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Cycles with a chord in dense graphs

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ABSTRACT

A cycle of order k is called a k-cycle. A non-induced cycle is called a chorded cycle. Let n be an integer with $n \geq 4$. Then a graph G of order n is chorded pancyclic if G contains a chorded k-cycle for every integer k with $4 \leq k \leq n$. Cream, Gould and Hirohata (Australas. J. Combin. 67 (2017), 463–469) proved that a graph of order n satisfying $\deg_G u + \deg_G v \geq n$ for every pair of nonadjacent vertices u, v in G is chorded pancyclic unless G is either $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \square K_2$, the Cartesian product of K_3 and K_2. They also conjectured that if G is Hamiltonian, we can replace the degree sum condition with the weaker density condition $|E(G)| \geq \frac{1}{4}n^2$ and still guarantee the same conclusion. In this paper, we prove this conjecture by showing that if a graph G of order n with $|E(G)| \geq \frac{1}{4}n^2$ contains a k-cycle, then G contains a chorded k-cycle, unless $k = 4$ and G is either $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \square K_2$. Then observing that $K_{\frac{n}{2},\frac{n}{2}}$ and $K_3 \square K_2$ are exceptions only for $k = 4$, we further relax the density condition for sufficiently large k.

1. Introduction

In this paper, we only consider finite simple graphs. A *k-cycle* is a cycle of order k. A graph G of order $n \geq 3$ is *pancyclic* if G contains a k-cycle for every k with $3 \leq k \leq n$.

To determine whether a given graph G is Hamiltonian is an NP-complete problem. Therefore, there is little hope in obtaining a criterion for the existence of a Hamiltonian cycle which can be described in a polynomial-time algorithm. The hardness of the problem also affects sufficient conditions. Many sufficient conditions for the existence of a Hamiltonian cycle make a graph G so dense that G is not only Hamiltonian but it also satisfies stronger cycle properties. This situation is highlighted by Bondy's Meta-Conjecture.

Bondy's Meta-Conjecture. *Almost all sufficient conditions for the existence of a Hamiltonian cycle make a graph pancyclic, possibly with a small number of well-described families of exceptional graphs.*

Bondy's Meta-Conjecture has long served as a driving force in the research of cycles in graphs. Bondy [3] himself proved a result to support it. For a non-complete graph G, we let $\sigma(G)$ denote the minimum degree sum over all pairs of nonadjacent vertices in G. If G is complete, we let $\sigma(G) = +\infty$. The classical Ore's Theorem states that every graph G of order $n \geq 3$ with $\sigma(G) \geq n$ is Hamiltonian. Bondy proved that under the same hypothesis, G is actually pancyclic unless n is even and G is $K_{\frac{n}{2},\frac{n}{2}}$. Moreover, he proved that once we have a Hamiltonian cycle, we no longer need the degree sum condition, but a simple density condition makes the graph pancyclic.

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Theorem A (*Bondy [3]*)**.** *Every Hamiltonian graph G of order n with at least $\frac{1}{4}n^2$ edges is pancyclic unless n is even and G is $K_{\frac{n}{2},\frac{n}{2}}$. In particular, if $|E(G)| > \frac{1}{4}n^2$, then G is pancyclic.*

Theorem A has shed a new light on the distribution of cycle lengths. When we require a Hamiltonian cycle, the density condition of Theorem A is not strong enough. The union of K_1 and K_{n-1} contains $\frac{1}{2}(n-1)(n-2)$ edges but it is not Hamiltonian. Even if we restrict ourselves to the class of k-connected graphs for some constant k, we still have $K_k \vee \left(kK_1 \cup K_{n-2k}\right)$, which is k-connected and contains $\frac{1}{2}n^2 - o(n^2)$ edges, but it is not Hamiltonian. However, once we have a Hamiltonian cycle in a graph G of order n, $|E(G)| > \frac{1}{4}n^2$ guarantees the existence of cycles of all possible lengths.

Inspired by Theorem A, several studies have been conducted concerning the relationship between the density of a graph and cycles of a variety of lengths. Note that a bipartite graph is not pancyclic and since $K_{\frac{n}{2},\frac{n}{2}}$ appears as an exception in Theorem A, it is natural to restrict ourselves to non-bipartite graphs to further pursue this line of research. Then Häggkvist, Faudree and Schelp [8] relaxed the density condition.

Theorem B (*Häggkvist, Faudree and Schelp [8]*)**.** *Every non-bipartite Hamiltonian graph of order n with more than $\frac{1}{4}(n-1)^2+1$ edges is pancyclic.*

Both Theorems A and B assume Hamiltonicity in the hypothesis. However, Brandt [4] proved that the existence of a Hamiltonian cycle is not related with the relationship between the density of a graph and the distribution of cycle lengths. Let $g(G)$ and $c(G)$ be the lengths of a shortest and a longest cycle in G, respectively.

Theorem C (*Brandt [4]*)**.** *A non-bipartite graph G of order n with more than $(n-1)^2/4+1$ edges contains a k-cycle for every integer k with $3 \le k \le c(G)$.*

A graph G is *weakly pancyclic* if G contains a k-cycle for every integer k with $g(G) \le k \le c(G)$. In the same paper, Brandt conjectured the density condition can be further relaxed if we consider weak pancyclicity.

Conjecture 1 (*Brandt [4]*)**.** *A non-bipartite graph G of order n with more than $(n-1)(n-3)/4+4$ edges is weakly pancyclic.*

Note that $(n-1)(n-3)/4+4 = n^2/4 - n + 19/4$. Bollobás and Thomason [2] gave a partial answer to this conjecture.

Theorem D (*Bollobás and Thomason [2]*)**.** *A non-bipartite graph G of order n with at least $\lfloor n^2/4 \rfloor - n + 59$ edges contains a k-cycle for every integer k with $4 \le k \le c(G)$.*

A *chord* of a cycle C is an edge joining two non-consecutive vertices of C. If there exists a chord of C, we say that C is a *chorded cycle*. A chorded cycle of order k is called a *chorded k-cycle*. A chord is one of the main tools in the study of cycle length distribution. Intuitively speaking, chords in a cycle enrich the cycle space of a graph and raise the chance of finding a cycle of required length or property. Actually, the proofs in [3] locate a cycle of desired length by using chords in a Hamiltonian cycle. In this sense, it is worth studying the distribution of chords in a cycle. And the first step in this direction is to find a chorded cycle.

Cream, Gould and Hirohata [6] studied a degree sum condition for a graph to have chorded cycles of all possible lengths. A graph G of order $n \ge 4$ is *chorded pancyclic* if G contains a chorded k-cycle for every integer k with $4 \le k \le n$. Cream et al. proved that a graph satisfying Ore's degree sum condition is not only pancyclic but also chorded pancyclic, except for a balanced complete bipartite graph, plus one more. For graphs G and H, let $G \,\square\, H$ denote the Cartesian product of G and H.

Theorem E (*Cream, Gould and Hirohata [6]*)**.** *A graph of order $n \ge 4$ with $\sigma(G) \ge n$ is chorded pancyclic unless G is $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$.*

Cream et al. also conjectured that if a graph G is Hamiltonian, we can replace the degree sum condition with the density condition $|E(G)| \ge \frac{1}{4}n^2$.

In this paper, we affirmatively answer the above conjecture and further clarify the relationship between the density of a graph and the existence of a chorded cycle of specified length. The following is the main theorem of this paper.

Theorem 1. *Let G be a graph of order n with $|E(G)| \ge \frac{1}{4}n^2$ and let k be a positive integer. If G contains a k-cycle, then it contains a chorded k-cycle unless $k = 4$ and G is either $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$.*

By combining this theorem with Theorem A, we affirmatively answer the conjecture of Cream et al.

Corollary 2. *A Hamiltonian graph G of order $n \ge 4$ with $|E(G)| \ge \frac{1}{4}n^2$ is chorded pancyclic unless G is $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$.*

Theorem 1 suggests that the existence of a chorded cycle in a dense graph is independent of Hamiltonicity and pancyclicity. In a dense graph, we can discuss the existence of a chorded cycle of a specified length from the existence of a cycle of the same length. In this setting, we may be able to obtain a refined density condition. For example, a bipartite graph does not have a chorded 4-cycle. Therefore, as long as we seek a chorded 4-cycle, it is difficult to improve the condition $|E(G)| \ge \frac{1}{4}n^2$ in Theorem 1. However, if $n \ge 6$, $K_{\frac{n}{2},\frac{n}{2}}$ contains a chorded 6-cycle. Also, when we require a chorded 5-cycle under the assumption of the existence of a 5-cycle, bipartite graphs are automatically ruled out. Furthermore, if we obtain a Hamiltonian cycle in a graph G of order $n \geq 4$, then $|E(G)| \geq n+1$ guarantees the existence of a chorded n-cycle. These observations suggest that by discussing the existence of a chorded cycle of an individual length, we may be able to refine Theorem 1. We also discuss this possibility in this paper.

We prove Theorem 1 in the next section. In Section 3, we improve Theorem 1 for sufficiently long cycles. In Section 4, we make several concluding remarks.

For standard graph-theoretic notation and terminology not explained in this paper, we refer the reader to [5]. Let G be a graph and let x be a vertex in G. We denote by $N_G(x)$ and $\deg_G x$ the neighborhood and the degree of x, respectively. If H is a subgraph of G, we define $N_H(x) = N_G(x) \cap V(H)$. Note that we use this notation even if $x \notin V(H)$. Let $A, B \subset V(G)$. If $x \notin A$, let $e_G(x, A)$ denote the number of edges between x and vertices in A. Note that since we only consider simple graphs, $e_G(x, A) = |N_{G[A]}(x)|$, where $G[A]$ is the subgraph induced by A. Moreover, if $A \cap B = \emptyset$, then we denote by $e_G(A, B)$ the number of edges joining a vertex in A and a vertex in B. In other words, $e_G(A, B) = \sum_{a \in A} e_G(a, B)$. We call a cycle C in a graph G an *induced cycle* or a *chordless cycle* if there does not exist a chord of C in G. We denote by $\overline{G}$ the complement of G. Let $T = v_0 v_1 v_2 \ldots v_l$ be a path or a cycle in a graph. For i, j with $0 \leq i \leq j \leq l$, we let $v_i \overrightarrow{T} v_j$ denote the subpath $v_i v_{i+1} v_{i+2} \ldots v_j$. The same path traversed in the opposite direction is denoted by $v_j \overleftarrow{T} v_i$. We define v_i^+ and v_i^- by $v_i^+ = v_{i+1}$ and $v_i^- = v_{i-1}$. Moreover, we let $v_i^{++} = v_{i+2}$ and $v_i^{+++} = v_{i+3}$. The vertices v^{--} and v^{---} are defined in a similar way. For a positive integer k, we let $v^{(k)+} = v_{i+k}$. If $X \subset V(T)$, we define X^+ by $X^+ = \{v^+ : v \in X\}$. We similarly define X^-, X^{++} etc.

2. Proof of Theorem 1

In this section, we prove Theorem 1. We first make a simple observation, which plays a crucial role in our inductive arguments.

Lemma 3. *Let n and k be integers with $k \geq 4$ and $n \geq k+1$. Assume that every graph G' of order $n-1$ with $|E(G')| > \frac{1}{4}(n-1)^2$ and with a k-cycle contains a chorded k-cycle. Let G be a graph of order n with $|E(G)| \geq \frac{1}{4}n^2$. Suppose G contains a k-cycle C but it does not contain a chorded k-cycle. Then $\deg_G x \geq \frac{1}{2}n$ for every $x \in V(G) - V(C)$.*

Proof. Assume $\deg_G(x) \leq \frac{n-1}{2}$ for some $x \in V(G) - V(C)$. Let $G' = G - x$. Then G' contains C. Moreover,

$$|E(G')| = |E(G)| - \deg_G x \geq \frac{1}{4}n^2 - \frac{n-1}{2} = \frac{1}{4}(n^2 - 2n + 2) > \frac{1}{4}(n-1)^2.$$

Thus, by the hypothesis, G' contains a chorded k-cycle, which is also a chorded k-cycle of G. This is a contradiction. $\square$

Note that in the hypothesis of Lemma 3, we assume $|E(G')| > \frac{1}{4}(n-1)^2$, where we do not allow the equality.

The proof of Theorem 1 is divided into three cases : $k = 4$, $k = 5$ and $k \geq 6$. First, we handle the case $k = 4$.

Theorem 4. *Let G be a graph of order n. If $|E(G)| \geq \frac{1}{4}n^2$ and G contains a 4-cycle, then either*

(1) *G contains a chorded 4-cycle,*
(2) *$G = K_{\frac{n}{2}, \frac{n}{2}}$, or*
(3) *$G = K_3 \square K_2$.*

In particular, if a graph G of order n with $|E(G)| > \frac{1}{4}n^2$ contains a 4-cycle, then G contains a chorded 4-cycle.

Proof. We proceed by induction on n. Since G contains a 4-cycle, $n \geq 4$. Let $C = v_1 v_2 v_3 v_4 v_1$ be a 4-cycle in G.

Suppose $n = 4$. By the hypothesis of the theorem, $|E(G)| \geq 4$. If $|E(G)| \geq 5$, then $E(C) \subsetneq E(G)$ and an edge in $E(G) - E(C)$ is a chord of C. If $|E(G)| = 4$, then $G = C$ and hence $G = K_{2,2}$. Thus, the theorem holds for $n = 4$.

Suppose $n = 5$. Let $C = v_1 v_2 v_3 v_4 v_1$ be a 4-cycle in G and suppose C does not contain a chord. Let $V(G) - V(C) = \{x\}$. Since $|E(G)| \geq \left\lceil \frac{5^2}{4} \right\rceil = 7$, x is adjacent with at least three vertices in C. We may assume $\{v_1, v_2, v_3\} \subset N_G(x)$. Then $x v_1 v_2 v_3 x$ is a 4-cycle with chord $x v_2$.

Now we assume $n \geq 6$. We further assume that G contains a 4-cycle $C = v_1 v_2 v_3 v_4 v_1$ but does not contain a chorded 4-cycle, and we will prove that $G = K_{\frac{n}{2}, \frac{n}{2}}$ or $G = K_3 \square K_2$. Let $H = G - V(C)$.

By the induction hypothesis, every graph G' of order $n-1$ with $|E(G')| > \frac{1}{4}(n-1)^2$ and with a 4-cycle contains a chorded 4-cycle. Hence by Lemma 3, $\deg_G x \geq \frac{1}{2}n$ for each $x \in V(H)$.

Claim 1. *For each $x \in V(H)$, $e_G(x, V(C)) \leq 2$.*

Proof. Assume $e_G(x, V(C)) \geq 3$ for some $x \in V(H)$. We may assume $\{v_1, v_2, v_3\} \subset N_G(x)$. Then $x v_1 v_2 v_3 x$ is a 4-cycle with chord $x v_2$. This is a contradiction. $\square$

By Lemma 3 and Claim 1, $\deg_H x \geq \frac{1}{2}n - 2 = \frac{1}{2}(n-4) = \frac{1}{2}|V(H)|$ for each $x \in V(H)$.
We consider three cases.

Case 1. $n = 6$.
In this case, $|V(H)| = 2$. Let $V(H) = \{x, y\}$. Since C is an induced cycle, we have

$$|E(G)| = |E(C)| + e_G(x, V(C)) + e_G(y, V(C)) + |E(H)| \leq 4 + 2 + 2 + 1 = 9.$$

On the other hand, $|E(G)| \geq 9$ by the hypothesis of the theorem. Thus, the equality holds, which yields $e_G(x, V(C)) = e_G(y, V(C)) = 2$ and $xy \in E(G)$.

First, suppose that neither $N_C(x)$ nor $N_C(y)$ consists of a pair of adjacent vertices in C. By symmetry, we may assume $N_C(x) = \{v_1, v_3\}$. If $N_C(y) = \{v_1, v_3\}$, then xv_1yv_3x is a 4-cycle with chord xy, contradicting the assumption. Therefore, $N_C(y) = \{v_2, v_4\}$. Then G is a complete bipartite graph with partite sets $\{x, v_2, v_4\}$ and $\{y, v_1, v_3\}$.

Next, suppose $N_C(x)$ consists of a pair of adjacent vertices in C. We may assume $N_C(x) = \{v_1, v_2\}$. If $yv_1 \in E(G)$, then xyv_1v_2x is a 4-cycle with chord xv_1, a contradiction. Hence $v_1 \notin N_G(y)$. Similarly, we have $v_2 \notin N_G(y)$, and hence $N_C(y) = \{v_3, v_4\}$. This yields $G = K_3 \,\square\, K_2$.

Case 2. $n = 7$.
In this case $|V(H)| = 3$ and $|E(H)| \leq 3$. Let $V(H) = \{x, y, z\}$. By Claim 1, we have

$$|E(G)| = |E(C)| + e_G(x, V(C)) + e_G(y, V(C)) + e_G(z, V(C)) + |E(H)| \leq 4 + 2 \cdot 3 + 3 = 13.$$

On the other hand, by the hypothesis of the theorem, we have $|E(G)| \geq 13$. Thus, the equality holds, which yields $e_G(x, V(C)) = e_G(y, V(C)) = e_G(z, V(C)) = 2$ and $H = K_3$.

Assume $N_C(x)$ does not consist of a pair of adjacent vertices in C. By symmetry, we may assume $N_C(x) = \{v_1, v_3\}$. If $v_1 \in N_G(y)$, then $xzyv_1x$ is a 4-cycle with chord xy, a contradiction. Hence $v_1 \notin N_G(y)$. Similarly, $v_3 \notin N_G(y)$ and hence $N_C(y) = \{v_2, v_4\}$. By applying the same argument to z instead of y, we also have $N_C(z) = \{v_2, v_4\}$. However, now yv_2zv_4y is a 4-cycle with chord yz, a contradiction. Therefore, $N_C(x)$ consists of a pair of adjacent vertices in C. By the same argument, we have that both $N_C(y)$ and $N_C(z)$ consist of a pair of adjacent vertices in C.

If $N_C(x) = N_C(y)$, then we may assume that $N_C(x) = \{v_1, v_2\}$ and xv_1yv_2x is a 4-cycle with chord v_1v_2, a contradiction. Hence $N_C(x) \neq N_C(y)$. By a similar argument, we have that $N_C(x)$, $N_C(y)$ and $N_C(z)$ are all different. Then by symmetry, we may assume $N_C(x) = \{v_1, v_2\}$, $N_C(y) = \{v_2, v_3\}$ and $N_C(z) = \{v_3, v_4\}$. However, now xv_2yzx is a 4-cycle with chord xy, a contradiction. Thus, the theorem follows for $n = 7$.

Case 3. $n \geq 8$.
In this case, $|V(H)| \geq n - 4 \geq 4$. Since $\delta(H) \geq \frac{1}{2}|V(H)|$, H is Hamiltonian by Dirac's Theorem. Moreover, since $|E(H)| \geq \frac{1}{2}\delta(H) \cdot |V(H)| \geq \frac{1}{4}|V(H)|^2$, either H is pancyclic or $H = K_{\frac{n-4}{2},\frac{n-4}{2}}$ by Theorem A. In either case, H contains a 4-cycle. Therefore, by the induction hypothesis, H contains a chorded 4-cycle, or H is either $K_{\frac{n-4}{2},\frac{n-4}{2}}$ or $K_3 \,\square\, K_2$. However, since G does not contain a chorded 4-cycle, the first possibility does not occur, and we have either $H = K_{\frac{n-4}{2},\frac{n-4}{2}}$ or $H = K_3 \,\square\, K_2$. In either case, $|E(H)| = \frac{1}{4}|V(H)|^2 = \frac{1}{4}(n-4)^2$. Furthermore, by Claim 1, $e_G(V(H), V(C)) \leq 2(n-4)$. These imply

$$|E(G)| \leq |E(C)| + e_G(V(H), V(C)) + |E(H)| \leq 4 + 2(n-4) + \frac{1}{4}(n-4)^2 = \frac{1}{4}n^2.$$

Then by the hypothesis of the theorem, the equality holds in the above. This yields that $e_G(x, V(C)) = 2$ for each $x \in V(H)$ and $E_G(V(H), V(C)) = 2(n-4)$.

First, suppose $H = K_{\frac{n-4}{2},\frac{n-4}{2}}$. Let $X = \left\{x_1, x_2, \ldots, x_{\frac{n-4}{2}}\right\}$ and $Y = \left\{y_1, y_2, \ldots, y_{\frac{n-4}{2}}\right\}$ be the partite sets of H. Take $v_i \in V(C)$ and suppose $|N_G(v_i) \cap X| \geq 2$. By symmetry, we may assume $\{x_1, x_2\} \subset N_G(v_i)$. Then $v_ix_1y_jx_2v_i$ is a 4-cycle in G for each j, $1 \leq j \leq \frac{n-4}{2}$. Since this cycle does not contain a chord by the assumption, $v_iy_j \notin E(G)$. This means $N_G(v_i) \cap Y = \emptyset$. Similarly, if $|N_G(v_i) \cap Y| \geq 2$, then $N_G(v_i) \cap X = \emptyset$. Therefore, if $N_G(v_i) \cap X \neq \emptyset$ and $N_G(v_i) \cap Y \neq \emptyset$, then we have $|N_G(v_i) \cap X| = 1$ and $|N_G(v_i) \cap Y| = 1$. Let $A = \{v_i \in V(C) : N_G(v_i) \cap X \neq \emptyset \text{ and } N_G(v_i) \cap Y \neq \emptyset\}$ and $B = V(C) - A$. By the definition, if $v_i \in A$, then $e_G(v_i, V(H)) = 2$ and if $v_i \in B$, then $N_G(v_i) \cap X = \emptyset$ or $N_G(v_i) \cap Y = \emptyset$, and hence $e_G(v_i, V(H)) \leq \frac{n-4}{2}$. Therefore,

$$\begin{aligned} e_G(V(H), V(C)) &\leq 2|A| + \frac{n-4}{2}|B| = \frac{n-4}{2}(|A| + |B|) + \left(2 - \frac{n-4}{2}\right)|A| \\ &= \frac{n-4}{2} \cdot 4 - \frac{1}{2}(n-8)|A| = 2(n-4) - \frac{1}{2}(n-8)|A|. \end{aligned}$$

Since $e_G(V(H), V(C)) = 2(n-4)$, we have either $n = 8$ or $A = \emptyset$. Moreover, $N_G(v_i) \cap V(H) = X$ or $N_G(v_i) \cap V(H) = Y$ for every $v_i \in B$.

Suppose $A = \emptyset$. By symmetry, we may assume $N_G(v_1) \cap V(H) = X$. If $N_G(v_2) \cap V(H) = X$, then $v_1x_1v_2x_2v_1$ is a 4-cycle with chord v_1v_2, a contradiction. Hence we have $N_G(v_2) \cap V(H) = Y$. By applying the same argument to v_2 and v_3, we have $N_G(v_3) \cap V(H) = X$. Similarly, we have $N_G(v_4) \cap V(H) = Y$. Therefore, G is a complete bipartite graph with partite sets $\{v_2, v_4\} \cup X$ and $\{v_1, v_3\} \cup Y$.

Next, suppose $A \neq \emptyset$. Then $n = 8$ and $V(H) = \{x_1, x_2, y_1, y_2\}$. In this case, H is a 4-cycle. Therefore, we can exchange the role of C and H and argue as above. As a result, we have $e_G(v_i, V(H)) = 2$ for each i with $1 \le i \le 4$.

Without loss of generality, we may assume $v_1 \in A$ and $N_G(v_1) \cap V(H) = \{x_1, y_1\}$. If $x_1 \in N_G(v_2)$, then $v_2x_1y_1v_1v_2$ is a 4-cycle with chord x_1v_1. If $y_1 \in N_G(v_2)$, then $v_2y_1x_1v_1v_2$ is a 4-cycle with chord v_1y_1. Hence we reach a contradiction in either case. Thus, we have $N_G(v_2) \cap \{x_1, y_1\} = \emptyset$. Since $e_G(v_2, V(H)) = 2$, this implies $N_G(v_2) = \{x_2, y_2\}$. We apply the same argument to v_4 instead of v_2 and we obtain $N_G(v_4) \cap V(H) = \{x_2, y_2\}$. However, we now see that $v_2x_2v_4y_2v_2$ is a 4-cycle with chord x_2y_2. This is a contradiction.

Finally, suppose $H = K_3 \,\square\, K_2$. In this case $n = 10$ and $e_G(V(H), V(C)) = 12$. Let $V(H) = \{x_1, y_1, z_1, x_2, y_2, z_2\}$, where both $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ induce K_3 and $\{x_1x_2, y_1y_2, z_1z_2\} \subset E(H)$. Since $e_G(V(H), V(C)) = 12$, $e_G(v_i, V(H)) \ge 3$ for some $v_i \in V(C)$. Then we have either $|N_G(v_i) \cap \{x_1, y_1, z_1\}| \ge 2$ or $|N_G(v_i) \cap \{x_2, y_2, z_2\}| \ge 2$. Without loss of generality, we may assume $\{x_1, y_1\} \subset N_G(v_1)$. Then $v_1x_1z_1y_1v_1$ is a 4-cycle with chord x_1y_1. This is a final contradiction, and the theorem follows. □

Next, we prove the case $k \ge 6$.

Theorem 5. *Let k and n be integers with $k \ge 6$ and $n \ge k$. Let G be a graph of order n. If $|E(G)| \ge \frac{1}{4}n^2$ and G contains a k-cycle, then G contains a chorded k-cycle.*

Proof. We proceed by induction on n. Let $C = v_1v_2 \dots v_kv_1$ be a k-cycle in G. If $n = k$, then $V(G) = V(C)$ and $|E(G)| \ge \frac{1}{4}k^2 > k$ since $k \ge 6$. Hence G contains an edge which is not an edge of C. This edge is a chord of C. Therefore, we may assume $n \ge k+1$.

Assume, to the contrary, that G does not contain a chorded k-cycle. By the induction hypothesis, every graph G' of order $n-1$ with $|E(G')| > \frac{1}{4}(n-1)^2$ and with a k-cycle contains a chorded k-cycle. Let $H = G - V(C)$. By Lemma 3, $\deg_G x \ge \frac{1}{2}n$ for every $x \in V(H)$.

Claim 1. *$e_G(x, V(C)) \le \frac{1}{2}k$ for each $x \in V(H)$.*

Proof. If $N_C(x) \cap N_C(x)^{++} = \emptyset$, then $|N_C(x)| + |N_C(x)^{++}| = |N_C(x) \cup N_C(x)^{++}| \le |V(C)| = k$. Since $|N_C(x)| = |N_C(x)^{++}| = e_G(x, V(C))$, we have $e_G(x, V(C)) \le \frac{1}{2}k$.

Suppose $N_C(x) \cap N_C(x)^{++} \neq \emptyset$. We may assume $\{v_1, v_3\} \subset N_C(x)$. Then $xv_3v_4 \dots v_kv_1x$ is a k-cycle in G. Since G does not contain a chorded k-cycle, $N_C(x) \cap \{v_4, v_5, \dots, v_k\} = \emptyset$, which yields $N_C(x) \subset \{v_1, v_2, v_3\}$ and hence $e_G(x, V(C)) \le 3 \le \frac{1}{2}k$. □

Since $|V(H)| = n - k$, we have $e_G(V(H), V(C)) \le \frac{k}{2}(n-k)$ by Claim 1.

Claim 2. *$\delta(H) \ge \frac{1}{2}|V(H)|$.*

Proof. Let $x \in V(H)$. Then $\deg_G x \ge \frac{1}{2}n$ by Lemma 3 and $e_G(x, V(C)) \le \frac{1}{2}k$ by Claim 1. Therefore,

$$\deg_H x = \deg_G x - e_G(x, V(C)) \ge \frac{1}{2}n - \frac{1}{2}k = \frac{1}{2}(n-k) = \frac{1}{2}|V(H)|,$$

and the claim follows. □

By Claim 2 and Dirac's Theorem, H is Hamiltonian if $|V(H)| \ge 3$.

Claim 3. *$|E(H)| > \frac{1}{4}|V(H)|^2$.*

Proof. Since C is an induced cycle,

$$\frac{1}{4}n^2 \le |E(G)| = |E(C)| + e_G(V(H), V(C)) + |E(H)| \le k + \frac{1}{2}k(n-k) + |E(H)|,$$

which yields

$$|E(H)| \ge \frac{1}{4}n^2 - k - \frac{1}{2}k(n-k) = \frac{1}{4}(n-k)^2 + \frac{1}{4}k(k-4) > \frac{1}{4}(n-k)^2 = \frac{1}{4}|V(H)|^2. \qquad \square$$

Claim 4. *$n \le 2k - 1$.*

Proof. Assume $n \ge 2k$. Then $|V(H)| = n - k \ge k \ge 6$ and H is Hamiltonian. Then by Claim 3 and Theorem A, H is pancyclic. In particular, H contains a k-cycle. Then by Claim 3 and the induction hypothesis, H contains a chorded k-cycle, which is also a chorded k-cycle of G. This is a contradiction. □

By Claim 4, $k \ge \frac{n+1}{2}$. Let $k = \frac{n+s}{2}$ and $|E(\overline{H})| = t$. Then $s \ge 1$ and $t \ge 0$. Moreover, since $|V(H)| = n - k$, $|E(H)| = \frac{1}{2}(n-k)(n-k-1) - t$. Therefore,

$$|E(G)| = |E(C)| + e_G(V(H), V(C)) + |E(H)| \le k + \frac{1}{2}k(n-k) + \frac{1}{2}(n-k)(n-k-1) - t.$$

On the other hand, $|E(G)| \geq \frac{1}{4}n^2$ by the hypothesis of the theorem. Therefore, we have $\frac{1}{4}n^2 \leq k + \frac{1}{2}k(n-k) + \frac{1}{2}(n-k)(n-k-1) - t$, which yields $n^2 - 2n - 4t \geq 2k(n-3) = 2(n-3)\cdot\frac{n+s}{2} = (n-3)(n+s)$. This implies $(s-1)n - 3s + 4t \leq 0$. Since $n \geq k+1 \geq 7$ and $s \geq 1$, we have $0 \geq 7(s-1) - 3s + 4t = 4(s+t) - 7$. Since $s \geq 1$, this is possible only if $s = 1$ and $t = 0$. Therefore, we have $n = 2k-1$ and $H = K_{k-1}$.

Suppose $|N_G(v_i) \cap V(H)| \geq 2$ for some $v_i \in V(C)$. Let $x, x' \in N_G(v_i) \cap V(H)$ with $x \neq x'$. Since H is a complete graph, H contains a Hamiltonian path P which starts at x and ends at x'. Then $v_i x \overrightarrow{P} x' v_i$ is a k-cycle with chord xx'. This contradicts the assumption. Thus, we have $e_G(v_i, V(H)) \leq 1$ for each $v_i \in V(C)$ and hence $e_G(V(H), V(C)) \leq V(C) = k$. Now we have

$$\frac{1}{4}(2k-1)^2 = \frac{1}{4}n^2 = |E(G)| = |E(C)| + e_G(V(H), V(C)) + |E(H)|$$
$$\leq k + k + \frac{1}{2}(n-k)(n-k-1) = 2k + \frac{1}{2}(k-1)(k-2),$$

which yields $2k^2 - 6k - 3 \leq 0$. However, in the range of $k \geq 6$, $2k^2 - 6k + 3$ is monotone increasing. Hence $2k^2 - 6k - 3 \geq 2\cdot 6^2 - 6\cdot 6 - 3 = 33 > 0$. This is a contradiction, and the theorem follows. $\square$

We now consider the remaining case : $k = 5$.

Theorem 6. *Let G be a graph of order n. If $|E(G)| \geq \frac{1}{4}n^2$ and G contains a 5-cycle, then G contains a chorded 5-cycle.*

Proof. We proceed by induction on n. By the hypothesis of the theorem, we have $n \geq 5$. Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-cycle of G.

If $n = 5$, then $V(G) = V(C)$. Moreover, $|E(G)| \geq 7$ by the hypothesis of the theorem. Then G contains an edge which is not an edge of C. This edge is a chord of C.

Suppose $n \geq 6$ and we assume that G does not contain a chorded 5-cycle. Let $H = G - V(C)$. By the induction hypothesis, every graph of order $n-1$ with more than $\frac{1}{4}(n-1)^2$ edges and with a 5-cycle contains a chorded 5-cycle. By Lemma 3, $\deg_G x \geq \frac{1}{2}n$ for each $x \in V(H)$.

Claim 1. *Let $x \in V(H)$. Then*

(1) $|N_C(x)| \leq 3$, *and*
(2) *if* $|N_C(x)| = 3$, *then $N_C(x)$ consists of three consecutive vertices in C.*

Proof. Suppose $|N_C(x)| \geq 3$. Then $N_C(x)$ contains a pair of non-adjacent vertices in C. Without loss of generality, we may assume $\{v_1, v_3\} \subset N_G(x)$. Then $xv_3v_4v_5v_1x$ is a 5-cycle in G. Since G does not contain a chorded 5-cycle, $\{v_4, v_5\} \cap N_G(x) = \emptyset$ and hence $N_C(x) = \{v_1, v_2, v_3\}$. This proves both (1) and (2). $\square$

Claim 2. *If $|N_C(x_1)| = |N_C(x_2)| = 3$ for distinct vertices x_1 and x_2 in H, then $|N_C(x_1) \cap N_C(x_2)| = 1$.*

Proof. By Claim 1, we may assume $N_C(x_1) = \{v_1, v_2, v_3\}$. If $|N_C(x_1) \cap N_C(x_2)| \geq 2$, then since $N_C(x_2)$ also consists of consecutive vertices in C, we may assume $\{v_1, v_2\} \subset N_C(x_2)$. Then $x_1v_1x_2v_2v_3x_1$ is a 5-cycle in G with chord v_1v_2, This contradicts the assumption. Therefore, we have $|N_C(x_1) \cap N_C(x_2)| = 1$. $\square$

By Claims 1 and 2, at most two vertices in H have three neighbors in C and the other vertices in H have at most two neighbors in C. This implies, $e_G(V(H), V(C)) \leq 3\cdot 2 + 2(|V(H)| - 2) = 6 + 2(n-7) = 2n - 8$.

We consider the cases $6 \leq n \leq 9$. First, suppose $n = 6$. In this case, $|E(G)| \geq 9$. On the other hand, $V(G) - V(C)$ consists of exactly one vertex and this vertex has at most three neighbors in C by Claim 1. This implies $|E(G)| \leq |E(C)| + 3 = 8$. This is a contradiction.

Suppose $n = 7$. In this case, the hypothesis of the theorem yields $|E(G)| \geq 13$. On the other hand, $|V(H)| = 2$ and hence $|E(H)| \leq 1$. Therefore, $|E(G)| \leq |E(C)| + e_G(V(H), V(C)) + |E(H)| \leq 5 + (2\cdot 7 - 8) + 1 = 12$. This is again a contradiction.

Suppose $n = 8$. Then $|V(H)| = 3$, and $|E(G)| \geq 16$ by the hypothesis of the theorem. Let $V(H) = \{x_1, x_2, x_3\}$. Since $|E(H)| \leq 3$, we have $16 \leq |E(G)| = |E(C)| + e_G(V(C), V(H)) + |E(H)| \leq 5 + (2\cdot 8 - 8) + 3 = 16$. Thus, the equality holds, which implies $H = K_3$. Also, we may assume $e_G(x_1, V(C)) = e_G(x_2, V(C)) = 3$ and $e_G(x_3, V(C)) = 2$. Moreover, by Claim 2 we may assume $N_C(x_1) = \{v_1, v_2, v_3\}$ and $N_C(x_2) = \{v_4, v_5, v_1\}$. Then $x_1x_2v_1v_2v_3x_1$ is a 5-cycle in G with chord x_1v_1, a contradiction.

Suppose $n = 9$. Then $|V(H)| = 4$ and $|E(H)| \leq 6$. Let $V(H) = \{x_1, x_2, x_3, x_4\}$. By the hypothesis of the theorem, $|E(G)| \geq 21$. On the other hand, $|E(G)| \leq |E(C)| + e_G(V(H), V(C)) + |E(H)| \leq 5 + (2\cdot 9 - 8) + 6 = 21$. Thus, the equality holds. This implies that $H = K_4$ and there are two vertices in H, say x_1 and x_2, that have three neighbors in C. By Claim 1, we may assume $N_C(x_1) \cap N_C(x_2) = \{v_1\}$. However, we now have a 5-cycle $v_1x_1x_3x_4x_2v_1$ with chord x_1x_4. This is a contradiction.

Now we assume $n \geq 10$.

Claim 3. *H is Hamiltonian.*

Proof. Assume H is not Hamiltonian. By Lemma 3 and Claim 1, $\deg_H x = \deg_G x - e_G(x, V(C)) \geq \frac{1}{2}n - 3 = \frac{n-6}{2} = \frac{|V(H)|-1}{2}$ for each $x \in V(H)$. By Dirac's Theorem, this implies that H contains a Hamiltonian path. Take a Hamiltonian path $P = x_1x_2 \ldots x_{n-5}$ so that $\deg_H x_1 + \deg_H x_{n-5}$ is as large as possible. By the assumption, $x_1x_{n-5} \notin E(G)$.

If $N_H(x_1)^- \cap N_H(x_{n-5}) \neq \emptyset$, let $x_p \in N_H(x_1)^- \cap N_H(x_{n-5})$, and $x_1x_{p+1} \overrightarrow{P} x_{n-5}x_p \overleftarrow{P} x_1$ is a Hamiltonian cycle of H, a contradiction. Thus, $N_H(x_1)^- \cap N_H(x_{n-5}) = \emptyset$. Since $N_H(x_1)^- \cup N_H(x_{n-5}) \subset V(H) - \{x_{n-5}\}$, we have

$$\begin{aligned}\deg_H x_1 + \deg_H x_{n-5} &= |N_H(x_1)| + |N_H(x_{n-5})| = |N_H(x_1)^-| + |N_H(x_{n-5})| \\ &= |N_H(x_1)^- \cup N_H(x_{n-5})| \leq n - 6.\end{aligned}$$

Since $\deg_H x_1 \geq \frac{1}{2}n - 3$ and $\deg_H(x_{n-5}) \geq \frac{1}{2}n - 3$, the equality holds in the above. This implies $e_G(x_1, V(C)) = e_G(x_{n-5}, V(C)) = 3$. Since $n \geq 10$, $x_1 \neq x_{n-5}$, and these are the only vertices in H having three neighbors in C. In particular, $\deg_H x_i \geq \frac{1}{2}n - 2$ for each i with $2 \leq i \leq n - 6$.

Since $n \geq 10$, $\deg_H x_1 \geq 2$. Thus, we can take $x_i \in N_G(x_1)$ with $i \geq 3$. Then $x_{i-1} \overleftarrow{P} x_1x_i \overrightarrow{P} x_{n-5}$ is a Hamiltonian path in H with end-vertices x_{i-1} and x_{n-5}. However, $\deg_H x_{i-1} + \deg_H x_{n-5} \geq \frac{1}{2}n - 2 + \frac{1}{2}n - 3 = n - 5 > \deg_H x_1 + \deg_H x_{n-5}$. This contradicts the choice of P. $\square$

Now we can complete the proof. Since $|E(G)| \geq \frac{1}{4}n^2$,

$$\begin{aligned}|E(H)| &= |E(G)| - |E(C)| - e_G(V(H), V(C)) \geq \frac{1}{4}n^2 - 5 - (2n - 8) = \frac{1}{4}(n^2 - 8n + 12) \\ &= \frac{1}{4}(n-5)^2 + \frac{1}{4}(2n - 13) > \frac{1}{4}(n-5)^2 = \frac{1}{4}|V(H)|^2.\end{aligned}$$

Since H is Hamiltonian and $|E(H)| > \frac{1}{4}|V(H)|^2$, H is pancyclic. Moreover, since $n \geq 10$, $|V(H)| \geq 5$ and hence H contains a 5-cycle. Then by the induction hypothesis, H contains a chorded 5-cycle, which is also a chorded 5-cycle in G. This is a final contradiction, and the theorem follows. $\square$

Theorems 4–6 complete the proof of Theorem 1.

3. Improving the density condition for large cycles

As we have seen in the introduction, as long as we require chorded pancyclicity in a graph G of order n, it seems to be difficult to relax the density condition $|E(G)| \geq \frac{1}{4}n^2$ since exceptional graphs appear at $|E(G)| = \frac{1}{4}n^2$. However, in these exceptions, only a chorded 4-cycle is missing. Therefore, if we only require a chorded k-cycle for large k, we may be able to improve the density condition. In this section, we study this possibility, and prove the following theorem.

Theorem 7. *Let k be an integer with $k \geq 8$ and let G be a graph of order n with $|E(G)| \geq \frac{1}{4}n^2 - n + 16$. If G contains a k-cycle, then G contains a chorded k-cycle.*

By combining this theorem with Theorem D, we obtain the following corollary.

Corollary 8. *A non-bipartite graph G of order n with at least $\left\lfloor \frac{n^2}{4} \right\rfloor - n + 59$ edges contains a chorded k-cycle for every integer k with $8 \leq k \leq c(G)$.*

We use the following theorem in the proof.

Theorem F (*Faudree et al.* [7]). *Let G be a graph of order $n \geq 3$ and minimum degree $\delta = \delta(G) < \frac{1}{2}n$. If $\left|\{v \in V(G) : \deg_G v < \frac{1}{2}n\}\right| \leq \delta - 1$, then G is Hamiltonian.*

Proof of Theorem 7. We proceed by induction on n. The hypothesis requires $n \geq k \geq 8$. For $8 \leq n \leq 16$, the theorem follows from Theorem 1. Suppose $n \geq 17$ and let G be a graph of order n with at least $\frac{1}{4}n^2 - n + 16$ edges and with a k-cycle C. We prove that G contains a chorded k-cycle.

Assume, to the contrary, that G does not contain a chorded k-cycle. Let $H = G - V(C)$.

Claim 1. *For every $x \in V(H)$, $\deg_G x \geq \frac{1}{2}n - 1$.*

Proof. Let $G' = G - x$. If $|E(G')| \geq \frac{1}{4}(n-1)^2 - (n-1) + 16$, then since G' contains C, G' contains a chorded k-cycle by the induction hypothesis. This contradicts the assumption. Therefore, we have $|E(G')| < \frac{1}{4}(n-1)^2 - (n-1) + 16$. Then

$$\deg_G x = |E(G)| - |E(G')| > \frac{1}{4}n^2 - n + 16 - \left(\frac{1}{4}(n-1)^2 - (n-1) + 16\right) = \frac{1}{2}n - \frac{5}{4}.$$

Since $\deg_G x$ is an integer, we have $\deg_G x \geq \frac{1}{2}n - 1$. $\square$

Let $A = \{x \in V(H) : N_C(x) \cap N_C(x)^{++} \neq \emptyset\}$ and $B = V(H) - A$. Also let $a = |A|$ and $b = |B|$. Note $a + b = n - k$.

Claim 2. *For each $x \in A$, $e_G(x, V(C)) \leq 3$.*

Proof. Since $x \in A$, there exists a vertex v in C with $\{v, v^{++}\} \subset N_G(x)$. Then $vxv^{++}\overrightarrow{C}v$ is a cycle of order k, and it is an induced cycle by the assumption. This implies $N_C(x) \subset \{v, v^+, v^{++}\}$ and $e_G(x, V(C)) \leq 3$. □

Claim 3.

(1) *For each $x \in V(H)$, $e_G(x, V(C)) \leq \frac{1}{2}k$, and*
(2) $\delta(H) \geq \frac{1}{2}|V(H)| - 1$.

Proof. (1) If $x \in A$, Claim 2 yields $e_G(x, V(C)) \leq 3 < \frac{1}{2}k$. If $x \in B$, then the function $f : N_C(x) \to V(C) - N_C(x)$ defined by $f(x) = x^{++}$ is an injection, which implies $|N_C(x)| \leq |V(C)| - |N_C(x)|$. This yields $e_G(x, V(C)) = |N_C(x)| \leq \frac{1}{2}|V(C)| = \frac{1}{2}k$.

(2) By (1) and Claim 1, we have $\deg_H(x) = \deg_G x - e_G(x, V(C)) \geq \frac{1}{2}n - 1 - \frac{1}{2}k = \frac{1}{2}(n-k) - 1 = \frac{1}{2}|V(H)| - 1$. □

Claim 4. *Suppose that there exist vertices $x_0 \in V(H)$ and $v \in V(C)$ with $\{v, v^{+++}\} \subset N_G(x_0)$. Then $N_C(x) \cap N_C(x)^+ \cap V(v^{(4)+}\overrightarrow{C}v) = \emptyset$ for each $x \in V(H) - \{x_0\}$.*

Proof. Assume $N_C(x) \cap N_C(x)^+ \cap V(v^{(4)+}\overrightarrow{C}v) \neq \emptyset$, and let u be a vertex in $v^{+++}\overrightarrow{C}v^-$ with $\{u, u^+\} \subset N_G(x)$. Then $uxu^+\overrightarrow{C}vx_0v^{+++}\overrightarrow{C}u$ is a cycle of order k with the chord uu^+. This is a contradiction. □

Claim 5. *If there exists a vertex x_0 in B with $N_C(x_0) \cap N_C(x_0)^{+++} \neq \emptyset$, then $e_G(x, V(C)) \leq \frac{k+2}{3}$ holds for every $x \in B - \{x_0\}$.*

Proof. By the assumption, there exists a vertex $v \in V(C)$ with $\{v, v^{+++}\} \subset N_G(x_0)$. Let $P = v^{+++}\overrightarrow{C}v$. Let $x \in B - \{x_0\}$ and $u \in N_P(x)$. If $u \neq v$, then $u^+ \notin N_G(x)$ by Claim 4. Moreover, if $u \notin \{v, v^-\}$, then $u^{++} \notin N_G(x)$ by the definition of B. These imply $|N_P(x)| \leq \frac{|V(P)|+2}{3} = \frac{1}{3}k$.

Suppose $\{v^+, v^{++}\} \subset N_G(x)$. Since $v^+ \in N_G(x)$ and $x \in B$, $\{v^-, v^{+++}\} \cap N_G(x) = \emptyset$. Similarly, since $v^{++} \in N_G(x)$, $\{v, v^{(4)+}\} \cap N_G(x) = \emptyset$. These imply $|N_P(x)| \leq \frac{|V(P)|-2}{3} = \frac{k-4}{3}$ and hence $|N_C(x)| \leq \frac{k-4}{3} + 2 = \frac{k+2}{3}$.

Next, suppose $|\{v^+, v^{++}\} \cap N_G(x)| = 1$. By symmetry, we may assume $\{v^+, v^{++}\} \cap N_G(x) = \{v^{++}\}$. Then $v \notin N_G(x)$ and hence $|N_P(x)| \leq \frac{|V(P)|+1}{3} = \frac{k-1}{3}$, which yields $|N_C(x)| \leq \frac{k-1}{3} + 1 = \frac{k+2}{3}$.

Finally, suppose $\{v^+, v^{++}\} \cap N_G(x) = \emptyset$. In this case, $|N_C(x)| = |N_P(x)| \leq \frac{k}{3}$. □

Claim 6. *If $x \in B$ and $N_C(x) \cap N_C(x)^{+++} = \emptyset$, then $e_G(x, V(C)) \leq \frac{2}{5}k$.*

Proof. Assume $e_G(x, V(C)) > \frac{2}{5}|V(C)|$. Then there exists a subpath $P = v_1v_2v_3v_4v_5$ of order 5 in C with $|N_G(x) \cap \{v_1, v_2, v_3, v_4, v_5\}| \geq 3$.

If $\{v_1, v_2\} \cap N_G(x) = \emptyset$, then $\{v_3, v_4, v_5\} \subset N_G(x)$. This contradicts $x \in B$. Hence we have $\{v_1, v_2\} \cap N_G(x) \neq \emptyset$. If $v_2 \in N_G(x)$, then $v_4 \notin N_G(x)$ since $x \in B$. Moreover, $v_5 \notin N_G(x)$ since $N_C(x) \cap N_C(x)^{+++} = \emptyset$. These yield $\{v_1, v_2, v_3\} \subset N_G(x)$, which again contradicts $x \in B$. Therefore, $v_2 \notin N_G(x)$ and hence $v_1 \in N_G(x)$. Since $v_3 \notin N_G(x)$ by the definition of B, we have $\{v_1, v_4, v_5\} \subset N_G(x)$. However, we have $v_4 \in N_C(x) \cap N_C(x)^{+++}$, a contradiction. □

Claim 7. *If $|V(H)| \geq 5$, then H is Hamiltonian.*

Proof. Let $x \in V(H)$. If $x \in A$, then by Claim 2, $e_G(x, V(C)) \leq 3$ and by Claim 1, we have $\deg_H x \geq \frac{1}{2}n - 1 - 3 = \frac{1}{2}(n-k) + \frac{1}{2}k - 4 \geq \frac{1}{2}|V(H)|$.

Suppose $x \in B$. If $N_G(x) \cap N_G(x)^{+++} = \emptyset$, then $e_G(x, V(C)) \leq \frac{2}{5}k$ by Claim 6 and

$$\deg_H x \geq \frac{1}{2}n - 1 - \frac{2}{5}k = \frac{1}{2}(n-k) - \frac{1}{2} + \frac{k-5}{10} > \frac{1}{2}(n-k) - \frac{1}{2},$$

which implies $\deg_H x \geq \frac{1}{2}(n-k) = \frac{1}{2}|V(H)|$.

Suppose $N_G(x) \cap N_G(x)^{+++} \neq \emptyset$. By Claim 3, $e_G(x, V(C)) \leq \frac{1}{2}k$ and $\deg_H x \geq \frac{1}{2}n - 1 - \frac{1}{2}k = \frac{1}{2}|V(H)| - 1$. Let $x' \in B - \{x\}$. Then $e_G(x', V(C)) \leq \frac{k+2}{3}$ by Claim 5 and

$$\deg_H x' \geq \frac{1}{2}n - 1 - \frac{k+2}{3} = \frac{1}{2}(n-k) - \frac{1}{2} + \frac{k-7}{6} > \frac{1}{2}(n-k) - \frac{1}{2},$$

which implies $\deg_H x' \geq \frac{1}{2}(n-k)$.

By the above observations, if $N_G(x) \cap N_G(x)^{+++} = \emptyset$ for every $x \in B$, then $\delta(H) \geq \frac{1}{2}|V(H)|$ and H is Hamiltonian by Dirac's Theorem. If $N_G(x_0) \cap N_G(x_0)^{+++} \neq \emptyset$ for some $x_0 \in B$, then x_0 is the only vertex that may have degree less than $\frac{1}{2}|V(H)|$. On the other hand, by Claim 3(2), $\delta(H) \geq \left\lceil \frac{1}{2}|V(H)| \right\rceil - 1 \geq 2$ since $|V(H)| \geq 5$. Hence we have $\left|\left\{v \in V(H) : \deg_H v < \frac{1}{2}|V(H)|\right\}\right| \leq 1 \leq \delta(H) - 1$ and H is Hamiltonian by Theorem F. $\square$

Claim 8.

(1) *If* $k \geq 10$, *then* $e_G(V(H), V(C)) \leq \max\left\{\frac{k+2}{3}(n-k) + \frac{k-4}{6}, \frac{2k}{5}(n-k)\right\}$.
(2) *If* $k = 9$, *then* $e_G(V(H), V(C)) \leq 3n - 26$.
(3) *If* $k = 8$, *then* $e_G(V(H), V(C)) \leq 3n - 23$.

Proof. (1) Suppose B contains a vertex x_0 with $N_C(x_0) \cap N_C(x_0)^{+++} \neq \emptyset$. Then Claims 2, 3 and 5 yield

$$\begin{aligned} e_G(V(H), V(C)) &= \sum_{x \in A \cup B} e_G(x, V(C)) \\ &= \sum_{x \in A} e_G(x, V(C)) + e_G(x_0, V(C)) + \sum_{x \in B - \{x_0\}} e_G(x, V(C)) \\ &\leq 3a + \frac{1}{2}k + \frac{k+2}{3}(b-1) = 3a + \frac{k+2}{3}b + \frac{k-4}{6}. \end{aligned}$$

Next, suppose $N_C(x) \cap N_C(x)^{+++} = \emptyset$ for every $x \in B$. Then Claims 2 and 6 yield

$$\begin{aligned} e_G(V(H), V(C)) &= \sum_{x \in A} e_G(x, V(C)) + \sum_{x \in B} e_G(x, V(C)) \\ &\leq 3a + \frac{2k}{5}b. \end{aligned}$$

Since $k \geq 10$, $\frac{k+2}{3} > 3$ and $\frac{2k}{5} > 3$. Thus, under the constraints of $a \geq 0, b \geq 0$ and $a+b = n-k$, both $3a + \frac{k+2}{3}b + \frac{k-4}{6}$ and $3a + \frac{2k}{5}b$ take the minimum value at $(a, b) = (0, n-k)$, and we have $e_G(V(H), V(C)) \leq \max\left\{\frac{k+2}{3}(n-k) + \frac{k-4}{6}, \frac{2k}{5}(n-k)\right\}$.

(2) If B contains a vertex x_0 with $N_G(x_0) \cap N_G(x_0)^{+++} \neq \emptyset$, then Claims 3(1) and 5 yield $e_G(x_0, V(C)) \leq 4$ and $e_G(x, V(C)) \leq 3$ for each $x \in B - \{x_0\}$. Thus, we have

$$\begin{aligned} e_G(V(H), V(C)) &= \sum_{x \in A} e_G(x, V(C)) + e_G(x_0, V(C)) + \sum_{x \in B - \{x_0\}} e_G(x, V(C)) \\ &\leq 3a + 4 + 3(b-1) = 3(a+b) + 1 = 3(n-9) + 1 = 3n - 26. \end{aligned}$$

On the other hand, if $N_G(x) \cap N_G(x)^{+++} = \emptyset$ for each $x \in B$, then Claim 6 yields $e_G(x, V(C)) \leq 3$ for each $x \in B$ and

$$\begin{aligned} e_G(V(H), V(C)) &= \sum_{x \in A} e_G(x, V(C)) + \sum_{x \in B} e_G(x, V(C)) \\ &\leq 3a + 3b = 3(a+b) = 3(n-9) = 3n - 27 < 3n - 26. \end{aligned}$$

(3) We can follow the same arguments as in the proof of (2) and obtain $e_G(V(H), V(C)) \leq 3(n-8) + 1 = 3n - 23$. $\square$

Claim 9. $|E(H)| > \frac{1}{4}|V(H)|^2$.

Proof. Assume $|E(H)| \leq \frac{1}{4}|V(H)| = \frac{1}{4}(n-k)^2$. Since C is an induced cycle of order k, we have

$$|E(G)| = |E(C)| + e_G(V(H), V(C)) + |E(H)| \leq e_G(V(H), V(C)) + \frac{1}{4}(n-k)^2 + k.$$

On the other hand, $|E(G)| \geq \frac{1}{4}n^2 - n + 16$ by the hypothesis. By combining these inequalities, we have $e_G(V(H), V(C)) + \frac{1}{4}(n-k)^2 + k \geq \frac{1}{4}n^2 - n + 16$, which yields

$$e_G(V(H), V(C)) \geq \frac{k-2}{2}(n-k) + \frac{1}{4}k^2 - 2k + 16. \tag{1}$$

If $k \geq 10$, then

$$\begin{aligned} &\frac{k-2}{2}(n-k) + \frac{1}{4}k^2 - 2k + 16 \\ &\quad = \frac{k+2}{3}(n-k) + \frac{k-4}{6} + \frac{k-10}{6}(n-k) + \frac{1}{4}k(k-10) + \frac{1}{3}k + \frac{50}{3} \\ &\quad > \frac{k+2}{3}(n-k) + \frac{k-4}{6} \end{aligned}$$

and

$$\begin{aligned}\frac{k-2}{2}(n-k)+\frac{1}{4}k^2-2k+16 &= \frac{2k}{5}(n-k)+\frac{k-10}{10}(n-k)+\frac{1}{4}k(k-10)+\frac{1}{2}k+16\\ &> \frac{2k}{5}(n-k).\end{aligned}$$

Hence by Claim 8, $e_G(V(H),V(C)) < \frac{k-2}{2}(n-k)+\frac{1}{4}k^2-2k+16$, which contradicts (1).

If $k=9$, then (1) yields $e_G(V(H),V(C)) \geq \frac{7}{2}n-\frac{53}{4} > 3n-26$, contradicting Claim 8(2). If $k=8$, then (1) yields $e_G(V(H),V(C)) \geq 3n-8 > 3n-23$, contradicting Claim 8(3). $\square$

Claim 10. $k \leq \frac{1}{2}n$.

Proof. Assume $k > \frac{1}{2}n$. Since $n \geq 17$, this implies $k \geq 9$.

Since C is an induced k-cycle and H is a graph of order $n-k$, we have

$$|E(G)| = |E(C)| + e_G(V(H),V(C)) + |E(H)| \leq k + e_G(V(H),V(C)) + \frac{1}{2}(n-k)(n-k-1).$$

On the other hand, by the hypothesis of the theorem, we have $|E(G)| \geq \frac{1}{4}n^2-n+16$. By combining these inequalities, we obtain $e_G(V(H),V(C)) + \frac{1}{2}(n-k)(n-k-1)+k \geq \frac{1}{4}n^2-n+16$, which implies

$$e_G(V(H),V(C)) \geq -\frac{1}{2}k^2+\left(n-\frac{3}{2}\right)k-\frac{1}{4}n^2-\frac{1}{2}n+16. \tag{2}$$

If $k=9$, then $n=17$ and (2) yields $e_G(V(H),V(C)) \geq \frac{137}{4}$. On the other hand, we have $e_G(V(H),V(C)) \leq 25$ by Claim 8(2). This is a contradiction. Therefore, we have $k \geq 10$.

By Claim 8(1), $e_G(V(H),V(C)) \leq \max\left\{\frac{k+2}{3}(n-k)+\frac{k-4}{6}, \frac{2k}{5}(n-k)\right\}$. This implies either

$$\frac{k+2}{3}(n-k)+\frac{k-4}{6} \geq -\frac{1}{2}k^2+\left(n-\frac{3}{2}\right)k-\frac{1}{4}n^2-\frac{1}{2}n+16 \tag{3}$$

or

$$\frac{2k}{5}(n-k) \geq -\frac{1}{2}k^2+\left(n-\frac{3}{2}\right)k-\frac{1}{4}n^2-\frac{1}{2}n+16. \tag{4}$$

Assume (3) holds. Then

$$-\frac{1}{6}k^2+\left(\frac{2}{3}n-1\right)k-\frac{1}{4}n^2-\frac{7}{6}n+\frac{50}{3} \leq 0. \tag{5}$$

Let $f_1(k) = -\frac{1}{6}k^2+\left(\frac{2}{3}n-1\right)k-\frac{1}{4}n^2-\frac{7}{6}n+\frac{50}{3} = -\frac{1}{6}\left(k-(2n-3)\right)^2+\frac{5}{12}n^2-\frac{19}{6}n+\frac{109}{6}$. Since $n \geq 17$, $n < 2n-3$. Thus, in the range of $\frac{1}{2}n < k \leq n$, $f_1(k)$ is monotone increasing, and

$$f_1(k) > f_1\left(\frac{1}{2}n\right) = \frac{1}{24}(n-20)^2 \geq 0.$$

This contradicts (5).

Next assume that (4) holds. Then we have

$$-\frac{1}{10}k^2+\left(\frac{3}{5}n-\frac{3}{2}\right)k-\frac{1}{4}n^2-\frac{1}{2}n+16 \leq 0. \tag{6}$$

Let $f_2(k) = -\frac{1}{10}k^2+\left(\frac{3}{5}n-\frac{3}{2}\right)k-\frac{1}{4}n^2-\frac{1}{2}n+16 = -\frac{1}{10}\left(k-(3n-\frac{15}{2})\right)^2+\frac{13}{20}n^2-5n+\frac{173}{8}$. Since $n \geq 17$, $n < 3n-\frac{15}{2}$. Hence in the range of $\frac{1}{2}n < k \leq n$, $f_2(k)$ is monotone increasing, and

$$f_2(k) > f_2\left(\frac{1}{2}n\right) = \frac{1}{40}n^2-\frac{5}{4}n+16 = \frac{1}{40}(n-25)^2+\frac{3}{8} > 0.$$

This contradicts (6). $\square$

We now complete the proof of Theorem 7. By Claim 10, $k \leq \frac{1}{2}n$, which implies $|V(H)| = n-k \geq k \geq 10$. Hence H is Hamiltonian by Claim 7. We also have $|E(H)| > \frac{1}{4}|V(H)|^2$ by Claim 9 and hence H is pancyclic by Theorem A. Then again since $|V(H)| \geq k$, H contains a k-cycle. Hence H contains a chorded k-cycle by Theorem 1, which is also a chorded k-cycle of G. $\square$

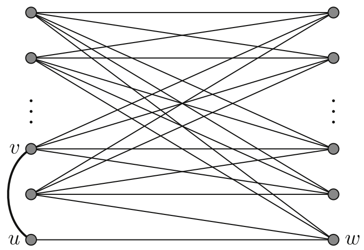

Fig. 1. The graph $H_5^{(n)}$.

4. Concluding remarks

We assume $k \geq 8$ in Theorem 7. We do not know whether the same conclusion holds for $6 \leq k \leq 7$. But we know that the conclusion does not hold for $k = 5$. Let n be an even integer n with $n \geq 30$. Let H be a copy of $K_{\frac{n}{2},\frac{n}{2}}$. Pick a pair of distinct vertices u, v in one partite set of H and pick a vertex w from the other partite set. Delete the edge vw, delete all the edges incident with u except for the edge uw and add the edge uv. Let $H_5^{(n)}$ be the resulting graph (see Fig. 1). Then $H_5^{(n)}$ is a graph of order n with $|E(H_5^{(n)})| = \frac{1}{4}n^2 - \frac{1}{2}n + 1 \geq \frac{1}{4}n^2 - n + 16$, and it contains a 5-cycle, but it does not contain a chorded 5-cycle.

For $k = 4$, no subgraph of $K_{\frac{n}{2},\frac{n}{2}}$ contains a chorded 4-cycle. Moreover, even if we restrict ourselves to non-bipartite graphs, the conclusion of Theorem 7 does not hold. Let $H_4^{(n)} = H_5^{(n)} + vw$. Then $H_4^{(n)}$ is not bipartite, $|E(H_4^{(n)})| \geq \frac{1}{4}n^2 - \frac{1}{2}n + 2 \geq \frac{1}{4}n^2 - n + 16$ if $n \geq 28$, $H_4^{(n)}$ contains a 4-cycle, but it does not contain a chorded 4-cycle.

We do not know the sharpness of Theorem 7. We actually suspect that we can further relax the density condition. But we currently do not know how to improve it.

One of the referees suggests we investigate the relationship between the theme of this paper and the famous Thomassen's Chord Conjecture.

Conjecture 2 (*Thomassen's Chord Conjecture [1,9]*)**.** *Every longest cycle in a* 3*-connected graph is a chorded cycle.*

The results in this paper do not seem to contribute to the solution of this conjecture. The assumption of Theorem 1 makes the graph G of order n have average degree at least $\frac{1}{2}n$, while there are infinitely many 3-regular 3-connected graphs. There is a large gap on the average degree between the classes of graphs considered in this paper and Conjecture 2. (Note that Conjecture 2 has been solved for 3-regular 3-connected graphs by Thomassen [10].)

On the other hand, in Theorem 7, we slightly relax the density condition of Theorem 1 by restricting ourselves to cycles of order at least 8. By further pursuing this line of research, we may be able to obtain some hint to tackle Conjecture 2. In particular, if we obtain a lower bound on the density which is linear in the order of the graph, then it could be a partial answer to the conjecture. However, we currently do not know whether this is a feasible approach.

Thomassen [11] has recently proved that every 3-connected graph of minimum degree at least 4 contains a longest cycle which is also a chorded cycle.

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