Minimum degree and even cycle lengths

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Abstract

A classic result of Dirac states that if G is a 2-connected graph of order n with minimum degree $\delta \geq 3$, then G contains a cycle of length at least $\min\{n, 2\delta\}$. In this paper, we consider the problem of determining the number of different odd or even cycle lengths that must exist under the minimum degree condition. We conjecture that there are $\delta - 1$ even cycles of different lengths, and when G is nonbipartite, that there are $\delta - 1$ odd cycles of different lengths. We prove this conjecture when $\delta = 3$. Related results concerning the number of different even cycle lengths supporting the conjecture are also included. In particular, we show that there are always at least $(\delta - 1)/2$ even cycles of different lengths.

1 Introduction

Results regarding cycles in graphs abound in the literature. Many of these results are concerned with Hamiltonian cycles, many with disjoint cycles and recently, more of a focus has turned to cycles with specified lengths. In this article we present a new conjecture and some supporting results regarding the number of even cycles of different lengths in a graph of order n having minimum degree $\delta(G) \geq d$ for an integer $d \geq 3$. Our work extends the following classic result of Dirac [2].

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Theorem 1 ([2])**.** *If G is a 2-connected graph of order $n \geq 4$ with $\delta(G) \geq d \geq 3$, then there exists a cycle in G of length at least* $\min\{n, 2d\}$.

We propose the following conjecture, which would be a direct extension of the result of Theorem 1 for graphs with at least $2d$ vertices.

Conjecture 1. *If G is a 2-connected graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$, then G contains at least $d-1$ even cycles of different lengths. Furthermore, if G is also nonbipartite, then G contains at least $d-1$ odd cycles of different lenths.*

The following theorem has been mentioned in several papers.

Theorem 2 ([3, 4, 5, 6])**.** *If G is a 2-connected graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$, then G contains a cycle of even length at least $2d$, and if G is also nonbipartite, then G contains a cycle of odd length at least $2d-1$.*

The result above was mentioned in a series of papers by H. J. Voss ([4], [5], and [6]) as well as a joint paper by H. J. Voss and C. Zuluaga ([3]). However, it has been difficult to find a proof of this result in these papers, and this result does not seem to be widely known. Thus, for completeness of our presentation, we include a proof of the even cycle case of Theorem 2.

We also provide supporting evidence for Conjecture 1. In particular, we show that a 2-connected graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$ will contain at least $\frac{d-1}{2}$ cycles of different even lengths.

Theorem 3. *If G is a 2-connected graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$, then G contains at least $(d-1)/2$ even cycles of different lengths.*

We also prove Conjecture 1 when $d = 3$ in the following results.

Theorem 4. *If G is a 2-connected graph of order $n \geq 6$ with $\delta(G) \geq 3$, then G contains at least two even cycles of different lengths.*

Theorem 5. *If G is a 2-connected nonbipartite graph of order $n \geq 6$ with $\delta(G) \geq 3$, then G contains at least two odd cycles of different lengths.*

Our notation generally follows the notation of Chartrand, Lesniak and Zhang in [1]. Let $(u, v)_P$ denote the vertices of the path P strictly between the vertices u and v. We use similar notation for cycles. Denote by P^- the path P traversed in the opposite direction of its definition. Other necessary terms and notation will be defined as needed.

2 Results

In this section we will prove our main results. We begin with a useful lemma.

Lemma 1. *If G is a graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$ which contains a Hamiltonian cycle, then G contains an even cycle of length at least $2d$.*

Proof. We may assume that n is odd, since otherwise any Hamiltonian cycle would give the desired even cycle length. Thus, $n \geq 2d+1$. Over all Hamiltonian cycles, choose one with the shortest possible chord, say $C : x_1, x_2, \ldots, x_n, x_1$. Let this shortest chord be xy and assume without loss of generality that $x = x_1$ and $y = x_t$. This chord determines two other cycles, a short cycle $C^* : x_1, x_2, \ldots, x_t, x_1$ and $C' : x_1, x_t, x_{t+1}, \ldots, x_n, x_1$. Let P denote the subpath $x_2, x_3, \ldots, x_{t-1}$ with $|P| = t-2$ and let P^1 denote the subpath of C from x_{t+1} to x_n with $|P^1| = s$. Note, $s + t = n$.

Case 1: Suppose t is odd (hence $t-2$ is odd).

Then $|C'|$ is even. If $t = 3$, then C' is the desired cycle, thus we assume that $t \geq 3$.

Consider any pair of vertices $x_i, x_{i+1} \in C'$. We claim that x_2 and x_{t-1} can collectively have at most two edges to $\{x_i, x_{i+1}\}$. Suppose this does not hold and first consider the case when $x_2x_i, x_{t-1}x_i, x_{t-1}x_{i+1} \in E(G)$. This gives the Hamiltonian cycle

$$x_{i+1}, x_{i+2}, \ldots, x_1, x_t, x_{t+1}, \ldots, x_i, x_2, \ldots, x_{t-1}, x_{i+1}.$$

However, the edge $x_{t-1}x_i$ is a shorter chord of this graph than x_1x_t is of C, a contradiction to our choice of C with shortest chord. Next, consider the case when $x_2x_{i+1}, x_{t-1}x_i, x_{t-1}x_{i+1} \in E(G)$. This gives the Hamiltonian cycle

$$x_{i+1}, x_{i+2}, \ldots, x_1, x_t, \ldots, x_i, x_{t-1}, x_{t-2} \ldots, x_2, x_{i+1}.$$

In this case the edge $x_{t-1}x_{i+1}$ is a shorter chord of this cycle than x_1x_t is of C, again a contradiction. By symmetry the other cases follow; thus, $\{x_i, x_{i+1}\}$ has at most two edges to $\{x_2, x_{t-1}\}$.

Since x_1x_t is a shortest chord, the vertices x_2 and x_{t-1} have at least $d-1$ adjacencies in C'. But as each consecutive pair of vertices in C' can receive at most two edges from $\{x_2, x_{t-1}\}$, this implies that C' has precisely $2(d-1)$ vertices; otherwise it would be the desired even cycle.

If $d = 3$, then $|C'| = 2(3-1) = 4$ and then C^* must be a 3-cycle, contradicting the fact that $t \geq 3$.

Now consider the vertices $A = \{x_{n-1}, x_n, x_t, x_{t+1}\}$. Each vertex in A has at most one edge to the pair $\{x_2, x_{t-1}\}$ or a shorter chord would exist. Furthermore, x_{n-1} and x_{t+2} have no edges to x_2 or x_{t-1} since otherwise either we have a shorter chord or, for example,

$$x_{t-1}, x_{n-1}, x_{n-2}, \ldots, x_t, x_1, x_2, \ldots, x_{t-1}$$

would be the desired long even cycle. Now

$$|C' - \{x_{n-1}, x_n, x_1, x_t, x_{t+1}, x_{t+2}\}| = 2(d-4),$$

while these $2(d-4)$ vertices must receive a total of $2(d-1)-4 = 2(d-3)$ edges from $\{x_2, x_{t-1}\}$, or more than two edges per consecutive pair, a contradiction. This completes Case 1.

Case 2: Suppose t is even.

Since t is even, $s-2 = |C'|$ is odd, so s is odd. First consider the subcase when $s \geq 2d-1$. It follows that, x_2 cannot be adjacent to consecutive vertices of C', or we could insert x_2 into C' to obtain an even cycle with at least $2d$ vertices. Let $\{z_1, z_2, \ldots, z_m\}$ be the vertices of C' adjacent to x_2, ordered using the reverse of the orientation of the cycle C' (Recall $m \geq d-1$). The path segment P^i between (but not including) any pair of vertices z_i and z_{i+1} (with indices taken modulo m) will be called even or odd depending on its cardinality. Since $|C'|$ is odd, there will be an odd number p of even paths, and C' will have a least $3p+2(m-p) = 2m+p \geq 2d-2+p$ vertices. If $p \geq 3$, then replacing one of the even path segments, say P^j, of C' with the path segment $z_j x_2 z_{j+1}$ will result in an even cycle with at least

$$3(p-1) + 2(m-(p-1)) = 2m + p - 1 \geq 2(d-1) + 2 = 2d$$

vertices, and we would have the desired cycle. Thus, we can assume that there is only one even path, say P^j. If $m \geq d$ or if one of the odd paths has strictly more than one vertex, then replacing the even path P^j with the path segment $z_j x_2 z_{j+1}$ will again result in an even cycle with at least $2d$ vertices, as desired.

Hence, we can assume that there are exactly $d-1$ paths, and only one of these is even (say P^j), and the all of the odd paths have exactly one vertex. Observe that x_{t-1} cannot be adjacent to any of the vertices $\{z_1, z_2, \ldots, z_{d-1}\}$, since then there is an odd path P^i that can be replaced in C' by $P : x_2, x_3, \ldots, x_{t-1}$, which is even. This would again create an even cycle of length at least $2d$.

Also, x_{t-1} cannot be adjacent to any of the vertices $z_j+1, z_j+3, \ldots$ in the even path P^j, since then the path P can replace a path of odd parity from $z_j+(2k-1)$ to z_{j+1} for the appropriate k, again resulting in the desired cycle. For the same reason, x_{t-1} cannot be adjacent to $z_{j+1}-1, z_{j+1}-3, \ldots$ in P^j. This implies that x_{t-1} is adjacent to at most $d-2$ vertices of C', a contradiction. Thus, we can assume that C' has at most $2d-3$ vertices and $|P| \geq 4$.

Consider x_{t-2}. Now, if $x_2 x_i \in E(G)$, then both $x_{t-2}x_{i+1}$ and $x_{t-2}x_{i-1}$ are not in $E(G)$, since this would yield an even cycle of length $n-1 \geq 2d$.

Thus, each adjacency of x_2 forces both a successor and a predecessor that is not adjacent to x_{t-2}. The vertex x_2 cannot be adjacent to alternating vertices on C', since $d_{C'}(x_2) \geq d-1$ and $|C'| \leq 2d-3$. Thus, the number of distinct successors and predecessors of the adjacencies of x_2 on C' will be at least d, and so x_{t-2} has at most $d-3$ adjacencies on C', which implies $d(x_{t-2}) \leq 2+d-3 < d$, a contradiction. This completes the proof of Lemma 1. $\square$

Let us recall the statement of Theorem 2.

Theorem 2 *If G is a 2-connected graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$, then G contains a cycle of even length at least $2d$, and if G is also nonbipartite, then G contains a cycle of odd length at least $2d-1$.*

We include a proof of the even case.

Proof. Let C be a cycle of maximum length in G. By Theorem 1, C has at least $2d$ vertices. Let P be a path of maximum length from C that is vertex disjoint from C except for the one common vertex, which we will denote by r. Let x denote the other end-vertex of P, and y the vertex of P adjacent to the vertex r of C.

Case 1: Assume that all of the adjacencies of x are on the path P (see Figure 1).

We can assume that P is chosen such that the end-vertex x has the longest chord on the path P. Since $d_P(x) \geq d$, the path P has length at least d. Since G is 2-connected there is a path Q from P to C whose interior vertices are disjoint from $C \cup P$. Let s be the initial vertex of Q on P. By the choice of x, s can be chosen to be closer to x than some of the adjacencies of x on P. Also let t be the final vertex of Q which is on C (see Figure 1). Thus, there are 3 paths P_1, P_2, P_3 from r to t. The first path P_1 contains x, all of the adjacencies of x on P, the vertex s, and all of the vertices of Q. The second path P_2 will contain all of the vertices of C between r and t, and the third path P_3 will contain all of the vertices of C between t and r (for some fixed orientation of the cycle C). The path P_1 has length at least d, since all of the d adjacencies of x are in P_1. This implies that each of P_2 and P_3 have length at least d, since their length must be at least as large as P_1, for otherwise P_1 could replace P_3 or P_2 on C to get a longer cycle. At least two of the three paths P_1, P_2, P_3 must have the same parity, and this results in a cycle of even length at least $2d$. This completes the proof of Case 1.

Case 2: Assume that x has $k < d$ adjacencies on P.

Note that since x has an adjacency on C, it follows that y is also the end-vertex of a longest path off C. Assume that both x and y each have at least $d-k$ adjacencies on C, and that P has at least $k+1$ vertices. Let $X = \{x_1, x_2, \cdots, x_{d-k}\}$ be $d-k$ of the adjacencies of x on C and

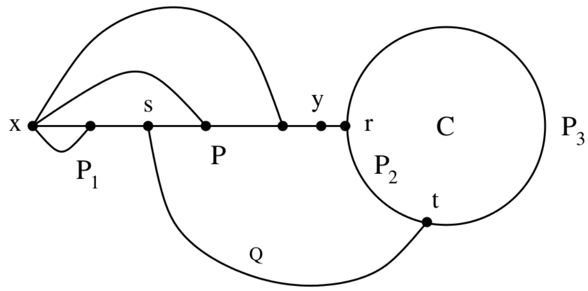

Figure 1: The longest chord from x to P.

$Y = \{y_1, y_2, \cdots, y_{d-k}\}$ be a corresponding set of $d-k$ adjacencies of y on C. Let $Z = X \cup Y$, and so $Z = \{z_1, z_2, \cdots, z_t\}$ with $t \geq d-k$. Since C is a cycle of maximum length, there is at least one vertex on C between z_i and z_{i+1}. Additionally, whenever two vertices $u \neq v$ are selected with $u \in X$ and $v \in Y$, it follows that there are at least $k+1$ vertices of C in a path between u and v along C. We refer to such an interval as a $(k+1)$-gap.

First suppose $|Z| = 1$ so $Z = \{z_1 = r\}$. Since G is 2-connected, there is a path Q from a vertex $q \in P$ to a vertex $c \in C$ that is internally disjoint from $P \cup C$ and vertex disjoint from z_1. Let x_q be the neighbor of x on P between q and y that is closest to q if such a vertex exists. Since x has $d-1$ edges to P, at least one of the paths z_1yPx_qxPqQc, z_1yPqQc or z_1xPqQc is a path of length at least d from z_1 to c. This path, along with the two paths in C provides three internally disjoint paths from z_1 to c, each of length at least d. We can then easily find an even cycle of length at least $2d$ as desired. Thus, we may assume $|Z| > 1$.

Note that there are at least two $(k+1)$-gaps. Since each $(k+1)$-gap of C can be replaced by P and each other gap between vertices of Z can be replaced by either x or y to yield another cycle of length at least $2d$, we see that the length of each $(k+1)$-gap must have the same parity as the length of P and all other gaps must have even length. If the number of $(k+1)$-gaps is even, then C is even as desired. Also if the length of P is even, then C is again even. Thus we may assume there are an odd number of $(k+1)$-gaps and each has odd length.

Since there are at least two $(k+1)$-gaps and the number must be odd, there are actually at least three $(k+1)$-gaps. With an odd number of $(k+1)$-gaps, there must be a $(k+1)$-gap with one end in $X \cap Y$. Suppose, without loss of generality, that $z_i \in X \cap Y$ and $z_{i+1} \in Y$. We may then replace this gap with the vertex y to create a new cycle of length at least $2d$ which has even length. $\square$

Now we give several results supporting our conjecture. Let us recall the statement of Theorem 3.

Theorem 3 *If G is a 2-connected graph of order $n \geq 2d$ with $\delta(G) \geq d \geq 3$, then G contains at least $(d-1)/2$ even cycles of different lengths.*

Proof. Let G be as given and let P be a longest path in G. Let v be an end-vertex of P and let $v_1, v_2, \ldots, v_{\delta-1}$ be the neighbors of v in order along P away from v arising from chords. Each vertex v_i produces a cycle of the form vPv_iv. If at least $\frac{\delta-1}{2}$ of these cycles are even, they are certainly all different lengths so we have the desired result. Suppose at least $\frac{\delta+1}{2}$ of the cycles are odd, and let i be the smallest index such that vPv_iv is an odd cycle.

Now for $j > i$, consider cycles of the form vv_iPv_jv. The lengths of these cycles are all different and since there were at least $\frac{\delta+1}{2}$ odd cycles before, there are at least $\frac{\delta-1}{2}$ even cycles of this form, completing the proof of Theorem 3. $\square$

Now we prove Conjecture 1 in the case $\delta = 3$. First, we consider the case even cycles case.

Theorem 4 *If G is a 2-connected graph of order $n \geq 6$ with $\delta(G) \geq 3$, then G contains at least two even cycles of different lengths.*

Proof. Let P be a longest path in G with the property that an end-vertex x has its third neighbor z as close to x along P as possible. Let y be the second neighbor of x on P and x_1 the first neighbor of x on P. If P is a path with initial vertex x and y is a vertex of P distinct from x, then the vertex immediately preceding y on P is denoted by y^-. Note that y^- and z^- are also end-vertices of a longest path. By our choice of x, the vertex y^- has no adjacencies (other then y and y^{--}) in the interval $[x, z^-]$. This implies that the third adjacency of y^- on P is at z or beyond z along P. Call this adjacency w.

Note that it is possible that $z^- = y$. In this case G has girth 3. But now note that the cycles x, P, y, x and xP, z, x differ in length by exactly 1. Further, the cycles x, P, y^-, w, P^-, y, x and x, P, y^-, w, P^-, z, x also differ in length by exactly 1 and these last two cycles are longer than the first two, or the 4-cycle x, y^-, y, z, x results and we would be done by Theorem 1. If y^- is adjacent to z^-, then x, y, y^-, z^-, x is a 4-cycle and by Theorem 1 we are done. Thus, we may assume that z^- is not y and z is not w.

We denote the order of the interval (x, y^-) as a, that of (y, z^-) as b and that of (z, w) as c. We can easily find cycles of lengths $a+3$, $b+4$, $a+5$, $a+b+5$, $b+c+5$, $a+c+4$, and $a+b+6$. We wish to show that this collection of cycles always contains two even cycles of different lengths.

Case 1. Assume that a is odd.

Then $a+3$ is even. This implies that c is even; otherwise $a+c+4$ would be even and different than $a+3$. But now b must be odd; otherwise $a+b+5$ is even and again different than $a+3$. Now $a+b+c+6$ is even and strictly greater than $a+3$. This completes Case 1.

Case 2: Assume that a is even.

If b is even, then $b+4$ is even. Note, if $b=0$, then a 4-cycle results and we are done by applying Theorem 1. Thus, we may assume $b \neq 0$. But b even implies that c is even; otherwise $b+c+5$ would be even and greater than $b+4$. Now a, b and c are all even; hence $a+b+c+6$ is even and greater than $b+4$.

Hence we may assume b is odd. Now $a+b+5$ is even. This implies that c is even; otherwise $a+b+c+6$ is even and greater than $a+b+5$. Thus, we have that a is even, b is odd and c is even. This implies that $a+b+5 = a+c+4 = b+c+5$ or we would have two different even lengths for cycles. This implies that $a = c = b+1$.

Claim: The vertex z^- has no adjacency in $[x, z^{--})$.

First assume that z^- has an adjacency, say p, in $[y, z^-)$. Let α be the order of the open interval (p, z^-). If α is even, then $x, \ldots y^-, y, \ldots, p, z^-, z, x$ is a cycle of length $a+b-\alpha+5$, which is even and different than $a+b+5$. If α is odd, then $x, y, \ldots, z^-, z, x$ is an even cycle of length $b-\alpha+4$ and is of different length than $a+b+5$.

Next suppose that z^- has an adjacency $p \in (x, y^-)$. Let the order of (p, y^-) be α. If α is odd, since a is even, $a-\alpha$ is odd. But then $x, z, z^-, p, \ldots, x$ is of order $a-\alpha+3$, which is even and less then $a+b+5$. If instead α is even, then $x, \ldots, p, z^-, \ldots, y, x$ has order $a+b+3-\alpha$, which is even and less than $a+b+5$.

Hence, the other adjacency of z^- must be beyond z along P. Let p be this adjacency. Now let α be the order of (z, p). We first assume that p precedes w (the neighbor of y^-) along P. Now if α is even, then $c-\alpha$ is even and the cycle $w, \ldots, p, z^-, z, x, y, y^-, w$ has length $c-\alpha+6$, which is even. Also, the cycle $p, \ldots, z, x, \ldots, y^-, y, \ldots, z^-, p$ has length $6+a+b+\alpha$, which is even and different than $c-\alpha+6$ since $a=c=b+1$ and $b>0$.

If, on the other hand, α is odd, then the cycle $x, \ldots, y^-, y, \ldots, z^-$, $p, \ldots, z, x$ has length $a+b+5+\alpha+1$, which is even and greater then $a+b+5$.

Finally we assume that p follows w along P. We let α be the order of the interval (w, p). Now if α is even, then the cycle $x, z, \ldots, p, z^-, \ldots, x$ has order $a+b+c+\alpha+7$, which is even and longer than $a+b+5$. If instead, α is odd, then $x, y, \ldots, z^-, p, \ldots, z, x$ has order $b+c+5+\alpha+1$, which is even and greater than $b+c+5$.

Thus, in all cases we obtain two even cycles of different lengths. This completes Case 2 and the proof of the Theorem. $\square$

Theorem 5 *If G is a 2-connected nonbipartite graph of order $n \geq 6$ with $\delta(G) \geq 3$, then G contains at least two odd cycles of different lengths.*

Proof. **Case 1:** The girth of G is 3.

Let a, b, c be the vertices of a triangle T in G. Since G is 2-connected, from any point $x \notin T$ there are vertex disjoint paths, to say, a and b of T. If either of the paths from x to a or b has length at least 2, then clearly that is an odd cycle with at least five vertices. Thus, x is adjacent to both a and b. Thus, each vertex in $G - T$ is adjacent to two of the vertices of T. Since $\delta(G) \geq 3$, there must be a $y \notin T$ that is adjacent to c and with no loss of generality also to a. This implies there is a cycle of length 5, which completes the proof of this case.

Case 2 The odd girth of G is at least 5.

Consider the smallest odd cycle, say C. There can be no chords in C, since any such chord would give an odd cycle of smaller length. If there is a path P that connects two consecutive vertices of C, then the number of vertices in P must be odd, for otherwise this would give a longer odd cycle than C. If there was another path Q with the same property relative to another pair of consecutive vertices of C, then the number of vertices in Q must also be odd. However, using both the paths P and Q along with C will give a longer odd cycle. Thus, there is at most one path with the same property as P.

Select a shortest path-chord P_1 in C, and assume that v_1 and v_3 are the end-vertices of P_1 and that P_1 has x interior vertices. Let v_2 be the predecessor of v_3 on C, and select the smallest odd path-chord P_2 starting with v_2. Let v_4 be the end-vertex of P_2 and y the number of interior vertices of P_2. Also, denote by a the number of vertices of C between v_1 and v_2, by b the number of vertices of C between v_3 and v_4, and by c the number of vertices of C between v_4 and v_1 on C. Thus, $a+b+c+4$ is odd, since it is the length of the cycle C. Thus, either all of a, b, c are odd, or exactly one of a, b, c is odd.

Using the path-chords and paths of C between v_1, v_2, v_3, v_4 there are seven different cycles of G, and they have the following lengths.

(1) $a+b+c+4$,	(2) $a+x+3$,	(3) $b+y+3$,
(4) $x+y+c+4$,	(5) $x+b+c+3$,	(6) $y+a+c+3$,
(7) $x+y+a+b+4$.		

In the remainder of the proof, various subcases where a, b, c, x, y are odd or even are considered.
Case A: Suppose all of a, b, c are odd.

Subcase A1: Suppose both x and y are odd.

This implies that the cycles $(1), (2), (3), (4)$ are all odd cycles, and thus must all have the same length or we done. Cycles (1) and (2) being even imply $x = b+c+1$, and likewise (1) and (3) imply $y = a+c+1$. Substituting for x and y in (4) and setting that equal to (1) gives $a+b+2c+2+c+4 = a+b+c+4$, a contradiction.

Subcase A2: Suppose both x and y are even.

This implies that the cycles $(1), (4), (5), (6)$ are all odd cycles. Equations (1) and (4) imply $x+y = a+b$, (1) and (5) imply $x = a+1$, and (1) and (6) imply $y = b+1$. The last two equations imply $x+y = a+b+2$, a contradiction.

Subcase A3: Suppose x is odd and y is even. (There is the symmetric case x is even and y is odd.)

This implies that the cycles $(1), (2), (6), (7)$ are all odd cycles. Cycles (1) and (2) imply $x = b+c+1$, and (1) and (6) imply $y = b+1$. Substituting for x and y in (7) and setting that equal to (1) gives $2b+c+2+a+b+4 = a+b+c+4$, a contradiction.

Case B: Suppose c is odd, and a, b are even.

Subcase B1: Suppose x and y are odd.

This implies $(1), (4), (5), (6)$ are all odd cycles. Equations (4) and (5) imply $y = b-1$, and substituting for y in (6) and setting it equal to (1) gives $b-1+a+c+3 = a+b+c+4$, a contradiction.

Subcase B2: Suppose x and y are even.

This implies $(1), (2), (3), (4)$ are all odd, and this was shown to lead to a contradiction in subcase A1.

Subcase B3: Suppose x is even and y is odd. (There is the symmetric case when x is odd and y is even.)

This implies $(1), (2), (6), (7)$ are all odd cycles. It was shown in subcase A3 that this leads to a contradiction.

Case C: Suppose a is odd, and b, c are even. (There is the symmetric case when b is odd, and a, c are even.)

Subcase C1: Suppose x and y are both odd.

This implies $(1), (2), (6), (7)$ are all odd cycles, and this was shown to imply a contradiction in subcase A3.

Subcase C2: Suppose x and y are both odd.

This implies $(1), (2), (5), (7)$ are all odd cycles. Moreover, (1) and (3) imply $y = a+c+1$, and (1) and (5) imply $x = a+1$. Substituting for x and y in (7) and setting this equal to (1) implies that $a+1+a+c+1+a+b+4 = a+b+c+4$, a contradiction.

Subcase C3: Suppose x is odd and y is even.

This implies $(1), (2), (3), (4)$ are all odd, and this was shown to lead to a contradiction in subcase A1.

Subcase C4: Suppose x is even and y is odd.

This implies $(1), (4), (5), (6)$ are all odd cycles, and this was shown to lead to a contradiction in subcase B1.

This completes the proof. $\square$

Dedication: The authors RG, MJ and CM would like to dedicate this paper to the memory of our friend, Ralph Faudree, who died suddenly during the completion of this project, one of the last projects of his illustrious career.

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