# DISJOINT CHORDED CYCLES OF THE SAME LENGTH[*]

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**Abstract.** Bollobás and Thomason showed that a multigraph of order $n$ and size at least $n + c \, (c \geq 1)$ contains a cycle of length at most $2(\lfloor n/c \rfloor + 1)\lfloor \log_2 2c \rfloor$. We show in this paper that a multigraph (with no loop) of order $n$ and minimum degree at least 5 contains a chorded cycle (a cycle with a chord) of length at most $300 \log_2 n$. As an application of this result, we show that a graph of sufficiently large order with minimum degree at least $3k + 8$ contains $k$ vertex-disjoint chorded cycles of the same length, which is analogous to Verstraëte's result: A graph of sufficiently large order with minimum degree at least $2k$ contains $k$ vertex-disjoint cycles of the same length.

**1. Introduction.** Hereinafter, the term "graph" means "simple graph," that is, a graph with no parallel edges and loops. A multigraph, on the other hand, may have parallel edges but no loops. Considering more than one vertex-disjoint cycle, Corrádi and Hajnal [3] showed that a graph with order at least $3k$ and minimum degree at least $2k$ contains $k$ vertex-disjoint cycles. Besides asking for the cycles to be vertex-disjoint, also requiring the cycles to be of the same length causes the problem to become much more intriguing. Häggkvist [7] conjectured that a graph of sufficiently large order and minimum degree at least 4 contains 2 vertex-disjoint cycles of the same length, and Thomassen [9] further speculated that the same result may hold for $k$ vertex-disjoint cycles if one replaces the minimum degree condition in Häggkvist's conjecture by $2k$. Thomassen's conjecture was confirmed by Egawa [5] for $k \geq 3$. Recently, Verstraëte [10] gave a simpler proof for Thomassen's conjecture for all $k \geq 2$. (When $k \geq 3$, Verstraëte's result is weaker than Egawa's in terms of the lower bound of the requested number of vertices.) We state Verstraëte's result below.

THEOREM 1.1. *Let $k$ be a natural number. Then there exists a positive integer $n_k$ such that if $G$ is a graph of order at least $n_k$ and minimum degree at least $2k$, then $G$ contains $k$ vertex-disjoint cycles of the same length.*

An edge joining two vertices of a cycle that is not an edge of the cycle is called a chord, and a cycle with a chord is called a chorded cycle. For a given nonnegative integer $s$, if a cycle contains $s$ chord(s), then we call the cycle an $s$-*chorded* cycle. Corresponding to studies of cycles, what can we say about chorded cycles? It was asked by Pósa [8] to find conditions implying a graph contains a chorded cycle. By considering a longest path, one can immediately see that minimum degree at least

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3 assures the existence of a chorded cycle. In 2008, as a generalization of Corrádi and Hajnal's result, Finkel [6] showed that a graph on at least $4k$ vertices and with minimum degree at least $3k$ contains $k$ vertex-disjoint chorded cycles. Combining Theorem 1.1 and Finkel's result, we show in this paper the following.

THEOREM 1.2. *For every natural number $k$, there exist positive integers $n_k$ and $n'_k$ such that for every graph $G$ with order $n$ and minimum degree at least $3k + 8$, the following hold:* (i) *if $n \geq n_k$, then $G$ contains $k$ vertex-disjoint chorded cycles of the same length; and* (ii) *if $n \geq n'_k$, then $G$ contains $k$ vertex-disjoint isomorphic chorded cycles of the same length.*

In fact we believe that the minimum degree condition $3k + 8$ in Theorem 1.2 can be replaced by $3k$, which is best possible by considering the complete bipartite graph $K_{3k-1,n-3k+1}$. We propose this as a conjecture.

CONJECTURE 1. *Let $k$ be a natural number. Then there exists a positive integer $n_k$ such that if $G$ is a graph of order at least $n_k$ and minimum degree at least $3k$, then $G$ contains $k$ vertex-disjoint (isomorphic) chorded cycles of the same length.*

The proof of Theorem 1.2 is essentially an application of the result below.

THEOREM 1.3. *Let $G$ be a multigraph of order $n$ and minimum degree at least $5$. Then $G$ contains a chorded cycle of length at most $300 \log_2 n$.*

About Theorem 1.3, we have the following remarks:

- Minimum degree 5 is best possible. For example, for any real number $\varepsilon > 0$, we consider the multigraph obtained from a cycle of length $n$ with double edges by adding $n^{1-\varepsilon}$ edges such that there are no chorded cycles of length larger than $300 \log_2 n$. (For instance, label the vertices on the cycle as $1, 2, \ldots, n$, and then add each edge connecting the vertices of labels $sn^\varepsilon$ and $tn^\varepsilon$, where $s$ and $t$ are positive integers with $s + t = n^{1-\varepsilon}+1$.)
- The factor 300 is not optimal; we make no attempt to optimize this constant. However, the $\log n$ term is best possible, since when $r \geq 10$, there are infinitely many $(r + 1)$-regular graphs of order $n$ and girth larger than $\log_r n$, as shown by Dahan [4].

In asking about the existence of a short chorded cycle for a simple graph, we make the following conjecture.

CONJECTURE 2. *Let $G$ be a graph of order $n$ and minimum degree at least $3$. Then $G$ contains a chorded cycle of length at most $\alpha \log_2 n$, where $\alpha > 0$ is a universal constant.*

If Conjecture 2 is true, then Theorem 1.3 follows from it. To see this, let $G$ be a multigraph of order $n$ and minimum degree at least 5. We may assume that $G$ has no triple edges. Otherwise, $G$ contains a chorded cycle of length 2. We replace each pair of double edges of $G$ by a single edge. The resulting graph is simple and has minimum degree at least 3, and the existence of a chorded cycle of length at most $\alpha \log_2 n$ in this simple graph gives a chorded cycle of the same length in the original multigraph.

Notice that for a simple graph, minimum degree at least 3 guarantees the existence of a chorded cycle, but average degree at least 3 does not. For example, let $G$ be the graph obtained from $K_n$ by subdividing each edge exactly once. The graph has average degree $4\binom{n}{2}/(\binom{n}{2} + n) \to 4$ as $n \to \infty$, yet it has no chorded cycles. From this example, we can also see that in finding a chorded cycle, suppressing degree 2 vertices fails, while that is the technique used by Bollobás and Thomason to prove Lemma 2.1, which is given in next section.

The remaining parts of this paper are organized as follows: Some notation and lemmas are introduced in section 2, Theorem 1.3 is proved in section 3, the proof of

Theorem 1.2 is given in section 4, and in section 5, we give an average degree condition for an $n$-vertex graph to contain an $s$-chorded cycle of length about $\log_2 n$, and we discuss the relation between chorded cycles and nested cycles.

**2. Notation and lemmas.** Let $G$ be a (multi)graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively, and by $e(G)$ the size of $G$. For a vertex $v \in V(G)$, denote by $d_G(v)$ the degree of $v$, which is the number of edges incident to $v$, and by $N_G(v)$ the set of vertices adjacent to $v$. Notice that $d_G(v)$ may be greater than $|N_G(v)|$ in case that $G$ is a multigraph. The minimum degree of $G$ is denoted by $\delta(G)$, and the average degree is denoted by $\overline{d}(G)$. For a subset $S \subseteq V(G)$, denote by $G[S]$ the subgraph of $G$ induced on $S$, and let $N_G(S) = \cup_{v \in S} N_G(v) - S$. When the graph $G$ in consideration is clear from the context, we omit the index $G$ in notation $d_G(v)$, $N_G(v)$, or $N_G(S)$. A path connecting two vertices $u$ and $v$ of $G$ is called a $(u, v)$-path, and the distance between $u$ and $v$ is the length of a shortest $(u, v)$-path, denoted by $dist(u, v)$. The girth of $G$ is the length of a shortest cycle, denoted by $g(G)$. We call the length of a shortest chorded cycle of $G$ the chorded girth of $G$ and denote it by $g_c(G)$. Define $g(G) = \infty$ if $G$ is acyclic, and define $g_c(G) = \infty$ if $G$ is acyclic or does not have a chorded cycle. If $G$ is a bipartite graph with partite sets $A$ and $B$, then we denote $G$ by $G(A, B)$.

The following lemmas will be used in our proofs.

LEMMA 2.1 (Bollobás and Thomason [1]). *Let $G$ be a multigraph of order $n$ and size at least $n + c \, (c \geq 1)$. Then $g(G) \leq 2(\lfloor n/c \rfloor + 1)\lfloor \log_2 2c \rfloor$.*

This lemma implies that if the size of $G$ is at least $(1 + \alpha)n$ for some positive constant $\alpha$, then $G$ contains a cycle of length at most $2(\lfloor 1/\alpha \rfloor + 1)\lfloor \log_2 2\alpha n \rfloor$, which is quite "short" compared to the order $n$ of $G$ when $n$ is sufficiently large. We will use this result to find a "short" chorded cycle in a multigraph of minimum degree at least 5 in section 3.

In order to find a chorded cycle, we use small complete bipartite graphs. The following lemma gives a sufficient condition for doing so.

LEMMA 2.2 (Verstraëte [10]). *Let $G(A, B)$ be a bipartite graph with $|A| > a|B|^b$. Suppose that $d(v) = b$ for all $v \in A$. Then for some $r \geq 1$, there exist sets $A_1, A_2, \ldots, A_r, W \subset A$ and sets $B_1, B_2, \ldots, B_r \subset B$ such that for each $i = 1, 2, \ldots, r$, $G(A_i, B_i) = G[A_i \cup B_i]$ is a complete bipartite graph with $|A_i| \geq a$ and $|B_i| = b$, the $B_i$ are distinct, the $A_i$ are disjoint, $A = A_1 \cup A_2 \cup \cdots \cup A_r \cup W$, and $|W| < a|B|^b$.*

Immediately, from the aforementioned lemma, we have the following remark.

*Remark.* Let $G(A, B)$ be a bipartite graph with $|A| > 3|B|^3 + 3(k - 1)$. Suppose that $d(v) \geq 3k$ for all $v \in A$. By applying Lemma 2.2, we can find a complete bipartite graph $G(A_1, B_1) \cong K_{3,3}$ with $A_1 \subseteq A$ and $B_1 \subseteq B$. By applying Lemma 2.2 on $G(A - A_1, B - B_1)$, we get another copy of $K_{3,3}$. Since $d_G(v) \geq 3k$, we can repeat this process $k$ times and thus get $k$ vertex-disjoint copies of $K_{3,3}$, that is, $k$ vertex-disjoint (isomorphic) chorded 6-cycles.

**3. Proof of Theorem 1.3.** In this section, we prove Theorem 1.3, that a multigraph of order $n$ with minimum degree at least 5 has chorded girth $g_c(G) \leq 300 \log_2 n$. For convenience, in what follows we say that a cycle of length at most $300 \log_2 n$ is a short cycle.

Let $G$ be a multigraph with order $n$, and let $\delta(G) = \delta \geq 5$. We may assume that $G$ has no triple edges. Otherwise, $g_c(G) = 2$, and we are done. Given a vertex $v_0 \in V(G)$, let $T$ be a tree (delete duplicated edges, if any, to make it a simple graph) obtained from applying the breadth first search (BFS) algorithm on $G$ starting from $v_0$, which is called a BFS tree of $G$. We will show that either there is a short chorded

cycle in $G$ or the order of $T$ grows exponentially with the depth of $T$. If the latter happens, it turns out that the order of $T$ exceeds the order of $G$, which gives a contradiction.

We suppose on the contrary that $g_c(G) > 300 \log_2 n$. This implies that $n > 300 \log_2 n$, since $G$ contains a chorded cycle when $\delta(G) \geq 3$.

For $i \geq 0$, let

$$V_i = \{v \in V(G) \mid dist(v_0, v) = i\}.$$

Clearly, $V_i$, the set of vertices at distance $i$ from $v_0$ in $G$, is also the set of vertices at distance $i$ from $v_0$ in $T$.

For $i \geq 1$ and each vertex $u \in V_i$, there is a vertex in $V_{i-1}$ which is adjacent to $u$ in $G$, and no vertex in $V_j$ is adjacent to $u$ for $j \leq i - 2$. Moreover, in $T$, there is a unique vertex in $V_{i-1}$ adjacent to $u$. We call the unique vertex the *parent* of $u$ and denote it by $u^-$. Denote

$$N^-(u) = N_G(u) \cap V_{i-1}, \ N^0(u) = N_G(u) \cap V_i, \ \text{and} \ N^+(u) = N_G(u) \cap V_{i+1}.$$

Clearly, $u^- \in N^-(u)$, and $N_G(u) = N^-(u) \cup N^0(u) \cup N^+(u)$. We may use notation $N^-(U)$, $N^0(U)$, or $N^+(U)$ when $U$ is a subset of $V_i$ containing at least two elements.

Let $u, v \in V_i$ be two distinct vertices. We denote by $T_{uv}$ the unique $(u,v)$-path in $T$. Notice that $T_{uv} \cap (\bigcup_{k \geq i} V_k) - \{u, v\} = \emptyset$, and the length of $T_{uv}$ is at most $2i$.

In order to show that the order of $T$ grows exponentially with $i$, we analyze the degree sum of vertices in $V_i$. To do this, we define two types of subgraphs of $G$ for each level $i$ $(i \geq 1)$ of $T$ as follows:

$$H_i = G[V_{i-1} \cup V_i] - E(G[V_{i-1}]) \quad \text{and} \quad K_i = G[V_i \cup V_{i+1}] - E(G[V_{i+1}]).$$

We will show that, after removing some vertices from $H_i$ and $K_i$, the two resulting graphs have no cycles of length at most $100 \log_2 n$ (showed in Claim 3.2A) and $100 \log_2 n - 2$ (showed in Claim 3.2B), respectively. We first classify the vertices which need to be removed.

Let $v \in V(G) - \{v_0\}$. Suppose $v \in V_j$ for some $j \geq 1$. We call $v$ a *branching vertex* of $G$ if all of the following hold:
- $d_{H_j}(v) = 2$ (notice that $d_{H_j}(v) = 2$ if and only if $d_{G[V_{j-1} \cup V_j]}(v) = 2$);
- $N^+(v) \neq \emptyset$ and $N^-(N^+(v)) = \{v\}$.

We will remove the following two vertex sets from $H_i$ and $K_i$, respectively:

$$X_i = \{v \in V_i \mid v \text{ is a branching vertex on a cycle of } H_i\},$$

$$Y_i = \{v \in V_{i+1} \mid v \text{ is a branching vertex on a cycle of } K_i\}.$$

Figure 1 depicts what a branching vertex in $X_i$ or $Y_i$ looks like. Let

$$\varepsilon = \frac{8}{393} \quad \text{and} \quad c_0 = 6 \left( \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 \right).$$

By direct computation, $c_0 = 300$. We claim the following.

CLAIM 3.1A. *For each $i$ with $1 \leq i \leq \frac{c_0}{3} \log_2 n$, if there is a cycle $C$ in $H_i - X_i$ such that $|C| \leq \frac{c_0}{3} \log_2 n$, then for any $v \in V(C) \cap V_i$ with $N_C(v) \cap V_{i-1} \neq \emptyset$, the following three statements hold:*
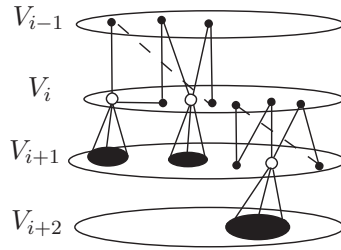(a) $N^-(v) \subseteq V(C)$;

FIG. 1. *Branching vertices in $H_i$ and $K_i$: white vertices.*

(b) $N^-(N^0(v) - V(C)) \subseteq V(C)$; *and*
(c) $N^-(N^-(N^+(v)) - V(C)) \subseteq V(C)$.

*Proof.* Let $v^* \in N_C(v) \cap V_{i-1}$.

To show (a), suppose on the contrary that there exists a vertex $w \in N^-(v) - V(C)$. Then we have $v^*, w \in V_{i-1}$. Recall that $T_{v^*w}$ is the unique $(v^*, w)$-path in the BFS tree $T$. Denote $C' = T_{v^*w} \cup wv \cup (C - vv^*)$, which is a cycle of length $|C'| \leq 2(i-1) + 1 + |C| - 1 < c_0 \log_2 n$. Notice that $vv^* \notin E(C')$, but $v, v^* \in V(C')$. This implies that $C' \cup vv^*$ is a short chorded cycle in $G$, showing a contradiction to the assumption that $g_c(G) > c_0 \log_2 n$.

To show (b), suppose on the contrary that there exists a vertex $w \in N^-(N^0(v) - V(C)) - V(C)$. Suppose that $w$ is adjacent to a vertex $w' \in N^0(v) - V(C)$. Then we have $v^*, w \in V_{i-1}$ and the cycle $C' = T_{v^*w} \cup ww'v \cup (C - vv^*)$ with $|C'| \leq 2(i-1) + 2 + |C| - 1 < c_0 \log_2 n$. Again, $vv^* \notin E(C')$, but $v, v^* \in V(C')$. Hence $C' \cup vv^*$ is a short chorded cycle in $G$, showing a contradiction.

To show (c), that $N^-(N^-(N^+(v)) - V(C)) \subseteq V(C)$, we may assume $N^-(N^+(v)) - V(C) \neq \emptyset$; otherwise we are done. Hence, under this assumption, suppose on the contrary that there exist $w' \in N^-(N^+(v)) - V(C)$ and $w \in N^-(w') - V(C)$. Let $w'' \in N^+(v)$ be a vertex adjacent to $w'$. Then we have $v^*, w \in V_{i-1}$ and the cycle $C' = T_{v^*w} \cup ww'w''v \cup (C - vv^*)$ with $|C'| \leq 2(i-1) + 3 + |C| - 1 \leq c_0 \log_2 n$. We see that $C' \cup vv^*$ is a short chorded cycle in $G$, which gives a contradiction. □

Let $C$ be a cycle and $v$ be a vertex described as in Claim 3.1A. Since $C$ is chordless by assuming $g_c(G) > c_0 \log_2 n$, $d_C(v) = 2$. Thus $d(v) = 2 + |N^0(v) - V(C)| + |N^+(v)|$. Since $\delta(G) \geq 5$, despite $G$ having multiedges or not, we have $(N^0(v) - V(C)) \cup N^+(v) \neq \emptyset$. If $N^0(v) - V(C) = \emptyset$, then $d_{H_i}(v) = 2$. Consequently, $N^+(v) \neq \emptyset$. Since $v$ is on a cycle $C$ of $H_i - X_i$, $v$ is not a branching vertex. Hence $N^-(N^+(v)) - \{v\} \neq \emptyset$. Let $w \in (N^0(v) - V(C)) \cup N^+(v)$. If $N^0(v) - V(C) = \emptyset$, we choose $w$ and further require that $N^-(w) - \{v\} \neq \emptyset$. We define a $(v, \overline{v})$-path $Q_{v\overline{v}}$ with $\overline{v} \in V(C) - \{v\}$ as follows:

(3.1)

$$Q_{v\overline{v}} = \begin{cases} vw\overline{v}, & \text{where } w \in N^0(v) - V(C) \text{ and } \overline{v} = w^-; \\ vw\overline{v}, & \text{where } N^0(v) - V(C) = \emptyset, w \in N^+(v), \\ & \text{and } \overline{v} \in (N^-(N^+(v)) - \{v\}) \cap V(C) \cap N^-(w); \\ vww'\overline{v}, & \text{where } N^0(v) - V(C) = \emptyset, w \in N^+(v), \\ & w' \in (N^-(N^+(v)) - V(C)) \cap N^-(w), \text{ and } \overline{v} \in N^-(w') \cap V(C). \end{cases}$$

By Claim 3.1A, notice that a $(v, \overline{v})$-path $Q_{v\overline{v}}$ always exists. We also note that $g_c(G) > c_0 \log_2 n$ holds, since we assumed that $G$ is a counterexample.

CLAIM 3.2A. *For each $i$ with $1 \leq i \leq \frac{c_0}{3} \log_2 n$, $g(H_i - X_i) > \frac{c_0}{3} \log_2 n$.*

*Proof.* Suppose on the contrary that Claim 3.2A fails. Let $C$ be a cycle in $H_i - X_i$ of length at most $\frac{c_0}{3} \log_2 n$. Since $G$ has no short chorded cycles, $C$ is chordless. We show Claim 3.2A by making the claims below.

*Claim* A1. If there is an edge $uv$ in $C$ with $u, v \in V_i$, then $N_C(u) \cup N_C(v) \nsubseteq V_i$.

*Proof.* Suppose on the contrary. As $C$ is chordless, $\{u^-, v^-\} \subseteq V_{i-1} - V(C)$. Then $C' = T_{uv} \cup (C - uv)$ is a cycle with $|C'| \le 2i + |C| - 1 < c_0 \log_2 n$, and $C' \cup uv$ is a short chorded cycle. $\square$

*Claim* A2. There is no edge of $C$ with both ends in $V_i$.

*Proof.* Suppose not, and let $uv$ be an edge of $C$ such that $u, v \in V_i$. We may assume that $N_C(u) \cap V_{i-1} \ne \emptyset$ by Claim A1.

Let $u^* = N_C(u) - \{v\}$. Then by Claim 3.1A, we have (a) $N^-(u) \subseteq V(C)$; (b) $N^-(N^0(u) - V(C)) \subseteq V(C)$; and (c) either $N^-(N^+(u)) \subseteq V(C)$, or if there exists a vertex $w' \in N^-(N^+(u)) - V(C)$, then $N^-(w') \subseteq V(C)$. As $\delta(G) \ge 5$ and $u$ is not a branching vertex, a $(u, \overline{u})$-path $Q_{u\overline{u}}$ defined in (3.1) exists. Recall that $u \ne \overline{u} \in V(C)$. We may also assume that $\overline{u} \notin \{v, u^*\}$. Otherwise, $C' = (C - u\overline{u}) \cup Q_{u\overline{u}}$ is a cycle with $|C'| \le |C| - 1 + 3 < c_0 \log_2 n$, and $C' \cup u\overline{u}$ is a short chorded cycle. If $\overline{u} \ne v^-$, let $P_{\overline{u}u^*}$ be the $(\overline{u}, u^*)$-path in $C - uv$. Then $C' = T_{u^*v^-} \cup v^-vu \cup Q_{u\overline{u}} \cup P_{\overline{u}u^*}$ is a cycle with $|C'| \le 2(i-1) + |C| - 1 + 3 \le c_0 \log_2 n$, and $C' \cup uu^*$ is a short chorded cycle. Thus $\overline{u} = v^-$. This implies that $N_C(v) = \{u, v^-\}$, combining the fact that $\overline{u} \in V(C)$ and $C$ is chordless. Applying Claim 3.1A on $v$, there exist a vertex $\overline{v} \in V(C) - \{v\}$ and a $(v, \overline{v})$-path $Q_{v\overline{v}}$. Again, we may assume that $\overline{v} \notin N_C(v) = \{u, v^-\}$. Notice that $\{v^-, \overline{v}\} \subseteq V_{i-1}$ and $\overline{u} = v^-$. We may assume that $Q_{\overline{v}v} \cap Q_{uv^-} = \emptyset$. Otherwise, $Q_{\overline{v}v} \cup Q_{uv^-} \cup C$ contains a short cycle with chord $uv$. Then $C' = Q_{\overline{v}v} \cup vu \cup Q_{uv^-} \cup T_{v^-\overline{v}}$ is a cycle with $|C'| \le 6 + 1 + 2(i-1) < c_0 \log_2 n$, and $C' \cup vv^-$ is a short chorded cycle. $\square$

Let $v \in V(C) \cap V_i$ be a vertex, and let $N_C(v) = \{u, u'\}$. By Claim A2, $\{u, u'\} \subseteq V_{i-1}$. Applying Claim 3.1A on $v$, we choose $\overline{v} \in V(C) - \{v\}$ and a $(v, \overline{v})$-path $Q_{v\overline{v}}$. We may assume $\overline{v} \notin \{u, u'\}$. Let $P_{\overline{v}u}$ be the $(\overline{v}, u)$-path in $C - uv$. Then $C' = u'v \cup Q_{v\overline{v}} \cup P_{\overline{v}u} \cup T_{uu'}$ is a cycle with $|C'| \le 3 + |C| - 1 + 2(i-1) \le c_0 \log_2 n$, and $C' \cup uv$ is a short chorded cycle. This gives a contradiction.

The proof of Claim 3.2A is finished. $\square$

We now repeat the previous discussions for the graph $K_i - Y_i$. A similar proof as for Claim 3.1A gives the following.

CLAIM 3.1B. *For each $i$ with $1 \le i \le \frac{c_0}{3} \log_2 n$, suppose there is a cycle $C$ in $K_i - Y_i$ such that $|C| \le \frac{c_0}{3} \log_2 n - 1$; then for any $v \in V(C) \cap V_{i+1}$ each of the following is true:*

(a) $N^-(v) \subseteq V(C)$;
(b) $N^-(N^0(v) - V(C)) \subseteq V(C)$; *and*
(c) $N^-(N^-(N^+(v)) - V(C)) \subseteq V(C)$.

Notice that in the above claim, $N_C(v) \cap V_i \ne \emptyset$ always. A $(v, \overline{v})$-path $Q_{v\overline{v}}$ can be defined similarly as in (3.1) for the vertex $v$ described as above.

CLAIM 3.2B. *For each $i$ with $1 \le i \le \frac{c_0}{3} \log_2 n$, $g(K_i - Y_i) > \frac{c_0}{3} \log_2 n - 2$.*

*Proof.* Suppose on the contrary that Claim 3.2B fails. Let $C$ be a cycle in $K_i - Y_i$ of length at most $\frac{c_0}{3} \log_2 n - 2$. Since $G$ has no short chorded cycle, $C$ is chordless.

Recall that $K_i = G[V_i \cup V_{i+1}] - E(G[V_{i+1}])$. We may assume that $E(C) \cap E(G[V_i]) = \emptyset$. Otherwise, let $uv \in E(C) \cap E(G[V_i])$. Then $C' = (C - uv) \cup T_{uv}$ is a cycle with $|C'| \le 2i + |C| - 1 < c_0 \log_2 n$, and $C' \cup uv$ is a short chorded cycle. Thus $V(C) \cap V_{i+1} \ne \emptyset$. Let $v \in V(C) \cap V_{i+1}$ and $N_C(v) = \{u, u'\}$. Clearly, $\{u, u'\} \subseteq V_i$. Applying Claim 3.1B on $v$, we choose $\overline{v} \in V(C) - \{v\}$ and a $(v, \overline{v})$-path

$Q_{v\overline{v}}$. We may assume $\overline{v} \notin \{u, u'\}$. Let $P_{\overline{v}u}$ be the $(\overline{v}, u)$-path in $C - uv$. Then $C' = u'v \cup Q_{v\overline{v}} \cup P_{\overline{v}u} \cup T_{uu'}$ is a cycle with $|C'| \leq 3 + |C| - 1 + 2i \leq c_0 \log_2 n$, and $C' \cup uv$ is a short chorded cycle. This gives a contradiction.

The proof of Claim 3.2B is finished.    □

We investigate some properties of $X_i$ and $Y_i$ in the following two claims. Recall that $g_c(G) > c_0 \log_2 n$ by the assumption.

CLAIM 3.3. *The following hold:*

(i) $|N_G(x) \cap V_{i+1}| \geq 2$ *for each* $x \in X_i$. *Moreover,* $|N_G(X_i) \cap V_{i+1}| \geq 2|X_i|$.

(ii) $|N_G(y) \cap V_{i+2}| \geq 2$ *for each* $y \in Y_i$. *Moreover,* $|N_G(Y_i) \cap V_{i+2}| \geq 2|Y_i|$.

*Proof.* Since the proof is similar, we show only statement (i) here. Notice that $N_G(x) = N_{H_i}(x) \cup (N_G(x) \cap V_{i+1})$. Since $x$ is a branching vertex, $d_{H_i}(x) = 2$. This in turns gives that $d_{G[V_{i+1}]}(x) \geq \delta(G) - 2 \geq 3$. Then as $G$ has no triple edges, we get $|N_G(x) \cap V_{i+1}| \geq 2$. As $X_i$ is the set of branching vertices in $V_i$, we have $|N^-(v)| = 1$ for each $v \in N_G(X_i) \cap V_{i+1}$. Hence $|N_G(X_i) \cap V_{i+1}| = \sum_{x \in X_i} |N_G(x) \cap V_{i+1}| \geq 2|X_i|$.    □

Let $X_i' = N_G(X_i) \cap V_{i+1}$. By the definition of $X_i$ and $Y_i$, for each vertex $x' \in X_i'$, $|N^-(x')| = 1$; for each vertex $y \in Y_i$, $|N^-(y)| = 2$. This gives the following.

CLAIM 3.4. $X_i' \cap Y_i = \emptyset$.

In fact, letting $Y_i' = N_G(Y_i) \cap V_i$, we would like to mention that $Y_i' \cap X_i = \emptyset$ also holds, although it will not be used in the proof.

Let $n_0 = |V_0| = 1$. For $i \geq 1$, denote

$$n_i = |V_i|, \qquad x_i = |X_i|, \quad \text{and} \quad y_i = |Y_i|.$$

By Claims 3.3 and 3.4, we have

(3.2)          $x_i \leq \dfrac{1}{2}(n_{i+1} - y_i), \quad \text{in particular} \quad x_i + y_i \leq n_{i+1}.$

We now show that the order of the BFS tree $T$ increases exponentially as its depth grows.

CLAIM 3.5. *For* $0 \leq i \leq \frac{c_0}{3} \log_2 n$, *either* $n_{i+1} \geq n_i$ *and* $n_{i+2} \geq \frac{12}{11}n_i$ *or* $n_{i+1} \geq \frac{12}{11}n_i$.

*Proof.* We claim that

$$e(H_i - X_i) \leq (1 + \varepsilon)(n_{i-1} + n_i - x_i) \quad \text{and} \quad e(K_i - Y_i) \leq (1 + \varepsilon)(n_i + n_{i+1} - y_i),$$

where $\varepsilon = \frac{8}{393}$ is defined previously. We show the first inequality first. Suppose on the contrary that $e(H_i - X_i) > (1 + \varepsilon)(n_{i-1} + n_i - x_i)$. This implies that $H_i - X_i$ contains a cycle. We may assume that $\varepsilon(n_{i-1} + n_i - x_i) \geq 1$. Otherwise, if $\varepsilon(n_{i-1} + n_i - x_i) < 1$, then $n_{i-1} + n_i - x_i < \frac{1}{\varepsilon}$. Hence the length of any cycle in $H_i - X_i$ is at most $n_{i-1} + n_i - x_i < \frac{1}{\varepsilon} < \frac{c_0}{3} \log_2 n$, showing a contradiction to Claim 3.2A. Hence, $\varepsilon(n_{i-1} + n_i - x_i) \geq 1$. By applying Lemma 2.1 on $H_i - X_i$, we have $g(H_i - X_i) \leq 2(\lfloor \frac{1}{\varepsilon} \rfloor + 1)\lfloor \log_2 2\varepsilon(n_{i-1} + n_i - x_i) \rfloor < \frac{c_0}{3} \log_2 n$. This shows a contradiction to Claim 3.2A. For the second inequality, assume it fails. Similarly, we may assume that $\varepsilon(n_i + n_{i+1} - y_i) \geq 1$. By applying Lemma 2.1 on $K_i - Y_i$, we have $g(K_i - Y_i) \leq 2(\lfloor \frac{1}{\varepsilon} \rfloor + 1)\lfloor \log_2 2\varepsilon(n_i + n_{i+1} - y_i) \rfloor < \frac{c_0}{3} \log_2 n - 2$ by $\varepsilon \leq 1/4$. Thus,

(3.3)          $e(H_i) \leq (1 + \varepsilon)(n_{i-1} + n_i - x_i) + 2x_i,$

(3.4)          $e(K_i) \leq (1 + \varepsilon)(n_i + n_{i+1} - y_i) + 2y_i.$

We use induction on $i$ to show the assertion. For $i = 0$, we have $n_0 = 1 = |\{v_0\}|$ and $n_1 \geq \lceil \frac{\delta}{2} \rceil \geq 3$. Notice that for each vertex $v \in V_1$, $d_{G[V_0 \cup V_1]}(v) \leq 2$. Otherwise,

we get a chorded cycle of length either 3 or 4 in $G$. As $G$ may have multiedges but no triple edges, we see that $n_2 \geq \lceil \frac{\delta-2}{2} \rceil \geq 2$. The arguments above give that either $n_1 \geq n_0$ and $n_2 \geq \frac{12}{11}n_0$ or $n_1 \geq \frac{12}{11}n_0$. For $i \geq 1$, we proceed by the induction hypothesis and consider the two cases as follows.

*Case* I. Suppose $y_i \geq \frac{6}{11}n_{i+1}$.

Then $n_{i+2} \geq \frac{12}{11}n_{i+1}$ by Claim 3.3. Thus, in order to show that $n_{i+1} \geq n_i$ and $n_{i+2} \geq \frac{12}{11}n_i$ it suffices to show that $n_i \leq n_{i+1}$. By the induction hypothesis, we have either $n_{i-1} \leq n_i$ and $n_{i-1} \leq \frac{11}{12}n_{i+1}$ or $n_{i-1} \leq \frac{11}{12}n_i$.

So suppose $n_{i-1} \leq n_i$ and $n_{i-1} \leq \frac{11}{12}n_{i+1}$. Then by inequality (3.2) with $x_i + y_i \leq n_{i+1}$,

$$\begin{aligned} \delta n_i &\leq e(H_i) + e(K_i) \leq 2(1+\varepsilon)n_i + \frac{11}{12}(1+\varepsilon)n_{i+1} + 2n_{i+1} \\ &= (2+2\varepsilon)n_i + (2 + \tfrac{11}{12} + \tfrac{11}{12}\varepsilon)n_{i+1}. \end{aligned}$$

By $\delta n_i \leq (2+2\varepsilon)n_i + (2 + \frac{11}{12} + \frac{11}{12}\varepsilon)n_{i+1}$, we get $n_i \leq n_{i+1}$, since $\varepsilon \leq \frac{1}{35}$. Suppose now $n_{i-1} \leq \frac{11}{12}n_i$. Then

$$\begin{aligned} \delta n_i &\leq e(H_i) + e(K_i) \leq 2(1+\varepsilon)n_i + \frac{11}{12}(1+\varepsilon)n_i + 2n_{i+1} \\ &= (2 + \tfrac{11}{12} + 2\varepsilon + \tfrac{11}{12}\varepsilon)n_i + 2n_{i+1}. \end{aligned}$$

Again, $n_{i+1} \geq n_i$ holds, since $\varepsilon \leq \frac{1}{35}$.

*Case* II. Suppose $y_i < \frac{6}{11}n_{i+1}$.

By inequality (3.2), $x_i + y_i = \frac{1}{2}(2x_i + y_i) + \frac{1}{2}y_i < \frac{1}{2}n_{i+1} + \frac{3}{11}n_{i+1} = \frac{8.5}{11}n_{i+1}$.

Suppose $n_{i-1} \leq n_i$ and $n_{i-1} \leq \frac{11}{12}n_{i+1}$. Then

$$\delta n_i \leq e(H_i) + e(K_i) \leq 2(1+\varepsilon)n_i + \frac{11}{12}(1+\varepsilon)n_{i+1} + \left(1 + \frac{8.5}{11} + \frac{2.5}{11}\varepsilon\right)n_{i+1},$$

and thus $n_{i+1} \geq \frac{12}{11}n_i$ as $\varepsilon = \frac{8}{393}$.

Suppose now that $n_{i-1} \leq \frac{11}{12}n_i$. Then

$$\delta n_i \leq e(H_i) + e(K_i) \leq 2(1+\varepsilon)n_i + \frac{11}{12}(1+\varepsilon)n_i + \left(1 + \frac{8.5}{11} + \frac{2.5}{11}\varepsilon\right)n_{i+1},$$

and thus $n_{i+1} \geq \frac{12}{11}n_i$. $\quad\square$

It is now time to give a contradiction by showing that the order of $T$ exceeds the order of $G$.

CLAIM 3.6. *For the BFS tree $T$, $|V(T)| > n$ holds.*

*Proof.* Suppose not. Let $t = \lfloor \frac{c_0}{3} \log_2 n \rfloor$. By Claim 3.5, the BFS tree $T$ rooted at $v_0$ has at least $t+2$ levels $V_0, V_1, \ldots, V_t, V_{t+1}$. Also it follows from Claim 3.5 that $n_{i+1} \geq n_i$ for $1 \leq i \leq t$, and $n_{i+2} \geq \frac{12}{11}n_i$ for $1 \leq i \leq t-1$. Then we get

$$\begin{aligned} n &\geq n_0 + n_1 + \cdots + n_t + n_{t+1} \\ &\geq n_0 + 2n_1 + 2n_3 + \cdots + 2n_{2\lceil t/2 \rceil - 1} \\ &> 2n_1\left(1 + \frac{12}{11} + \left(\frac{12}{11}\right)^2 + \cdots + \left(\frac{12}{11}\right)^{\lceil t/2 \rceil - 1}\right) \\ &= 22n_1\left(\left(\frac{12}{11}\right)^{\lceil t/2 \rceil} - 1\right). \end{aligned}$$

Thus,

$$\frac{n}{22n_1} + 1 > \left(\frac{12}{11}\right)^{t/2}.$$

Since $n_1 \geq \lceil \frac{\delta}{2} \rceil \geq 3$ and $t \geq \frac{c_0}{3} \log_2 n - 1$, this implies that

$$\frac{2\log_2(\frac{n}{66} + 1)}{\log_2 \frac{12}{11}} > \frac{c_0}{3} \log_2 n - 1.$$

Since $\log_2 \frac{12}{11} > \frac{1}{8}$, we get

$$48 \log_2 \left(\frac{n}{66} + 1\right) > c_0 \log_2 n - 3 \geq 48 \log_2 n - 3.$$

This implies that $-3 < 48\log_2(\frac{n+66}{66n}) < 48\log_2(\frac{1}{4}) = -96$, showing a contradiction. $\square$

The proof of Theorem 1.3 is then complete.

**4. Proof of Theorem 1.2.** The proof idea is similar to the proof for Theorem 1.1 by Verstraëte.

We pick

$$n_k = \min\left\{n \in \mathbb{N} : n > 28(k-1)^3\left(\frac{301^2}{2}\right)^3 (\log_2 n)^6 + 27(k-1)\right\},$$

and

$$n'_k = \min\left\{n \in \mathbb{N} : n > 28(k-1)^3\left(\frac{301^3}{6}\right)^3 (\log_2 n)^9 + 27(k-1)\right\}.$$

Let $\mathcal{C}$ be a collection of vertex-disjoint chorded cycles of length at most $c_0 \log_2 n = 300 \log_2 n$ such that $|\mathcal{C}|$ is maximal. Let

$$V_2 = \bigcup_{C \in \mathcal{C}} V(C).$$

For the proof of statement (i), we may assume that $\mathcal{C}$ does not contain $k$ vertex-disjoint chorded cycles of the same length. This gives that

$$(4.1) \qquad |V_2| \leq \sum_{\ell=4}^{\lfloor 300 \log_2 n \rfloor} (k-1)\ell < (k-1)\frac{301^2}{2}(\log_2 n)^2.$$

For the proof of statement (ii), we may assume that $\mathcal{C}$ does not contain $k$ vertex-disjoint isomorphic chorded cycles of the same length. For a fixed cycle length $\ell$, we have at most $\ell/2$ types of nonisomorphic chorded cycles of length $\ell$ (considering only cycles with exactly one chord). Thus, the number of vertices in a maximal collection of vertex-disjoint isomorphic chorded cycles of length at most $300 \log_2 n$ is

$$(4.2) \qquad |V_2| \leq \sum_{\ell=4}^{\lfloor 300 \log_2 n \rfloor} (k-1)\ell^2/2 < (k-1)\frac{301^3}{6}(\log_2 n)^3.$$

Let

$$V_1 = V - V_2, \ Y = \{v \in V_1 \,|\, d_{V_2}(v) \geq 3k\}, \text{ and } Z = \{v \in V_1 \,|\, d_{V_2}(v) < 3k\}.$$

We may assume that $|Y| \leq 3|V_2|^3 + 3(k-1)$. Otherwise, by the remark immediately after Lemma 2.2, we can find $k$ vertex-disjoint copies of $K_{3,3}$, thus getting $k$ vertex-disjoint (isomorphic) chorded 6-cycles. Hence

$$|Z| = |V_1| - |Y| = n - |V_2| - |Y| \geq 8|Y|.$$

Let $G' = G[Y \cup Z]$. For each vertex $z \in Z$, $d_{G'}(z) \geq (3k+8) - (3k-1) = 9$. Then,

$$\sum_{x \in V(G')} d_{G'}(x) \ \geq \ \sum_{z \in Z} d_{G'}(z) \ \geq \ 9|Z| \ \geq \ 8(|Y| + |Z|),$$

and hence the average degree of $G'$ is at least 8. Since deleting a vertex of degree at most 4 in $G'$ does not decrease the average degree of the resulting graph, we know $G'$ contains a subgraph $H$ of minimum degree at least 5. Applying Theorem 1.3 on $H$, we can find a chorded cycle of length at most $300 \log_2 n$, which is disjoint with cycles in $\mathcal{C}$. This shows a contradiction to the choice of $\mathcal{C}$. Hence, we know that there exist $k$ vertex-disjoint (isomorphic) chorded cycles of the same length in $G$. The proof is finished. □

**5. $s$-chorded cycles and nested cycles.** In our previous manuscript, we asked the question about finding a minimum degree condition for a sufficiently large graph to contain $k$ vertex-disjoint $s$-chorded cycles of the same length. Following a discussion similar to that for the chorded cycles in this paper, to answer the question, it suffices to show a result similar to Theorem 1.3 for $s$-chorded cycles. The referee gave a sketch on showing that there exists a function $f$ such that every $n$-vertex graph of average degree at least $f(s)$ contains an $s$-chorded cycle of length about $\log_2 n$. To thank the referee as well as to make this paper complete, we give the detailed proof of the result mentioned above in this section. We first show a minimum degree version of the result for bipartite graphs.

THEOREM 5.1. *If $H$ is a bipartite graph of order $n$ and minimum degree at least $3 \cdot (9/2)^s$, then $H$ contains an $s$-chorded cycle of length at most $2(s+1)\log_2 n$.*

Notice that in the result above, the minimum degree condition is not optimal.

*Proof.* We apply induction on $s$. By the standard argument, the theorem holds for $s = 0$. So we assume that $s \geq 1$ and assume the theorem is true for $s-1$. Define the BFS tree $T$ for $H$ the same way as in the proof of Theorem 1.3. Suppose $T$ is started at $v_0 \in V(H)$. For each $i \geq 0$, let

$$V_i = \{v \in V(H) \,|\, dist(v_0, v) = i\}.$$

Again, for $u, v \in V_i$, denote by $T_{uv}$ the unique $(u, v)$-path in $T$.

For each $i$ with $0 \leq i \leq \log_2 n - 1$, let $H_i = H[V_i \cup V_{i+1}]$.

*Claim.* If $\overline{d}(H_i) \leq 6 \cdot (9/2)^{s-1}$ for each $i$ with $0 \leq i \leq \log_2 n - 1$, then $|V_{i+1}| \geq 2|V_i|$ for each $i$ with $0 \leq i \leq \log_2 n - 1$.

We use induction on $i$. Since $\delta(H) \geq 3 \cdot (9/2)^s \geq 13$, the claim trivially holds for $i = 0$. Assume now $i \geq 1$, and assume the claim is true for $i - 1$. Suppose on the contrary that $|V_{i+1}| < 2|V_i|$. We then claim that $\overline{d}(H_i) > 6 \cdot (9/2)^{s-1}$. This gives a contradiction. To show that $\overline{d}(H_i) > 6 \cdot (9/2)^{s-1}$, it suffices to show that $2 \sum_{v \in V_i} |N_H(v) \cap V_{i+1}| \geq 18 \cdot (9/2)^{s-1}|V_i|$ under the assumption that $|V_{i+1}| < 2|V_i|$. Notice that $\sum_{v \in V_i} |N_H(v) \cap V_{i-1}| \leq 9/2 \cdot (9/2)^{s-1}|V_i|$. Otherwise, we get $2|E(H_{i-1})| > 6 \cdot (9/2)^{s-1}(|V_i| + |V_{i-1}|)$ by using $|V_i| \geq 2|V_{i-1}|$ given by the induction hypothesis, which in turn gives a contradiction to the assumption that $\overline{d}(H_{i-1}) \leq 6 \cdot (9/2)^{s-1}$. Hence

$$\sum_{v \in V_i} |N_H(v) \cap V_{i+1}| \geq \delta(H)|V_i| - \sum_{v \in V_i} |N_H(v) \cap V_{i-1}|$$
$$\geq (3 \cdot (9/2)^s - 9/2 \cdot (9/2)^{s-1})|V_i| = 9 \cdot (9/2)^{s-1}|V_i|.$$

Let $t = \lfloor \log_2 n \rfloor$. If the claim above holds for each $i$ with $0 \leq i \leq \log_2 n - 1$, then we know that $T$ has at least $t + 1$ layers $V_0, \ldots, V_t$. Hence, by the claim, we have

$$|V(T)| = |V_0| + |V_1| + \cdots + |V_t|$$
$$\geq 1 + (1 + \cdots + 2^{t-1})|V_1| > n \quad (\text{as } |V_1| \geq 13),$$

giving a contradiction. Hence, there exists some $i$ with $0 \leq i \leq \log_2 n - 1$ such that $\overline{d}(H_i) \geq 6 \cdot (9/2)^{s-1}$. Then $H_i$ contains a subgraph $H_i'$ of minimum degree at least $3 \cdot (9/2)^{s-1}$. By the induction hypothesis, $H_i'$ contains an $(s-1)$-chorded cycle $C$ of length at most $2s \log_2 n$. We may assume that $C$ contains exactly $s - 1$ chords. Then as $3 \cdot (9/2)^{s-1} \geq s + 2$, for each edge $uv \in E(C)$ with $u \in V_i$ and $v \in V_{i+1}$, there exists $v' \in N_{H_i'}(v) - V(C)$. Notice that $v' \in V_i$ as $H_i'$ is bipartite. Then $C' = (C - uv) \cup T_{uv'} \cup vv'$ is a cycle of length at most $2s \log_2 n + 2i \leq 2(s+1) \log_2 n$, and $C'$ contains one more chord $uv$ than $C$ does. So $C'$ is the desired $s$-chorded cycle.

The proof is finished. □

Since every graph of size $m$ contains a bipartite graph of size at least $m/2$ and every graph of average degree $\overline{d}$ contains a subgraph of minimum degree at least $\overline{d}/2$, we obtain the following result by Theorem 5.1.

COROLLARY 5.1. *If $G$ is a graph of order $n$ and average degree at least $12 \cdot (9/2)^s$, then $G$ contains an $s$-chorded cycle of length at most $2(s+1) \log_2 n$.*

Applying Corollary 5.1, similar to the proof of Theorem 1.2, the following result can be proved.

THEOREM 5.2. *For every pair of natural numbers $k$ and $s$, there exists a positive integer $n_{k,s}$ such that if $G$ is a graph with order $n \geq n_{k,s}$ and minimum degree at least $\lceil \sqrt{s+1} + 1 \rceil k + 12 \cdot (9/2)^s$, then $G$ contains $k$ vertex-disjoint (isomorphic) $s$-chorded cycles of the same length.*

A sequence of cycles $C_1, C_2, \ldots, C_k$ is *nested* if the cycles are pairwise edge-disjoint and $V(C_1) \subseteq V(C_2) \subseteq \cdots \subseteq V(C_k)$. It was proved in [2] that there exists a function $f(k)$ such that every graph of average degree at least $f(k)$ contains $k$ nested-cycles $C_1, C_2, \ldots, C_k$. As all these cycles are edge-disjoint, one immediately sees that $C_k$ contains at least $|V(C_1)| + \cdots + |V(C_{k-1})|$ chords. However, the length of the cycles is not controllable. So this result is not applicable in proving Theorem 5.1. In fact, as pointed out by the referee, there are $n$-vertex graphs with arbitrarily large average degree such that every subgraph of order $O(\log n)$ does not contain two nested cycles.

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