JOURNAL OF COMBINATORICS,
INFORMATION & SYSTEM SCIENCES

# Traceability in the Square of a Tree

RONALD J. GOULD

*Department of Mathematics and Computer Science,*
*Emory University, Atlanta, GA–30322*

A characterization of trees whose square contains a hamiltonian path is given. Further, a complete description is given of which vertices may be the initial vertex of a hamiltonian path in the square of a tree.

Terms not defined in this paper can be found in [1]. A graph is *traceable* if it contains a path through all its vertices. Such a path is called a *hamiltonian path*. For a graph $G$ we denote the vertex set as $V(G)$, edge set as $E(G)$, and the distance between two vertices $u$ and $v$ as $d(u, v)$. The *subdivision graph* $S(G)$ of a graph $G$ is obtained by removing each edge $uv(u, v \in V(G))$ of $G$ and inserting a new vertex $w$ and the edges $uw$, $vw$. The *neighbourhood* $N(v) = \{u | uv \in E(G)\}$ while the *closed neighbourhood* $N[v] = N(v) \cup \{v\}$.

The *square* of a connected graph $G$ is a graph $G^2$ where $V(G^2) = V(G)$ and such that $uv \in E(G)$ if and only if $1 \leqslant d(u, v) \leqslant 2$ in $G$. Since $G^2$ contains $G$ as a subgraph, it follows that $G^2$ is hamiltonian (traceable) whenever $G$ is hamiltonian (traceable).

Nash-Williams and Plummer independently conjectured that the square of a 2-connected graph is hamiltonian. In 1974 Fleischner [3] proved this conjecture to be correct.

THEOREM A [3]. *Let $G$ be a 2-connected graph. Then $G^2$ is hamiltonian.*

Various theorems have employed Theorem $A$ to obtain stronger results. For example, it has been shown [2] that the square of a 2-connected graph is hamiltonian connected.

THEOREM B [2]. *Let $G$ be a 2-connected graph. Then $G^2$ is hamiltonian connected.*

Recent work has centered on forbidden subgraphs and hamiltonian properties in squares (see [4], [6]).

Harary and Schwenk [5] determined all trees with a hamiltonian square and investigated some properties of such trees.

THEOREM C [5]. *Let $t$ be a tree of order $p \geqslant 3$. The following statements are equivalent:*

(1) *$t^2$ is hamiltonian.*

(2) *$t$ does not contain $S(K(1, 3))$ as a subgraph.*

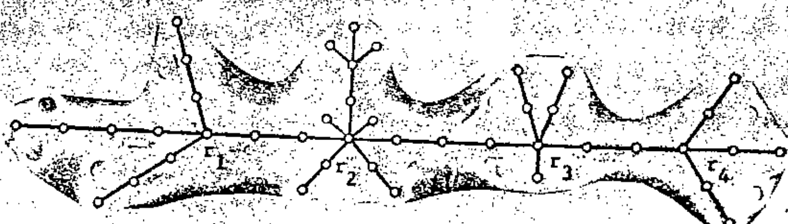
(3) *$t$ minus its end vertices is a path.*

THEOREM D [5]. *If $t$ is a tree and $t^2$ is hamiltonian, then any hamiltonian cycle contains exactly two edges of $t$, and these are the terminal edges of a longest path. Moreover, the terminal edgest of every longes path lie on some hamiltonian cycle.*

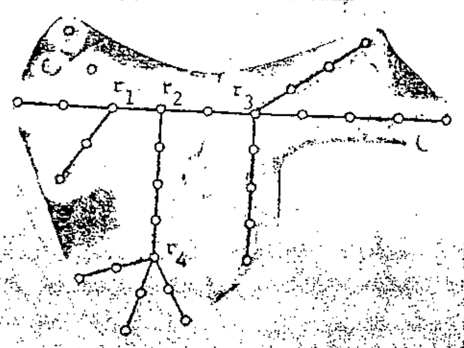
The object of this paper is to explore the concept of traceability in the square of a tree. Some terminology will be helpful.

A tree $t = t(n_1, n_2, \ldots, n_j)$, $j \geqslant 3$, with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_j$ is called *star-like* if it is isomorphic to the graph formed by joining an end-vertex of each of the paths $P_{n_1}, P_{n_2}, \ldots, P_{n_j}$ to a vertex (which we commonly denote $r$).

If $t$ is a tree, a *root* of $t$ is any vertex $r$ such that deg $r > 3$. A *branch* of $r$ is any component of $t - r$, for some root $r$. The *initial vertex* of a branch is the unique vertex of the branch adjacent to the root. An *$i$-root* is a root with exactly $i$ nontrivial branches, that is, an $i$-root is a root of an induced $S(K(1, i))$ but not an $S(K(i + 1))$. We say the tree $t$ has a *trunk* if there is a path $T$ in $t$, whose initial and terminal vertices are $i$-roots, $i \geqslant 3$, and which contains each $i$-root with $i \geqslant 3$ (see Figure 1). The path $T$ constitutes the trunk of $t$.




(a) a tree $t$ with a trunk




(b) a tree $t_1$ with no trunk

FIGURE 1

152

LEMMA 1. *If $t$ is star-like with $n$-root $r(n \geqslant 3)$, then no hamiltonian path in $t^2$ can begin and end in the same branch at $s$.*

*Proof.* Let $t$ be star-like with nontrivial branches $B_1, B_2, \ldots, B_n$, $n \geqslant 3$. Without loss of generality, assume $t^2$ has a hamiltonian path beginning in $B_1$ which proceeds through $B_2, B_3, \ldots, B_n$ and eventually concludes in $B_1$. Note that each nontrivial branch after $B_1$ can be entered and exited only once. If the branches are encountered in the order $B_1$, $B_2, \ldots, B_n$, $B_1$, there are $n$ branch changes and $n \geqslant 3$. But in order to reenter $B_1$, either its initial vertex $u_1$ or $r$ must not yet have been traversed. However, entering and exiting both of the branches $B_2$ and $B_3$ requires the use of both the initial vertex of the branch and the root $r$, and since $n \geqslant 3$ it will be impossible to reenter $B_1$.

Clearly we cannot begin in a trivial branch for this would imply $r^2$ was hamiltonian; however, this contradicts Theorem $C$. ∎

COROLLARY 2. *If $t$ is a tree and $r$ is an $i$-root of $t(i \geqslant 3)$, then no hamiltonian prth in $t^2$ can begin and end in the same branch at $r$.*

LEMMA 3. *If $t$ is a tree containing an $i$-root $r(i \geqslant 3)$, then $t^2$ is not traceable from $r$.*

*Proof.* Let $t$ be a tree with $i$-root $r(i \geqslant 3)$ and let $B_1, B_2, \ldots, B_i$ be the nontrivial branches of $t$ at $r$. If there is a hamiltonian path in $t^2$ with initial vervex $r$, then each branch at $r$ can be entered and exited exactly once. But entering and exiting any such branch requires traversing its initial vertex and $r$. Hence, at most two nontrivial branches about $r$ can be traversed. But $r$ has at least three nontrivial branches. Thus $t^2$ is not traceable from $r$. ∎

LEMMA 4. *If $t$ is star-like with root $r$, then $t^2$ is traceable from:*

(1) *every vertex, if $t$ has no $i$-root, $i \geqslant 3$.*

(2) *$V(t) - \{r\}$, if $r$ is a 3–root.*

(3) *$V(t) - N[r]$, if $r$ is a 4–root.*

(4) *no vertex, if $r$ is an $i$-root, $i \geqslant 5$.*

*Proof.* Let $t$ be star-like with root $r$. If $t$ contains on $i$-root, $i \geqslant 3$, then $t$ contains on induced $S(K(1, 3))$, so by Theorem $C$, $t^2$ is hamiltonian. Therefore, $t^2$ is traceable from each vertex; thus (1) holds.

If $r$ is a 3-root, let $B_1$, $B_2$, and $B_3$ be the nontrivial branches at $r$ and let $v_1, v_2, \ldots, v_t$ be the trivial branches at $r$, if any exists. Let $u_{ij}$ be on the branch $B_i$ and such that $d(r, u_{ij}) = j$.

By Lemma 3, $t^2$ is not traceable from $r$. Let $u_{1,s}$ be an arbitrary vertex on $B_1$. We construct a hamiltonian path in $t^2$ with initial vertex $u_{1,s}$ in the following manner.

Let the path $P_1$ in $B_1$ be

152

$$u_{1s}, u_{1(s+2)}, u_{1(s+4)}, \ldots, u_{1(n_1)}, u_{1(n_1-1)}, u_{1(n_1-3)}, \ldots, u_{1(s-1)}, u_{1(s-3)}, \ldots, u_{11}$$

if $s$ and $n_1$ have the same parity, and let $P_1$ be

$$u_{1s}, u_{1(s+2)}, u_{1(s+4)}, \ldots, u_{1(n_1-1)}, u_{1(n_1)}, u_{1(n_1-2)}, u_{1(n_1-4)}, \ldots, u_{1(s-1)}, u_{1(s-3)}, \ldots, u_{11}$$

if $s$ and $n_1$ are of opposite parity. Similarly, define a path $P_2$ in $B_2$ to be

$$u_{21}, u_{23}, u_{25}, \ldots, u_{2(n_2)}, u_{2(n_2)}, u_{2(n_2-3)}, \ldots, u_{22}$$

if $n_2$ is odd, and let $P_2$ be defined as

$$u_{21}, u_{23}, u_{25}, \ldots, u_{2(n_2-1)}, u_{2(n_2)}, u_{2(n_2-2)}, u_{2(n_2-4)}, \ldots, u_{22}$$

if $n^2$ is even. Note that an analogous path $P_3$ with initial vertex $u_{31}$ and terminal vertex $u_{32}$ exists in $B_3$.

We now construct a hamiltonian path in $t^2$ with initial vertex $u_{1,s}$ to be

$$P: P_1, P_2, r, v_1, v_2, \ldots, v_t, P_3$$

if there are nontrivial branches and to be

$$P: P_1, P_2, r, P_3, \text{ otherwise.}$$

Since $u_{1,s}$ was an arbitrary vertex in $B_1$, $t^2$ is traceable from each vertex of $B_1$ and because $B_1$ was an arbitrary nontrivial branch, $t^2$ is traceable from each vertex in $B_1$, $B_2$, and $B_3$. Note that $P$ also shows $t^2$ is traceable from $v_t$ and since, if trivial branches exist, they may be initially traversed in any order, followed by the vertices of $B_1$, $r$, and the vertices of $B_2$ and $B_3$, we conclude that $t^2$ is traceable from each vertex of $V(t) - \{r\}$.

Next suppose $r$ is a 4-root with nontrivial branches $B_1$, $B_2$, $B_3$, and $B_4$ and let $u_{ij}$ be the vertex on branch $B_i$ a distance $j$ from $r$. Again let $v_2, v_2, \ldots, v_t (t \geqslant 0)$ be the vertices which constitute the trivial branches at $r$, if any. Again, by Lemma 3, no hamiltonian path can begin at $r$. Now suppose a hamiltonian path begins with a vertex in $N(r)$, then before a vertex in the second nontrivial branch is traversed, the vertex $r$ must be traversed. However, it is then impossible to enter and exit the three remaining nontrivial branches. This $G$ is not traceable from any vertex in $N[r]$.

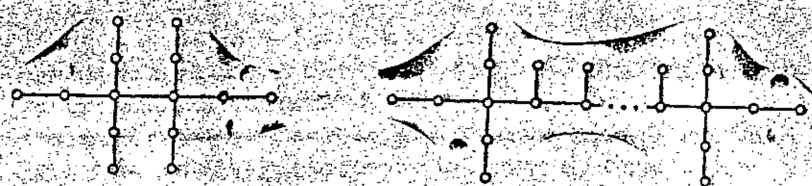
An argument similar to that given when $r$ is a 3-root shows any other vertex of $t$ can be initial vertex of a hamiltonian path in $t^2$.

Finally, suppose $r$ is an $i$-root ($i \geqslant 5$), with nontrivial branches $B_1$, $B_2, \ldots, B_i$. Since $r$ must be traversed within the first three branch changes, it is impossible to traverse vertices in five or more nontrivial branches. Thus $t^2$ is not traceable if $r$ is an $i$-root where $i \geqslant 5$.

Corollary 5. *If $t$ is a tree and $t$ contains $S(K(1, 5))$ as an induced subgraph, then $t^2$ is not traceable.*

We next turn our attention to exactly which trees $t$ have a traceable square and which vertices can be the initial vertices of a hamiltonian path in $t^2$.

Define the graph $F_0$ to be the graph obtained by taking two copies of $S(K(1, 3))$ and joining the roots in each copy by an edge [see Fig. 2(*a*)]. Define the graph $F_i$, $i \geqslant 1$, to be the graph obtained by taking a copy of $F_0$ and $i$ vertices, $w_1, w_2, \ldots, w_i$. Subdivide the edge joining the two roots of $F_0$ a total of $i$ times and join each of the vertices to one and only one of $w_1, w_2, \ldots, w_i$ [see Fig. 2(b)].



(a) the graph $F_0$ (b) the graph $F_i$, $i \geqslant 1$

Fig. 4.2

THEOREM 6. *If $t$ is a tree, then $t^2$ is traceable if and only if $t$ contains a trnnk and $t$ does not contain an $S(K(1, 5))$ or any $F_i$, $i = 0, 1, 2, \ldots$ as a subgraph.*

*Proof.* Let $t$ be a tree such that $t^2$ is traceable and suppose $t$ does not contain a trunk. Then there exist roots $r_1$, $r_2$, $r_3$ each with three or more nontrivial branches, which do not lie on a path. Hence, there exists a root $r$ such that the $r - r_i$ paths $P_i$ $(i = 1, 2, 3)$ are all disjoint, except for $r$. Let $B_2$ be the branch at $r$ containing $r_2$ and without loss of generality assume $B_2$ is intermediate to a hamiltonian path $P$ in $t^2$; that is, the path $P$ neither begins nor ends in $B_2$.

Then either the vertices of $B_2$ must all be traversed after $B_2$ is entered, in which case Corollary 2 is violated, or $B_2$ must be exited, entered, and exited again. But this is also impossible, as no branch can be entered and exited twice. Hence $t$ contains a trunk, which we denote by $T$.

Now note that $t$ contains no $S(K(1, 5))$, by Corollary 5, so suppose $t$ contains $F_i$, for some $i \geqslant 0$.

*Case* 1. Suppose $t$ contains $F_0$. Let $r_1$ and $r_2$ be the roots of $F_0$ and suppose $B_1$, $B_2$, and $B_3$ are branches at $r_1$ not containing $r_2$, while $B_4$, $B_5$, and $B_6$ are branches at $r_2$ not containing $r_1$. Without loss of generality, suppose vertices in $B_1$ are traversed prior to those in $B_2, B_3, \ldots, B_6$. In particular, then, $P$ cannot end in $B_1$, $B_2$, $B_3$, or at $r_1$; otherwise, $P$ would begin and end in the same branch at $r_2$ (that containing $r_1$, $B_1$, $B_2$, $B_3$), coutradicting Corollary 2.

Also, the branch $B^*$ at $r_1$ containing $B_4$, $B_5$, and $B_6$ cannot be entered

152

twice; hence, the branches $B_1$, $B_2$, $B_3$, and the vertex $r_1$ must be completely traversed before $B^*$ is entered. However, as $B_1$, $B_2$, $B_3$, and $B^*$ are all branches at $r_1$, this implies that $r_1$ is not the last vertex traversed before $B^*$ is entered. Hence $r_2$ is the first vertex traversed in $B^*$. But then it is impossible to traverse the remaining three branches at $r_2$, which implies $t^2$ is not traceable, contradicting the hypothesis. Thus, $F_0$ is not a subgraph of $t$.

*Case* 2. Suppose $t$ contains some $F_i$, $i \geqslant 1$. Let $r_1$ and $r_2$ be the 4-roots of $F_i$, while $v_1, v_2, \ldots, v_i$ are the 2-roots of $F_i$ lying consecutively on the $r_1 - r_2$ path in $F_i$. Let $w_j$ be adjacent to $v_j$ in $F_i$ ($j = 1, 2, \ldots, i$). An argument analogous to that of Case 1 shows $v_1$ is the first vertex traversed in the branch at $r_1$ containing $r_2$. But then $w_1, v_2, w_2, v_3, \ldots, v_i, w_i$ must be traversed in order. Hence $r_2$ must be traversed prior to any vertex of $B_4$, $B_5$, or $B_6$. But then it is impossible to completely traverse these branches, contradicting the fact that $t^2$ is traceable. Hence, $t$ does not contain any $F_i$, $i \geqslant 1$.

Now suppose $t$ is a tree containing a trunk that contains no $S(K(1, 5))$ or $F_i$, $i \geqslant 0$. Let $T: v_1, v_2, \ldots, v_k$ be the trunk of $t$. Since $t$ has a trunk, each branch $B$ disjoint from $T$ contains no $S(K(1, 3))$. Hence, by Theorem C, $B^2$ is hamiltonian. Further, by Theorem D, each terminal edge is contained in some hamiltonian cycle; that is, we can find a hamiltonian cycle of $B^2$ containing an edge of $t$ incident with the initial vertex of $B$. Thus, each such branch can be entered, traversed, and exited; as the end vertices of the hamiltonian path in $B^2$, created by removing a terminal edge that lies on a hamiltonian cycle, are at a distance 1 or 2 from the root of $B$.

Let $B_1$ be a branch at $v_1$. Let $x_1$ be a vertex in $B_1$ a distance 2 from $v_1$. By previous remarks there exists a hamiltonian path in $B_1^2$ beginning at $x_1$ and ending at $x_2$, the initial vertex of $B_1$. Next, traverse all trivial branches at $v_1$ and $x_3$, the initial vertex of $B_2$. Again, there is a hamiltonian path in $B_2^2$ with initial vertex $x_3$ and terminal vertex $x_4$, where $d(x_4, v_1) = 2$. Next, traverse $v_1$. Finally, if a third branch, disjoint from $T$, exists at $v_1$, we traverse it, ending ut the initial vertex and then traverse $v_2$. If no third branch exists at $v_1$, we again traverse $v_2$. Since $t$ contains no $F_0$, there are at most two nontrivial branches at $v_2$ which can be traversed. Continue these arguments at $v_3, v_4, \ldots, v_k$. Since $t$ contains no $F_i$. We may enter and traverse one of the branches at $v_k$ before traversing $v_k$ itself. This leaves at most two remaining nontrivial branches which clearly can be traversed. Hence $t^2$ is traceable. ∎

Lemma 7. *Let $t$ be a tree that contains no $S(K(1, 3))$. Then for each vertex $v$, there exists a hamiltonian path in $t^2$ with initial vertex $v$ and whose terminal vertex in an end-vertex of $t$.*

*Proof.* By Theorem C, $t$ minus its end-vertices is a path, say $P: v_1$,

2E

$v_2, \ldots, v_k$. For each vertex $v_i$ $(i = 1, 2, \ldots, k)$, let $W_i = \{x \deg x = 1$ and $xv_i \in E(t)\}$.

*Case* 1. Suppose $v = v_t$, $1 \leqslant t \leqslant k$. We construct a hamiltonian path in $t^2$ by traversing in other $v_t$, the vertices of $W_{t-1}$, $v_{t-2}$, $W_{t-3}$, and so on until either $v_1$ or the vertices of $W_1$ have been traversed, say $v_1$. Then traverse the vertices of $W_1$, $v_2$, $W_3$, and so on until the vertices of $W_t$ have been traversed. (Otherwise, traverse $v_1$, $W_2$, $v_3$, etc.) In either case, we now traverse vertices in the order $v_{t+1}$, $W_{t+1}$, $v_{t+2}$, $W_{t+2}, \ldots, v_k$, $W_k$. If $W_k \neq \phi$, we have ended with a vertex of degree 1. If $W_k = \phi$, then $\deg v_k = 1$ and $v_k$ terminates the path.

*Case* 2, Suppose $v \in W_t$, $1 \leqslant t \leqslant k$. Then traverse the remaining vertices of $W_t$, $v_{t-1}$, $W_{t-2}$, and so on as before. Using an approach similar to Case 1, we can then construct a path that ends with a vertex of $W_k$, if $W_k \neq \phi$, or at $v_k$, otherwise. ∎

Suppose the tree $t$ contains a trunk $T$. By an *end branch* of $t$ we mean a branch $B$ at an end-vertex of $T$ such that $V(B) \cap V(T) = \phi$.

THEOREM 8. *Let $t$ be a tree that contains* 3*-roots but no i-roots, $i \geqslant 4$, and suppose $t^2$ is traceable. Then $t^2$ is traceable from the vertex $v$ if and only if $v$ lies in an end branch of $t$.*

*Proof.* Let $t$ be a tree that contains 3–roots but no $i$-roots, $i \geqslant 4$. Further suppose $t^2$ is traceable. Then $t$ contains a trunk $T$. If $T$ consists of a single 3–root $r$, then either $t$ is star-like or each nontrivial branch that is not a path contains on $S(K(1, 3))$. If $t$ is star-like, the result follows from Lemma 4. Otherwise, let $v$ be in a branch $B$ at $r$. By Lemma 7, there is a hamiltonian path in $B^2$ with initial vertex $v$ and terminal vertex $x$, where $d(x, r) \leqslant 2$. We can then traverse $r$ and all trivial branches at $r$. Now, by the proof techniques of Theorem 6, we know we can traverse the vertices of the two remaining nontrivial branches. Thus, $t^2$ is traceable from $v$. Note that, by Lemma 3, $t^2$ is not traceable from $r$. Thus, the result holds.

Next, suppose $T$ contains a nontrivial trunk $v_1, v_2, \ldots, v_l$ $(l \geqslant 2)$. If $v$ does not lie in an end branch of $t$, but $v$ is traceable from $v$, then the branches at one end-vertex of $T$ must be traversed prior to those at the other. But this implies that a hamiltonian path in $S(K(1, 3))$ can begin and end in the same branch, contradicting Lemma 1. Thus, $t^2$ is traceable from $v$ if and only if $v$ lies in an end branch of $t$. ∎

COROLLARY 9. *Let $t$ be a tree such that $t^2$ is traceable. Then $t^2$ is traceable from a vertex $v$ if and only if $v$ lies in an end branch of $t$ and $v$ is not adjacent to a 4–root.*

*Proof.* If $t$ is star-like, we apply Lemma 4. If $t$ contains only 3-roots, then the result follows by Theorem 8. If $t$ contains exactly one *i-root* $r$, $i \geqslant 3$, and $r$ is a 4–root, then an argument analogous to that in theorem 8 can be used; however, as in Lemma 4, $v$ cannot be adjacent to $r$. For the general case, an analogous proof technique again applies, noting that we must begin in an end branch of $t$ and that if $v$ is the initial vertex of an end branch at a 4-root, we cannot traverse all branches as thet root. ∎

REFERENCES

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster (1979), *Graph and Digraphs*, Prindle, Weber & Schmidt, Boston.

[2] G. Chartrand, A.M. Hobbs, H.A. Jung, S.F. Kapoor, and C. St. J.A. Nash-Williams (1974), "The square of a block is ramiltonian connected'.' *J. Combinatorial Theory*, 61B, 290–292.

[3] H. Fleischner (1974), "The square of every two-connected graph is Hamiltonian", *J. Combinatorial Theory*, 16B 29–34.

[4] R. Gould and M.S. Jacobson, "Forbidden subgraphs and Hamiltonian properties in the square of a connected graph," *J. Graph Theory* (to appear).

[5] F. Harary and A Schwenk (1971), "Trees with Hamiltonian square," *Mathematika*, 18, 138–140.

[6] M.M. Matthews and D.P. Sumner, "Hamiltonian results in $K_{1,3}$-Free graphs, *J. Graph Theory* (to appear).

[Received : Sept. 1983]

7 · E