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DEGREE SETS AND GRAPH FACTORIZATIONS

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Recently there has been considerable interest in the concepts of degree sets and factorizations of graphs. In this paper we look into each of these ideas. Any definitions not provided here may be found in [1].

We begin by defining the concept of factors of a graph. A *factor* of a graph G is a (possibly empty) spanning subgraph of G. If $G_1, G_2, \ldots, G_k$ are graphs such that $V(G_1) = V(G_2) = \ldots = V(G_k)$ and the edge sets are mutually disjoint, then the *edge sum* of the graphs $G_1, G_2, \ldots, G_k$ is the graph

$$G = G_1 \oplus G_2 \oplus \ldots \oplus G_k,$$

where

$$V(G) = V(G_i), \quad i = 1, 2, \ldots, k,$$

and

$$E(G) = \bigcup_{i=1}^{k} E(G_i).$$

If the graph G is expressed as the edge sum of its factors, then this edge sum is called a *factorization* of G.

The *degree set* D of a graph G is the set of degrees of the vertices of G. Kapoor et al. showed in [2] that if D is a finite set of positive integers, then D is the degree set of a graph G. Kapoor et al. [2] determined also the minimum order of such a graph. We now extend the concept of degree sets to factorizations of graphs. Let $D_1, D_2, \ldots, D_k$ $(k \geqslant 2)$ be a finite sequence of sets where each D_i $(1 \leqslant i \leqslant k)$ is a set of positive integers. Let $\mu(D_1, D_2, \ldots, D_k)$ denote the minimum order of any graph G such that G has a factorization $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$, where the degree set of G_i is D_i $(1 \leqslant i \leqslant k)$. The object of this paper is to determine $\mu(D_1, D_2, \ldots, D_k)$ for certain sequences of sets $D_1, D_2, \ldots, D_k$.

In the process of finding $\mu(D_1, D_2, \ldots, D_k)$ it is convenient to be able to construct graphs containing a specified number of edge disjoint hamiltonian cycles. The following lemma helps to produce such graphs.

Let K_r denote the complete graph of order r and let $\bar{G}$ be the complement of the graph G.

LEMMA. *Let $n \geqslant 3$. Then for each k, $0 \leqslant k \leqslant [(n-1)/2]$, there is a graph G_k of order n containing k edge disjoint hamiltonian cycles which can be extended to k edge disjoint hamiltonian cycles in G_k+K_1, $G_k+\bar{K}_2$, and $G_k+\bar{K}_3$.*

Proof. To prove this lemma, we begin with the well-known result that for $n = 2k+1$, $k \geqslant 1$, the complete graph K_n contains k edge disjoint hamiltonian cycles. In fact, we construct a set of k such cycles. Since n is odd, these cycles form a factorization of K_n. We then extend them to k edge disjoint hamiltonian cycles of K_{n+1}. These cycles together with the 1-factor of those edges not used in the cycles form a factorization of K_{n+1}. We extend now the set of cycles thus obtained to k edge disjoint hamiltonian cycles of $K_n+\bar{K}_2$, which finally are extended to k edge disjoint hamiltonian cycles of $K_n+\bar{K}_3$. For any of these graphs, we can delete enough hamiltonian cycles so that the remaining graph has the proper number of such cycles.

If $k = 1$, the result is obvious. Thus we assume that $k \geqslant 2$, and let

$$V(G) = V(K_{2k+1}) = \{v, v_0, v_1, \ldots, v_{2k-1}\}.$$

For each i, $0 \leqslant i \leqslant k-1$, define the edge set of the hamiltonian cycles C_i of K_n to be

$$E(C_i) = \{vv_i, vv_{i+k}\} \cup \{v_{i-j}v_{i+j+1}\colon 0 \leqslant j \leqslant k-1\} \cup \cup \{v_{i+j}v_{i-j}\colon 1 \leqslant j \leqslant k-1\},$$

where all subscripts are expressed modulo $2k$. Then each C_i $(0 \leqslant i \leqslant k-1)$ is a hamiltonian cycle of K_n and

$$K_n = K_{2k+1} = C_0 \oplus C_1 \oplus \ldots \oplus C_{k-1}.$$

We now extend these k edge disjoint hamiltonian cycles of $K_n = K_{2k+1}$ to those of $K_{2k+2} = K_{2k+1}+K_1$.

Let $V(K_1) = \{u_1\}$ and $G = K_{2k+1}+K_1$. If i is odd and $0 \leqslant i \leqslant k-1$, delete the edge $v_i v_{i+1}$ from C_i and insert the edges $u_1 v_i$ and $u_1 v_{i+1}$ to obtain the hamiltonian cycle $C_{i,1}$ of the graph G. If k and i are even $(0 < i \leqslant k-1)$, delete the edge $v_{i+k} v_{i+k+1}$ from C_i and insert the edges $u_1 v_{i+k}$ and $u_1 v_{i+k+1}$ to construct the hamiltonian cycle $C_{i,1}$. For k even and $i = 0$, delete the edge vv_0 and insert the edges $u_1 v$ and $u_1 v_0$ to obtain the hamiltonian cycle $C_{0,1}$. On the other hand, for k odd and i even $(0 \leqslant i \leqslant k-1)$, delete the edge $v_{i+k} v_{i+k+1}$ from C_i and insert the edges $u_1 v_{i+k}$ and $u_1 v_{i+k+1}$ to obtain the hamiltonian cycle $C_{i,1}$. Let F be the 1-factor of G with $V(F) = V(G)$ and

$$E(F) = E(G) - \bigcup_{i=0}^{k-1} E(C_{i,1}).$$

Then $G = C_{0,1} \oplus \ldots \oplus C_{k-1,1} \oplus F$. Thus, we have extended the k edge disjoint hamiltonian cycles C_i $(0 \leqslant i \leqslant k-1)$ of K_{2k+1} to k edge disjoint hamiltonian cycles $C_{i,1}$ $(0 \leqslant i \leqslant k-1)$ of $G = K_{2k+1}+\bar{K}_1$.

Using this construction we can extend these k edge disjoint hamiltonian cycles of $G = K_{2k+1}+K_1 = K_{2k+2}$ to those of $H = K_{2k+1}+\bar{K}_2$. Let u_2 be the vertex added to G to obtain H. If k is odd and $0 < i < k-1$, delete the edge $v_{i+k}v_{i+k+1}$ from $C_{i,1}$ and insert the edges $u_2 v_{i+k}$ and $u_2 v_{i+k+1}$ to obtain the hamiltonian cycle $C_{i,2}$ of H. If $k-1$ is odd, delete the edge vv_{2k-1} of $C_{k-1,1}$ and insert the edges $u_2 v$ and $u_2 v_{2k-1}$ to construct the hamiltonian cycles $C_{k-1,2}$ of H. On the other hand, if i is even and $0 \leqslant i \leqslant k-1$, delete the edge $v_i v_{i+1}$ from $C_{i,1}$ and insert the edges $u_2 v_i$ and $u_2 v_{i+1}$ to obtain the hamiltonian cycle $C_{i,2}$ of H. Then H contains the k edge disjoint hamiltonian cycles $C_{i,2}$, $0 \leqslant i \leqslant k-1$.

We use again the construction provided above to extend these k edge disjoint hamiltonian cycles $C_{i,2}$ $(0 \leqslant i \leqslant k-1)$ of $H = K_{2k+1}+\bar{K}_2$ to those of $H_1 = K_{2k+1}+\bar{K}_3$. Let u_3 be the vertex added to H to obtain H_1. Let t stand for $\{k/2\}$, where $\{x\}$ is the least integer greater than or equal to x. For each i, $0 \leqslant i \leqslant k-1$, delete the edge $v_{t+i}v_{t+k+i}$ from $C_{i,2}$ and add the edges $u_3 v_{t+i}$ and $u_3 v_{t+k+i}$ to obtain the hamiltonian cycle $C_{i,3}$ of H_1. Then H_1 contains the k edge disjoint hamiltonian cycles $C_{i,3}$, $0 \leqslant i \leqslant k-1$. This completes the proof of the lemma.

Our first theorem provides $\mu(D_1, D_2, \ldots, D_k)$ when all D_i are singleton sets.

THEOREM 1. *Let $d_1, d_2, \ldots, d_k$ be positive integers, let $s = \sum_{i=1}^{k} d_i$, and let $D_i = \{d_i\}$, $1 \leqslant i \leqslant k$. Then*

$$\mu(D_1, D_2, \ldots, D_k) = \begin{cases} s+2 & \text{if } s \text{ is even, but some } d_i \text{ is odd,} \\ s+1 & \text{otherwise.} \end{cases}$$

Proof. If a graph G can be factored as $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$, where the degree set of G_i is D_i, $1 \leqslant i \leqslant k$, then each of the graphs G_i must be regular of degree d_i, $1 \leqslant i \leqslant k$, and so G must be regular of degree s. Hence $|V(G)| \geqslant s+1$.

Case 1. Assume that s is odd. In this case we show that there exists a graph G such that $|V(G)| = s+1$, and G can be factored as $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$, where the degree set of G_i is D_i, $1 \leqslant i \leqslant k$. Since $s+1$ is even, K_{s+1} is 1-factorable and we define G_i to consist of the edge sum of d_i distinct 1-factors of K_{s+1}, $1 \leqslant i \leqslant k$. Then $K_{s+1} = G$

$= G_1 \oplus G_2 \oplus \ldots \oplus G_k$, the degree set of G_i is D_i, $1 \leqslant i \leqslant k$, and so $\mu(D_1, D_2, \ldots, D_k) = s+1$.

Case 2. Suppose that s is even and that each d_i is even. Then K_{s+1} is 2-factorable and contains $s/2$ disjoint hamiltonian cycles. Since each d_i is even, let G_i be the edge sum of exactly $d_i/2$ distinct hamiltonian cycles, $1 \leqslant i \leqslant k$. Then the degree set of G_i is D_i, $1 \leqslant i \leqslant k$, and $G = K_{s+1} = G_1 \oplus G_2 \oplus \ldots \oplus G_k$. Thus, $\mu(D_1, D_2, \ldots, D_k) = s+1$.

Case 3. Assume that s is even and that at least one d_i, say d_j, is odd. Then $s+1$ vertices cannot suffice, since the factor G_j would have odd order and would be regular of odd degree. Thus any such graph must have order at least $s+2$.

Since $s+2$ is even, K_{s+2} is 1-factorable and has $s+1$ distinct 1-factors. For each i, $1 \leqslant i \leqslant k$, let G_i be the edge sum of d_i distinct 1-factors of K_{s+2}. Then we can write $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$, where the degree set of G_i is D_i, and so $\mu(D_1, D_2, \ldots, D_k) = s+2$.

This completes the proof of Theorem 1.

The determination of the numbers $\mu(D_1, D_2, \ldots, D_k)$ appears to be an extremely difficult problem. The remainder of this paper deals with special cases. We shall prove the results in four propositions, four corollaries, and summarize them in a theorem.

PROPOSITION 1. *Let* k, m, n *be positive integers with* $m < n$, *let* $D_1 = \{m, n\}$, *and let* $D_2 = \{2k\}$. *Then*

$$\mu(D_1, D_2) = n+2k+1. \tag{1}$$

Proof. Suppose that $G = G_1 \oplus G_2$, where G_i has degree set D_i, $i = 1, 2$. Then G_2 is regular of degree $2k$ and G_1 has at least one vertex of degree n, so that $|V(G)| \geqslant n+2k+1$. Thus any graph having the desired properties must have order at least $n+2k+1$. We now show that such a graph having order $n+2k+1$ always exists.

Case 1. Suppose that m is odd. Let $H_1 = K_1$. Since

$$(m+2k-1)/2 \leqslant [(n+2k-1)/2]$$

and since K_{n+2k} contains $[(n+2k-1)/2]$ edge disjoint hamiltonian cycles, the construction used in the Lemma implies that there is a graph H_2 of order $n+2k$, which is the edge sum of $(m+2k-1)/2$ edge disjoint hamiltonian cycles, and that the graph $G = H_1 + H_2$ also contains $(m+2k-1)/2 \geqslant k$ edge disjoint hamiltonian cycles.

The vertex of H_1 has degree $n+2k$ in G, while each vertex of H_2 has degree $m+2k$ in G. Let G_2 be the edge sum of any k of the $(m+2k-1)/2$ hamiltonian cycles and let $G_1 = G - E(G_2)$. Then $G = G_1 \oplus G_2$ and the degree set of G_i is D_i, $i = 1, 2$. Thus, in this case, $\mu(D_1, D_2) = n+2k+1$.

Case 2. Assume that m is even. Let $H_1 = K_2$. Let $s = (m+2k-2)/2$. Since $s \leqslant [(n+2k-2)/2]$, the construction used in the Lemma implies that there is a graph H_2 of order $n+2k-1$ that is the edge sum of s edge disjoint hamiltonian cycles and such that $G = H_1 + H_2$ also contains s edge disjoint hamiltonian cycles. Further, each of the two vertices of H_1 has degree $n+2k$ in G, while each vertex of H_2 has degree $m+2k$ in G.

Let the graph G_2 be the edge sum of k of these s hamiltonian cycles and let $G_1 = G - E(G_2)$. Then $G = G_1 \oplus G_2$ and the degree set of G_i is D_i, $i = 1, 2$. Thus, in this case, $\mu(D_1, D_2) = n+2k+1$.

COROLLARY 1. *Let $d_1, d_2, \ldots, d_t$, m, n be positive integers such that $m < n$ and $d = \sum_{i=1}^{t} d_i$. Let $D_i = \{d_i\}$ for $1 \leqslant i \leqslant t$ and let $D_{t+1} = \{m, n\}$. If d is even and n is odd, then*

$$\mu(D_1, D_2, \ldots, D_{t+1}) = n+d+1. \qquad (2)$$

Proof. Let $G = G_1 \oplus G_2 \oplus \ldots \oplus G_{t+1}$ be a graph such that the degree set of G_i is D_i, $1 \leqslant i \leqslant t+1$. Then G must have order at least $n+d+1$.

By the construction used in the proof of Proposition 1, let $G = H \oplus G_{t+1}$, where G has order $n+d+1$, H is the edge sum of $d/2$ hamiltonian cycles of G, and G_{t+1} has degree set D_{t+1}. Since H is the edge sum of $d/2$ edge disjoint hamiltonian cycles of G and G has even order, we can write H as the edge sum of d 1-factors of G (by dividing each hamiltonian cycle into two 1-factors). Then, let G_i, $1 \leqslant i \leqslant t$, be the edge sum of d_i of these 1-factors. Therefore, G_i is the regular graph of degree d_i for each i, $1 \leqslant i \leqslant t$. Thus

$$G = G_1 \oplus G_2 \oplus \ldots \oplus G_t \oplus G_{t+1},$$

where the degree set of G_i is D_i, $1 \leqslant i \leqslant t+1$.

PROPOSITION 2. *Let k, m, n be positive integers with n odd and $m < n$. Let $D_1 = \{m, n\}$ and $D_2 = \{2k-1\}$. Then* (1) *holds true.*

Proof. As before, any graph G having the desired properties must have order at least $n+(2k-1)+1 = n+2k$. However, if $G = G_1 \oplus G_2$, where the degree set of G_i is D_i, $i = 1, 2$, and $|V(G)| = n+2k$, then the factor G_2 would be a regular graph of degree $2k-1$ and G_2 would have odd $(n+2k)$ order. Since this is impossible, any graph having the desired properties must have order at least $n+2k+1$. We now show that such a graph exists.

By the construction used in Proposition 1, there exists a graph G of order $n+2k+1$ such that $G = G_1 \oplus G_2$, where G_2 is the edge sum of k edge disjoint hamiltonian cycles of G, the degree set of G_1 is D_1, and the degree set of G_2 is $\{2k\}$. Since $n+2k+1$ is even, let F be the 1-factor of G_2 obtained by deleting every other edge of one of the hamiltonian cycles of G_2. Put $G_2^* = G_2 - E(F)$.

The graph $G^* = G_1 \oplus G_2^*$ has order $n+2k+1$ and G_1 has degree set D_1 while the degree set of G_2^* is D_2. Thus there is a graph with the desired properties.

This completes the proof of Proposition 2.

COROLLARY 2a. *Let $d_1, d_2, \ldots, d_t, m, n$ be positive integers such that $m < n$ and $d = \sum_{i=1}^{t} d_i$. Let $D_i = \{d_i\}$ for $1 \leqslant i \leqslant t$ and let $D_{t+1} = \{m, n\}$. If d and n are both odd, then*

$$\mu(D_1, D_2, \ldots, D_{t+1}) = n+d+2. \tag{3}$$

Proof. Assume that $G = G_1 \oplus G_2 \oplus \ldots \oplus G_{t+1}$ is a graph of order $n+d+1$ with the degree set of G_i being D_i, $1 \leqslant i \leqslant t+1$. The graph $H = G_1 \oplus G_2 \oplus \ldots \oplus G_t$ is a regular graph of degree d with odd order. Since d is odd, this is impossible. Hence the order of G must be at least $n+d+2$.

The construction of Proposition 2 provides a graph $G = H \oplus G_{t+1}$ of order $n+d+2$ such that H is the edge sum of $(d-1)/2$ edge disjoint hamiltonian cylces and a 1-factor. Since the order of H is even, H is the edge sum of d 1-factors. Let G_i, $1 \leqslant i \leqslant t$, be the edge sum of d_i of these 1-factors. Then G_i, $1 \leqslant i \leqslant t$, is regular of degree d_i and has order $n+d+2$. Thus $G = G_1 \oplus G_2 \oplus \ldots \oplus G_{t+1}$, where the degree set of G_i is D_i, $1 \leqslant i \leqslant t+1$.

COROLLARY 2b. *Let $d_1, d_2, \ldots, d_t, m, n$ be positive integers such that $m < n$ and $d = \sum_{i=1}^{t} d_i$. If d and n are both even but at least one of the d_i's is odd, then* (3) *holds true.*

Proof. Assume that $G = G_1 \oplus G_2 \oplus \ldots \oplus G_{t+1}$ is a graph of order $n+d+1$ with the degree set of G_i being D_i, $1 \leqslant i \leqslant t+1$. Suppose that d_1 is odd. Then G_1 is a regular graph of degree d_1 with order $n+d+1$. Since both d_1 and $n+d+1$ are odd, this is impossible. Thus, the order of G must be at least $n+d+2$.

The construction of Proposition 2 provides a graph $G = H \oplus G_{t+1}$ of order $n+d+2$ such that H is the edge sum of $d/2$ edge disjoint hamiltonian cycles and the degree set of G_{t+1} is D_{t+1}. Thus, since $n+d+2$ is even, H is the edge sum of d edge disjoint 1-factors of G. For each i, $1 \leqslant i \leqslant t$, let G_i be the edge sum of d_i of these edge disjoint 1-factors. Then $G = G_1 \oplus G_2 \oplus \ldots \oplus G_t \oplus G_{t+1}$, where G has order $n+d+2$, and G_i has the degree set D_i for $1 \leqslant i \leqslant t+1$.

PROPOSITION 3. *Let k, m, n be positive integers with n even, $1 < m < n$, and $1 < k$. Let $D_1 = \{m, n\}$ and $D_2 = \{2k-1\}$. Then*

$$\mu(D_1, D_2) = n+2k.$$

Proof. If G is any graph having these properties, then

$$|V(G)| \geqslant n+(2k-1)+1 = n+2k.$$

We now show that such a graph exists.

Case 1. Assume that m is odd (that is, $m \geqslant 3$). Let $H_1 = K_2$. By the proof of the Lemma, we can construct a graph H_2 of order $n+2k-2$, that is, the edge sum of $(m+2k-3)/2$ edge disjoint hamiltonian cycles, such that $G = H_1 + H_2$ also contains $(m+2k-3)/2$ edge disjoint hamiltonian cycles.

Each vertex of H_1 has degree $n+2k-1$ in G, while each vertex of H_2 has degree $m+2k-1$ in G. Since G contains $(m+2k-1)/2 \geqslant 1$ hamiltonian cycles and G has even order, we can construct a 1-factor F in G. Let G_2 be the edge sum of F and $k-1$ hamiltonian cycles of G and let $G_1 = G - E(G_2)$. Then $G = G_1 \oplus G_2$, the degree set of G_i is D_i, $i = 1, 2$, and G has order $n+2k$.

Case 2. Suppose that m is even. As before, any such graph must have order at least $n+2k$. Let $H_1 = K_2$. Using the proof of the Lemma, we can construct a graph H_2^* of order $n+2k-2$, that is, the edge sum of $(m+2k-2)/2$ hamiltonian cycles, such that the graph $G^* = H_1 + H_2^*$ also contains $m+2k-2$ edge disjoint hamiltonian cycles. Since $n+2k-2$ is even, we can construct a 1-factor F^* in H_2^* by choosing every other edge of some hamiltonian cycle.

Let $H_2 = H_2^* - E(F^*)$ and $G = H_1 + H_2$. Each vertex of H_1 has degree $n+2k-1$ in G, while each vertex of H_2 has degree $m+2k-1$ in G. Let F be the complement of F^* with respect to the hamiltonian cycle used to define F^*. Then F is also a 1-factor of H_2^* and H_2.

Let G_2 be the edge sum of $k-1$ hamiltonian cycles of G (other than the one used to define F) together with F and the edges of H_1. Then each vertex of G_2 has degree $2k-1$. Let $G_1 = G - E(G_2)$. Then each vertex of H_1 has degree n in G_1, while each vertex of H_2 has degree m in G_1. Let $G = G_1 \oplus G_2$. Then G has order $n+2k$ and the degree set of G_i is D_i, $i = 1, 2$. Thus $\mu(D_1, D_2) = n+2k$.

This completes the proof of Proposition 3.

COROLLARY 3. *Let $d_1, d_2, \ldots, d_t, m, n$ be positive integers with $1 < m < n$ and $d = \sum_{i=1}^{t} d_i$. Let $D_i = \{d_i\}$ for $1 \leqslant i \leqslant t$ and let $D_{t+1} = \{m, n\}$. If $d \geqslant 3$, d is odd, and n is even, then* (2) *holds true.*

Proof. By the construction used in the proof of Proposition 3, there is a graph G of order $n+d+1$ with $G = H \oplus G_{t+1}$, where H is the edge sum of $(d-1)/2$ edge disjoint hamiltonian cycles and a 1-factor, and the degree set of G_{t+1} is D_{t+1}. Since $n+d+1$ is even, H is the edge sum of d edge disjoint 1-factors. For each i, $1 \leqslant i \leqslant t$, let G_i be the edge sum of d_i of these 1-factors of G. Then we write $G = G_1 \oplus G_2 \oplus \ldots \oplus G_{t+1}$, where the degree set of G_i is D_i, $1 \leqslant i \leqslant t+1$.

We now consider the last case, namely, where $D_1 = \{1, n\}$, n is an even positive integer, and $D_2 = \{1\}$.

PROPOSITION 4. *Let $D_1 = \{1, n\}$, where n is a positive even integer, and let $D_2 = \{1\}$. Then*

$$\mu(D_1, D_2) = \begin{cases} 4 & \text{if } n = 2, \\ n+4 & \text{if } n \geqslant 4. \end{cases}$$

Proof. The graph $K_4 - x$, the complete graph of order 4 with an edge removed, provides an example to show that $\mu(\{1\}, \{1, 2\}) = 4$. Thus, we assume that $n \geqslant 4$.

As before, if G is any graph with these properties, then G must have order at least $n+2$. Suppose G is a graph such that $G = G_1 \oplus G_2$, where G has order $n+2$, and G_i has degree set D_i, $i = 1, 2$. Then G_1 is a graph of order $n+2$ with degree set $D_1 = \{1, n\}$. Clearly, this is not possible. Thus, G must have order at least $n+3$. However, then G_2, which has odd order, must have a 1-factor. This is again impossible, and so G must have order at least $n+4$. We now construct such a graph.

Let $H = K_{n+2}$ and let $V(H) = \{v_1, v_2, \ldots, v_{n+2}\}$. Define the graph G by letting

$$V(G) = V(H) \cup \{u, w\} \quad \text{and} \quad E(G) = E(H) - \{v_1 v_2\} \cup \{v_1 u, uw, wv_2\}.$$

Then G has order $n+4$, the degree of each vertex v_i is $n+1$, and the degrees of u and w are 2. Since n is even, let G_2 be the 1-factor consisting of $uw, v_2 v_3, v_4 v_5, \ldots, v_n v_{n+1}, v_{n+2} v_1$. Let $G_1 = G - E(G_2)$. Then the degree set of G_1 is $D_1 = \{1, n\}$, while the degree set of G_2 is $D_2 = \{1\}$. This completes the proof of Proposition 4.

We now summarize the results of the propositions and corollaries in the following

THEOREM 2. *Let $d_1, d_2, \ldots, d_t, m, n$ be positive integers such that $m < n$ and $d = \sum_{i=1}^{t} d_i$. Let $D_i = \{d_i\}$ for $1 \leqslant i \leqslant t$ and let $D_{t+1} = \{m, n\}$. Then*

$$\mu(D_1, D_2, \ldots, D_{t+1}) = \begin{cases} 4 & \text{if } t = 1,\ d_1 = m = 1, \text{ and } n = 2, \\ n+4 & \text{if } t = 1,\ d_1 = m = 1,\ n \text{ is even, and } n \geqslant 4, \\ n+d+1 & \text{if } n \text{ and } d \text{ have opposite parity, } d \geqslant 3, \text{ and } m > 1, \\ n+d+1 & \text{if } n \text{ is odd and all of the } d_i\text{'s are even,} \\ n+d+2 & \text{if } n \text{ and } d \text{ are both even and at least one } d_i \text{ is odd,} \\ n+d+2 & \text{if } n \text{ and } d \text{ are both odd.} \end{cases}$$

We conclude with two conjectures:

CONJECTURE 1 (**P 1277**). Let $n \geqslant 3$. Then for each k, $0 \leqslant k \leqslant [(n-1)/2]$, there is a graph G_k of order n such that G_k contains k edge disjoint hamiltonian cycles and these k hamiltonian cycles can be extended to k edge disjoint hamiltonian cycles in $G_k + \bar{K}_n$.

CONJECTURE 2 (**P 1278**). Let $n_1, n_2, \ldots, n_t$, and k be positive integers such that $n_1 < n_2 < \ldots < n_t$. Let $D_1 = \{n_1, n_2, \ldots, n_t\}$ and $D_2 = \{k\}$. Then

$$\mu(D_1, D_2) = \begin{cases} 4 & \text{if } t = 2,\ n_1 = k = 1, \text{ and } n_2 = 2, \\ n_2+4 & \text{if } t = 2,\ n_1 = k = 1,\ n_2 \text{ is even and } n_2 \geqslant 4, \\ n_t+k+2 & \text{if both } n_t \text{ and } k \text{ are odd}, \\ n_t+k+1 & \text{otherwise.} \end{cases}$$

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