would bring together vertices that bear the same letter.

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