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On the Ramsey Number of Trees Versus Graphs with Large Clique Number

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ABSTRACT

Chvátal established that $r(T_m, K_n) = (m-1)(n-1)+1$, where T_m is an arbitrary tree of order m and K_n is the complete graph of order n. This result was extended by Chartrand, Gould, and Polimeni who showed K_n could be replaced by a graph with clique number n and order $n+1$ provided $n \geq 3$ and $m \geq 3$. We further extend these results to show that K_n can be replaced by any graph on $n+2$ vertices with clique number n, provided $n \geq 5$ and $m \geq 4$. We then show that further extensions, in particular to graphs on $n+3$ vertices with clique number n are impossible. We also investigate the ramsey number of trees versus complete graphs minus sets of independent edges. We show that $r(T_m, K_n - tK_2) = (m-1)(n-t-1)+1$ for $m \geq 3$, $n \geq 6$, where T_m is any tree of order m except the star, and for each t, $0 \leq t \leq [(n-2)/2]$.

INTRODUCTION

For graphs G and H, the *ramsey number* $r(G,H)$ is the smallest positive integer p such that if every edge of the complete graph K_p is arbitrarily colored red or blue, then there exists either a red G (a subgraph isomorphic to G, all of whose edges are colored red) or a blue H. Equivalently, $r(G,H)$ is the smallest positive integer p such that if $K_p = R \otimes B$ is an arbitrary factorization of K_p (i.e., R and B have order p and $E(R) \cup E(B)$ partitions $E(K_p)$) then $G \subseteq R$ or $H \subseteq B$. A *(G,H)-blocking pattern* of K_t is a factorization $K_t = R \oplus B$ such that $G \nsubseteq R$ and $H \nsubseteq B$. The *clique number* of a graph G is the maximum order of a complete subgraph of G. If

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$S \subseteq V(G)$, the *subgraph induced* by S, denoted $\langle S \rangle$, is the subgraph with vertex set S and whose edge set consists of those edges of G incident with two elements of S. We denote by $G_1 - G_2$ the graph obtained by deleting the edges of G_2 from the graph G_1. Note that $G - K_2$ is also denoted $G - e$.

A well known result is the following:

Theorem A. (Chvátal [1]). If T_m is a tree of order m and n is a positive integer then $r(T_m, K_n) = (m-1)(n-1)+1$.

A result related to Theorem A was given in [3].

Theorem B [3]. For each tree T_m of order $m \geq 3$ and each integer $n \geq 4$, $r(T_m, K_n - e) = (m-1)(n-2)+1$ and hence, $r(T_m, G) = (m-1)(n-2)+1$ for each graph G of order n and clique number $n-1$.

We shall also require the following results:

Theorem C [2]. Let G be a graph of order n. Then

$$r(P_3, G) = \begin{cases} n, & \text{if } \bar{G} \text{ has a 1-factor,} \\ 2n - 2\beta_1(\bar{G}) - 1, & \text{otherwise,} \end{cases}$$

where $\beta_1(\bar{G})$ denotes the edge independence number of the complement of G.

Theorem D [7]. If P_m is a path of order $m \geq 4$ and G_n is a graph of order $n+2$ with clique number $n (n \geq 3)$ then

$$r(P_m, G_n) = (m-1)(n-1)+1$$

Theorem E [5]. If $l, t \geq 1$, $m \geq 2$, and

$$l > (t-1) - \left[\frac{t-1}{m-1}\right](m-1)$$

then

$$r(T_m, K_l + \bar{K}_t) = \left(l + \left[\frac{t-1}{m-1}\right]\right)(m-1)+1.$$

We note that Theorem E is an extension of Theorem B in that we may consider $K_l + \bar{K}_t$ to be $K_{l+t} - K_t$.

Theorem F [6]. If $g_1, g_2, \ldots, g_k \in \mathscr{G}(H)$ and $G = \cup_{i=1}^{k} g_i$, where $\mathscr{G}(H) = \{g \mid g$ is connected and $r(g, H) = (|V(g)| - 1)(\chi(H) - 1) + 1\}$ then,

$$r(G, H) = \max_{i \leq j \leq c(G)} \left\{ (j-1)(\chi(H) - 2) + \sum_{i=j}^{c(G)} ik_i \right\},$$

where $c(G)$ denotes the order of the largest component of G and k_i is the number of components of G of order i.

The purpose of this article is to investigate the ramsey number of trees versus complete graphs minus a set of independent edges. We also further extend Theorem B, and show that in general this extension cannot be improved.

Theorem 1. If $m \geq 3$, $n \geq 3$, and T_m is any tree of order m other than $K_{1,m-1}$ $(m \geq 4)$ then

$$r(T_m, K_{2n-1} - (n-2)K_2) = (m-1)n + 1.$$

Proof. By Theorem A, $r(T_m, K_{2n-1} - (n-2)K_2) \geq (m-1)n + 1$. We prove the reverse inequality by induction on n and m. The case for $n = 3$ follows from Theorem B. The case $m = 3$ follows from Theorem C while the case for $m = 4$ is a simple induction on n with the anchor cases of $n = 3$ and $n = 4$ following from Theorems B and D, respectively.

Assume $r(T_m, K_{2n-1} - (n-2)K_2) = (m-1)n + 1$ for a fixed but arbitrary integer $n \geq 3$ and for each $m \geq 3$. We prove $r(T_m, K_{2n+1} - (n-1)K_2) = (m-1)(n+1) + 1$ for every $m \geq 3$. As previously noted, this is true if $m = 3$ and 4. Hence, we assume $r(T_{m'}, K_{2n+1} - (n-1)K_2) = (m'-1)(n+1) + 1$ for all $m \geq m' \geq 3$ for a fixed $m \geq 4$ and show $r(T_{m+1}, K_{2n+1} - (n-1)K_2) = m(n+1) + 1$.

Let $T = T_{m+1}$ be an arbitrary tree (not a star) of order $m + 1$ and assume that a $(T, K_{2n+1} - (n-1)K_2)$-blocking pattern exists. Let v be an end vertex of T and u be the vertex of T adjacent to v, such that $T - v$ is a tree of order m that is not $K_{1,m-1}$. By the induction hypothesis $r(T - v, K_{2n+1} - (n-1)K_2) = (m-1)(n+1) + 1$. Let S denote the set of vertices of $K_{m(n+1)+1}$ that do not belong to the red $T - v$. Since $|S| = mn + 1$ and $r(T, K_{2n-1} - (n-2)K_2) = mn + 1$, we see that $\langle S \rangle \cap B \supseteq K_{2n-1} - (n-2)K_2$.

If any edge joining u to S is red, a red T results, hence u is blue adjacent to S. Now consider $H = K_{2n} - (n-2)K_2$ (formed with the $K_{2n-1} - (n-2)K_2$ contained in $\langle S \rangle \cap B$ and the vertex u) and let $S' = V(K_{m(n+1)+1}) - H$.

Then $|S'| = (m-2)(n+1)+3$ and since $r(T_{m-1}, K_{2n+1}-(n-1)K_2) = (m-2)(n+1)+1$, we see that $T_{m-1} \subseteq \langle S' \rangle \cap R$, where $T_{m-1} \neq K_{1,m-2}$ except if $m = 4$. (We note that since $T_{m+1} (m \geq 5)$ is not a star, it is possible to find vertices u and v such that $T_{m+1} - u - v$ is a tree other than a star, that is $T_{m+1} - u - v = T_{m-1}$.)

Case 1. Suppose u and v (as noted above) are adjacent to distinct vertices x and y in T_{m-1}. If either x or y is blue adjacent to H we are done. So each must be red adjacent to some vertex of H. If they are red adjacent to distinct vertices, a red T_{m+1} results. Thus x and y must be red adjacent to the same vertex of H. Let this vertex be p. If p is not an end vertex of a red edge in the coloring of the vertices of H, then $\langle H \cup \{x\} \rangle \cap B \supseteq K_{2n+1} - (n-1)K_2$. Thus p is an end vertex of an independent red edge in H. But then $\langle (V(H) - p) \cup \{x, y\} \rangle \cap B \supseteq K_{2n+1} - (n-2)K_2 \supseteq K_{2n+1} - (n-1)K_2$.

Case 2. Suppose u and v (as noted above) are adjacent to the same vertex w in T_{m+1}. As above, the remaining vertices contain a blue $K_{2n} - (n-2)K_2$. Call this set of vertices H. The vertex w has at most one red edge to H for otherwise a red T_{m+1} results. Further, this edge is not incident with any vertex $p \in H$, where p is not an end vertex of a red edge in $\langle H \rangle \cap R$. Let $wv \in E(R)$, $v \in H$. Let $H' = (H - \{v\}) \cup \{w\}$. Clearly, $\langle H' \rangle \cap B \supseteq K_{2n} - (n-3)K_2$. Consider $S = V(K_{m(n+1)+1}) - H'$. Since $|S| = (m-2)(n+1) + 3$, it follows from the induction hypothesis that $T_{m-1} \subseteq \langle S \rangle \cap R$. Repeat the above process. As in the argument above, there exists a vertex w' in the red T_{m-1} such that if w' were adjacent to two distinct vertices a and b $(a, b \notin V(T_{m-1}))$ then a red T_{m+1} would result. Thus, either a red T_{m+1} results or we find a set of vertices $H'' = (H' - \{v'\}) \cup \{w'\}$ with $\langle H'' \rangle \cap B \supseteq K_{2n} - (n-4)K_2$. We proceed with this process until a red T_{m+1}, a blue $K_{2n+1} - (n-1)K_2$, or a blue K_{2n} results. If a blue K_{2n} results, we are guaranteed that a red T_{m+1} or a blue $K_{2n+1} - (n-1)K_2$ will result on the next repetition of the process. ■

Theorem 2. If $m \geq 3$, $n \geq 3$, and T_m is any tree of order m, except $K_{1,m-1}$ when $m \geq 4$, then

$$r(T_m, K_{2n} - (n-1)K_2) = (m-1)(n+1) + 1.$$

The proof is analogous to that of the previous theorem and is not included. We note that a similar argument holds if we remove t independent edges from K_n $(0 \leq t \leq [(n-2)/2])$. This is summarized in the following theorem.

Theorem 3. If $m \geq 3$, $n \geq 6$, and T_m is any tree of order m, except $K_{1,m-1}$ when $m \geq 4$, then

$$r(T_m,\ K_n - tK_2) = (m-1)(n-t-1)+1$$

for each t, $0 \le t \le [(n-2)/2]$.

The following construction shows that stars cannot be included in the set of trees in the previous theorems.

Let $G = kC_m(k \ge 1,\ m \ge 5)$. It is clear that $K_{1,3} \not\subseteq G$. If $H = K_n - tK_2$, where

$$n > k\left(2\left[\frac{m}{3}\right] + \left[\frac{m}{3} - \left[\frac{m}{3}\right] + \frac{1}{3}\right]\right)$$

and $t \ge [m/3]k$ then $H \not\subseteq \bar{G}$. It follows that in these cases

$$r(K_{1,3},\ K_n - tK_2) > mk.$$

This bound is an improvement on the bound obtainable from Chvátal's formula for many values of n. For example, the clique number of $K_{25} - 11K_2$ is 14. From Theorem A we see that $r(K_{1,3},\ K_{25} - 11K_2) \ge 40$, but by considering $m = 5$ and $k = 8$ in the above construction, we may conclude that $r(K_{1,3},\ K_{25} - 11K_2) > 40$. As a last example, the clique number of $K_{34} - 16K_2$ is 18. Theorem A gives us that $r(K_{1,3}, K_{34} - 16K_2) \ge 52$, but by considering $m = 7$ and $k = 8$ in the above construction, we may conclude that $r(K_{1,3}, K_{34} - 16K_2) > 56$.

We now note another extension of Theorem B. The proof when $T_m = K_{1,m-1}$ follows standard inductive techniques.

Corollary 4. If $m \ge 4$, $n \ge 5$, and T_m is any tree of order m, then

$$r(T_m,\ K_{n+2} - 2K_2) = (m-1)(n-1)+1.$$

Theorem 5. If G is a graph of order $n+2$ $(n \ge 5)$ and clique number n and $m \ge 4$, then

$$r(T_m,\ G) = (m-1)(n-1)+1.$$

Proof. From Theorem E (with $l = n-1$, $t = 3$, and $m \ge 4$) we see that $r(T_m,\ K_{n+2} - K_3) = (m-1)(n-1)+1$, when $n \ge 5$. Theorem 3 and its corollary imply $r(T_m,\ K_{n+2} - 2K_2) = (m-1)(n-1)+1$ for $m \ge 4$ and $n \ge 6$ (while $n = 5$ can be shown by a simple induction on m). Now if G is a graph of order $n+2$ and clique number n then $G \subseteq K_{n+2} - 2K_2$ or $G \subseteq K_{n+2} - K_3$. Theorem A shows that $r(T_m,\ G) \ge (m-1)(n-1)+1$ and the result follows. ■

Corollary 6. If F is a forest with all components of order at least four and G_n is any graph of order $n+2$ with clique number $n (n \geq 5)$ then

$$r(F, G_n) = \max_{1 \leq j \leq c(F)} \left\{ (j-1)(n-2) + \sum_{i=j}^{c(F)} ik_i \right\}.$$

Proof. Theorem 5 implies that $T_m \in \mathcal{G}(G_n)$ when $m \geq 4$ and $n \geq 5$. Since $\chi(G_n) = n$ we may invoke Theorem F. ■

It is natural to ask what other graphs G with clique number n follow this pattern? Although a complete solution to this question has eluded us, we offer the following observations.

Let V_n be the graph obtained by joining a single vertex of K_n to $n-1$ other distinct independent vertices. Let W_n be the graph obtained by joining with an edge each of $n-1$ distinct independent vertices to distinct vertices in a copy of K_n. Let X_n denote the graph obtained by identifying one end vertex of the path P_n with a vertex of K_n. Let Y_n denote the graph obtained by identifying each of the end vertices of the path P_n with a distinct vertex of K_n. Let Z_n denote the graph obtained by identifying both end vertices of P_n with a single vertex of K_n.

Theorem 7. If T_m is a tree of order $m \geq 3$, then $r(T_m, G) = (m-1)(n-1) + 1$, where $G \in \{V_n, W_n, X_n, Y_n \mid n \geq 3\} \cup \{Z_n \mid n \geq 4\}$.

The proof for any of the classes above follows closely that of Theorem B.

We now show there are some limitations to the graphs with clique number n that follow Chvátal's formula.

Proposition 8. If $n \geq 2$ then $r(T_m, K_{2n} - nK_2) \geq (m-1)(n-1) + 2$.

Proof. The blocking pattern for $K_{(m-1)(n-1)} + 1$ comes from $R = (n-1)K_{m-1} \cup \{p\}$ for some distinct vertex p. ■

Theorem 9. If $m \geq 3$ and $n \geq 2$, $r(P_m, K_{2n} - nK_2) = (m-1)(n-1) + 2$.

Proof. As a consequence of a result in [4], $r(P_m, C_4) = m + [4/2] - 1 = m + 1 (m \geq 4)$. Further, by Theorem C, since $K_{2n} - nK_2$ has a 1-factor, $r(P_3, K_{2n} - nK_2) = 2n$. So we assume $r(P_m, K_{2(n-1)} - (n-1)K_2) = (m-1)(n-2) + 2$ for a fixed but arbitrary $n \geq 3$ and each $m \geq 3$. We wish to show $r(P_m, K_{2n} - nK_2) = (m-1)(n-1) + 2$ by induction on m. So suppose there exists a $(P_{m+1}, K_{2n} - nK_2)$-blocking pattern for $K_{m(n-1)+2}$. By the induction hypothesis $P_m \subseteq R$. Further, in the remaining $m(n-2)+2$ vertices $K_{2(n-1)} - (n-1)K_2 \subseteq B$. But then by examining the endpoints of the red P_m we see that either $P_{m+1} \subseteq R$ or $K_{2n} - nK_2 \subseteq B$. Equality follows from Proposition 8. ■

We note that a similar argument shows (see [7])

$$r(P_m, K_{2n-1} - (n-1)K_2) = (m-1)(n-1) + 1.$$

Burr, Faudree, Rousseau, and Schelp [private communication] have shown that

$$r(K_{1,m-1}, K_{1,s_2,s_3,\ldots,s_k}) = (k-1)(r(K_{1,m-1}, K_{1,s_2}) - 1) + 1,$$

when $s_2 \leq s_3 \leq \cdots \leq s_k$ and m is sufficiently large. In particular, this says for m large and even that $r(K_{1,m-1}, K_{2n-1} - (n-1)K_2) = m(n-1) + 1$. Thus, in this case, the Chvátal formula does not hold.

Finally, we show that Theorem 5 is the best extension possible of Theorem B.

Lemma 10. If $G = K_{j+5t} - tC_5$ then G has clique number $j + 2t$ and $\chi(G) = j + 3t$, thus for any connected graph H,

$$r(G, H) \geq (|V(H)| - 1)(j + 3t - 1) + 1.$$

Observe that if $G = K_{n+3} - C_5$ then G has order $n + 3$ and clique number n (≥ 4) and by Lemma 10 we see $r(T_m, K_{n+3} - C_5) \geq (m-1)n + 1$. This observation shows that if G is a graph of order $n + 3$ with clique number n, then it is not necessarily the case that $r(T_m, G) = (m-1)(n-1) + 1$.

Theorem 10. For $m, n \geq 4$, $r(T_m, K_{n+3} - C_5) = (m-1)n + 1$.

Proof. By Theorem 3, $(m-1)n + 1 = (T_m, K_{n+3} - 2K_2) \geq r(T_m, K_{n+3} - C_5)$. The Theorem follows from the observation above. ■

Thereom 11. For $m, n \geq 4$, $r(T_m, K_{n+5t} - tC_5) = (m-1)(n + 3t - 1) + 1$.

Proof. Lemma 10 implies that $r(T_m, K_{n+5t} - tC_5) \geq (m-1)(n + 3t - 1) + 1$. Since $K_{n+5t} - tC_5 \subseteq K_{n+5t} - 2tK_2$ it is clear that $r(T_m, K_{n+5t} - tC_5) \leq r(T_m, K_{n+5t} - 2tK_2) = (m-1)(n + 3t - 1) + 1$. ■

CONCLUSION

The possible directions are numerous. Some interesting problems which occur to us include the following:

(1) Determine all values of m, n, and t such that

$$r(K_{1,m}, K_n - tK_2) = m(n - t - 1) + 1.$$

(2) Determine for other values of m, n, and t

$$r(K_{1,m}, K_n - tK_2).$$

(3) Can similar formulas be found for the removal of multiple copies of other graphs? In particular other complete graphs or paths.

(4) Ultimately, determine all graphs G and trees T_m such that

$$r(T_m, G) = (m - 1)(\chi(G) - 1) + 1.$$

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