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Weak saturation numbers for multiple copies

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ABSTRACT

For a fixed graph F, a graph G is F-*saturated* if there is no copy of F in G, but for any edge $e \notin G$, there is a copy of F in $G+e$. The minimum number of edges in an F-saturated graph of order n will be denoted by $\mathbf{sat}(n, F)$. A graph G is weakly F-saturated if G contains no copy of F and there is an ordering of the missing edges of G so that if they are added one at a time, each edge added creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n will be denoted by $\mathbf{wsat}(n, F)$. The graphs of order n that are weakly F-saturated will be denoted by $\mathbf{wSAT}(n, F)$, and those graphs in $\mathbf{wSAT}(n, F)$ with $\mathbf{wsat}(n, F)$ edges will be denoted by $\underline{\mathbf{wSAT}}(n, F)$. The precise value of $\mathbf{wsat}(n, F)$ for many families of vertex disjoint copies of connected graphs such as small order graphs, trees, cycles, and complete graphs will be determined.

1. Introduction

Only finite graphs without loops or multiple edges will be considered. Notation will be standard, and generally follows the notation of [2]. For a graph G, the vertex set $V(G)$ and the edge set $E(G)$ will be represented by just G when it is clear from the context.

For a fixed graph F, a graph G is F-*saturated* if there is no copy of F in G, but for any edge $e \notin G$, there is a copy of F in $G + e$. The collection of F-saturated graphs of order n is denoted by $\mathbf{SAT}(n, F)$, the *saturation number*, denoted as $\mathbf{sat}(n, F)$, is the minimum number of edges in a graph in $\mathbf{SAT}(n, F)$, and the *extremal number*, denoted by $\mathbf{ex}(n, F)$, is the maximum number of edges in a graph in $\mathbf{SAT}(n, F)$. A graph G is *weakly* F-*saturated* if G contains no copy of F and there is an ordering of the missing edges of G so that if they are added one at a time, they form a complete graph and each edge added creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n will be denoted by $\mathbf{wsat}(n, F)$. The graphs of order n that are weakly F-saturated will be denoted by $\mathbf{wSAT}(n, F)$, and those graphs in $\mathbf{wSAT}(n, F)$ with $\mathbf{wsat}(n, F)$ edges will be denoted by $\underline{\mathbf{wSAT}}(n, F)$. Clearly $\mathbf{wsat}(n, F) \le \mathbf{sat}(n, F)$ as any F-saturated graph is also weakly F-saturated.

There are several general results on saturated and weakly saturated graphs and hypergraphs in print. These include a paper by Kászonyi and Zs Tuza [7] on sparse saturated graphs, results by Sidorowicz [10] and Borowiecki and Sidorowicz [1], Faudree, Gould, and Jacobson [6] on weakly saturated graphs, and papers by Erdős, Füredi, and Tuza [3] on saturated r-uniform hypergraphs, and Pikhurko [9] on weakly saturated hypergraphs. A survey of such results can be found in a paper by J. Faudree, R. Faudree, and Schmitt [5].

No general theory has been developed for weak saturation numbers $\mathbf{wsat}(n, G)$ for graphs comparable to the extensive theory that has been developed for the extremal number $\mathbf{ex}(n, G)$ for graphs. However, it is clear that the minimum degree $\delta(G)$ is a critical parameter in the determination of $\mathbf{wsat}(n, G)$ as the following result indicates.

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Theorem 1 ([6,10]). *Let G be a graph with p vertices, q edges, and minimal degree δ. Then,*

$$q - 1 + (\delta - 1)(n - p)/2 \leq \mathbf{wsat}(n, G) \leq (\delta - 1)n + (p - 1)(p - 2\delta)/2$$

for any $n \geq p$.

Thus, for n sufficiently large, Theorem 1 gives the following corollary.

Corollary 1 ([6,10]). *If G is a graph with p vertices, q edges, and minimal degree δ, then for n sufficiently large, there are constants $c_1 = c_1(p, q)$ and $c_2 = c_2(p, q)$ such that*

$$(\delta - 1)n/2 + c_1 \leq \mathbf{wsat}(n, G) \leq (\delta - 1)n + c_2.$$

One approach to developing a comprehensive theory for weak saturation numbers is to develop many examples of the weak saturation number for classes of graphs, so that the critical graphical parameters related to the weak saturation number can be identified from these examples. In this paper the precise value of $\mathbf{wsat}(n, F)$ for many families of disjoint copies of connected graphs such as small order graphs, trees, cycles, and complete graphs will be determined.

2. Results

We will begin with complete graphs, since $\mathbf{wsat}(n, K_t)$ is known for all $t \geq 3$.

Theorem 2. *For the complete graph K_t for $t \geq 3$ with $k \geq 1$ and $n \geq (k + 1)t$,*

$$\mathbf{wsat}(n, kK_t) = (t - 2)n + k - (t^2 - 3t + 4)/2.$$

Proof. Let B be the balanced bipartite graph with parts B_1 and B_2 each of order $t - 2$, in which the degree sequence of the vertices in each of the B_i for $i = 1, 2$ is $1, 2, \ldots, t - 2$. Thus, B has $\binom{t-1}{2}$ edges. Consider the graph H which is obtained from $(k - 1)K_t \cup K_{t-2}$ by adding a copy of the edges of B between consecutive copies of the complete graphs K_t in some order and ending with K_{t-2}. Thus $k - 1$ copies of B will be added. Let G be the graph obtained from H by adding an independent set with $n - (kt - 2)$ vertices each with $t - 2$ adjacencies in the K_{t-2} of H. Thus, G has n vertices and $(k-1)\binom{t}{2} + (k-1)\binom{t-1}{2} + \binom{t-2}{2} + (n - kt + 2)(t - 2) = (t - 2)n + k - (t^2 - 3t + 4)/2$ edges. The graph $K_{t-2} + \overline{K}_{n-kt+2}$ is K_t saturated. Edges can be added to the graph $2K_t$ with the bipartite graph B added between the copies of K_t, to get a new copy of K_t at each step and ending with a complete graph K_{2t}. Thus, it is straightforward to verify that G is kK_t-weakly saturated. Hence $\mathbf{wsat}(n, kK_t) \leq (t - 2)n + k - (t^2 - 3t + 4)/2$.

To show a lower bound, consider a graph $H \in \mathbf{wsat}(n, kK_t)$. In some order delete edges from a K_t in H until no K_t remains. Denote this graph by H'. Thus, at least $k - 1$ edges will be deleted to form H'. Clearly, H' is K_t weakly saturated, and so by the result in [8], H' has at least $\binom{t-2}{2} + (n - t + 2)(t - 2)$ edges, and so

$$|E(H)| \geq \binom{t-2}{2} + (n - t + 2)(t - 2) + k - 1 = (t - 2)n + k - (t^2 - 3t + 4)/2.$$

This completes the proof of Theorem 2. □

We need the following definition for the next result.

Definition 1. A graph G with p vertices and q edges is *minimum weakly saturated*, if $\mathbf{wsat}(n, G) = q - 1$ for any $n \geq 2p$ and is *rooted minimum weakly saturated* if there a vertex of G, called the root, such that the root remains the same in each of the copies of G obtained when edges are added to obtain the complete graph.

Theorem 3. *Let G be a connected minimum weakly saturated graph with p vertices and q edges. Then, for any integer $k \geq 1$ and $n \geq (k + 1)p$, $\mathbf{wsat}(n, kG) = kq - 1$ and*

$$(k - 1)G \cup (G - e) \cup \overline{K}_{n-pk} \in \underline{\mathbf{wSAT}}(n, kG),$$

where e is any edge of G.

Proof. Any graph in $\underline{\mathbf{wSAT}}(n, kG)$ must contain $(k - 1)G \cup G - e$ for some $e \in G$, and so $\mathbf{wsat}(n, kG) \geq kq - 1$. Since the graph G is minimum weakly saturated, edges can be added to the graph $(G - e) \cup \overline{K}_{n-kp}$ to obtain a complete graph $K_{n-p(k-1)}$. Then, since G is minimum weakly saturated, the remainder of the copies of G can be completed. This completes the proof of Theorem 3. □

Corollary 2. *Let H be an arbitrary connected graph with p vertices and q edges and let T be a rooted minimum saturated tree with p' vertices and with root v. If G is the graph obtained from H by attaching the tree T at the root v, then* $\mathbf{wsat}(n, G)$

$= k(q + p' - 1) - 1$ *for* $n \geq (k + 1)(p + p')$ *and*

$$(k-1)G \cup (G-e) \cup \overline{K}_{n-(p+p'-1)k} \in \underline{\mathbf{wSAT}}(n, kG),$$

where e is any edge of G.

Examples of rooted minimum saturated trees can be found in [6]. One such example is any path, and in particular, a path of length 2 rooted at an end vertex. Thus, any graph H obtained from a connected graph by appending a path of length 2 will be a minimum saturated graph and will satisfy the result of Corollary 2.

It was proved in [6] that almost all trees, if counted as labeled trees, are minimum weakly saturated. Thus, almost all trees T, counted as labeled trees, have the property that $\mathbf{wsat}(n, kT) = k(p-1) - 1$ if T has p vertices and n is sufficiently large.

Theorem 4. *Let G be a connected minimum weakly saturated graph with no cut edge, and with p vertices, q edges and* $\delta(G) = 2$. *Then, for any integer* $k \geq 1$ *and* $n \geq (k+1)p$,

$$\mathbf{wsat}(n, kG) = n + k(q - p + 1) - 2.$$

Also, where e is any edge of G, the graph F obtained from $(k-1)G \cup (G-e) \cup \overline{K}_{n-kp}$ *by adding* $n - kp + k - 1$ *edges to make F connected is in* $\underline{\mathbf{wSAT}}(n, kG)$.

Proof. Let e be an arbitrary edge of G, and consider the graph H obtained from $(k-1)G \cup (G-e) \cup \overline{K}_{n-kp}$ by adding an edge between consecutive copies of $G - e$ and G for a total of $k - 1$ additional edges and also a path of $n - kp$ edges containing $\overline{K}_{n-kp}$ and a vertex of $G - e$. Thus, H contains $(kq - 1) + (k - 1) + (n - kp) = n + k(q - p + 1) - 2$ edges and is connected. Since G is minimum weakly saturated, each of the copies of G closes into complete graphs and $G - e \cup \overline{K}_{n-kp}$ will close into a complete graph with $n - (k-1)p$ vertices. Since H is connected, H will close to a complete graph, since $\delta(G) = 2$. Thus, the graph H is kG weakly saturated and $\mathbf{wsat}(n, G) \leq n - k(q - p + 1) - 2$.

If $H \in \underline{\mathbf{wSAT}}(n, kG)$, then H must contain the graph $(k-1)G \cup (G-e)$ which has $kq - 1$ edges, and H must be connected, which requires and additional $(k-1)+(n-kp)$ edges. Thus, H has at least $(kq-1)+(k-1)+(n-kp) = n+k(q-p+1)-2$ edges. This completes the proof of Theorem 4. □

Theorem 5. *If* $t \geq 3$, $k \geq 1$ *and* $n > (k+1)t$, *then*

$$\mathbf{wsat}(n, kC_t) = \begin{cases} n + k - 2 & \text{if } t \text{ is odd} \\ n + k - 1 & \text{if } t \text{ is even.} \end{cases}$$

Proof. We begin with the case when t is odd. Let H be the graph obtained from the path $P = (x_1, x_2, \ldots, x_n)$ by adding the $k - 1$ chords $x_1x_t, x_{t+1}x_{2t}, \ldots, x_{t(k-2)+1}x_{(k-1)t}$, which has $n + k - 2$ edges. It was proved in [1] that $P_m \in \mathbf{wSAT}(n, C_t)$ for $m \geq 2t + 1$. Thus, the subpath of P of the last $n - (k-1)t$ vertices closes to a complete graph. Then this can be applied again to the last $n - (k-2)t$ vertices to obtain two cycles of length t disjoint from the first $k - 2$ cycles of length t. This continues until H is complete. Thus, $\mathbf{wsat}(n, kC_t) \leq n + k - 2$. To verify that $n + k - 2$ is a lower bound, consider a graph $H \in \mathbf{wsat}(n, kC_t)$. Note that H must be connected, since $\delta(C_t) \geq 2$, and H must contain at least $k - 1$ cycles, and so H must have at least as many as $(n-1) + k - 1$ edges. This completes the proof of the odd case.

Now, we consider the case when t is even. Let H be the graph obtained from the path $P = (x_1, x_2, \ldots, x_n)$ by adding the $k - 1$ chords $x_1x_t, x_{t+1}x_{2t}, \ldots, x_{t(k-2)+1}x_{(k-1)t}$ along with the chord $x_{n-2}x_n$. The graph H has $n + k - 1$ edges. Let P_m^* denote a path with m vertices with one 2-chord at the end of the path. It was proved in [1] that $P_m^* \in \mathbf{wSAT}(m, C_t)$ for $n \geq 2t$. Thus, the subpath of P of the last $n - (k-1)t$ vertices along with the 2-chord closes to a complete graph. Then this can be applied again to the last $n - (k-2)t$ vertices to obtain two cycles of length t disjoint from the first $k - 2$ cycles of length t. This continues until H is complete. Thus, $\mathbf{wsat}(n, kC_t) \leq n + k - 1$. To verify that $n + k - 1$ is a lower bound, consider a graph $H \in \mathbf{wsat}(n, kC_t)$. Since H must be connected because $\delta(C_t) \geq 2$, and H must contain at least $k - 1$ even disjoint cycles, H must have as many as $(n-1) + k - 1$ edges. If H has precisely $n + k - 2$ edges, then H will be a bipartite graph. Any edge added must be on an even cycle, and thus leaves the graph bipartite. Such a graph cannot close to a complete graph. Hence, H must have at least $n + k - 1$ edges. This completes the proof of the even case and Theorem 5. □

Theorem 6. *If* $t \geq 3$, $k \geq 1$ *and* $n > (k+1)(t+1)$, *then* $\mathbf{wsat}(n, kK_{1,t}) = (k-1)t + \binom{t}{2}$ *and*

$$(k-1)K_{1,t} \cup K_t \cup \overline{K}_{n-k(t+1)+1} \in \mathbf{wSAT}(n, kK_{1,t}).$$

Proof. The graph $H = (k-1)K_{1,t} \cup K_t \cup \overline{K}_{n-k(t+1)+1} \in \mathbf{wSAT}(n, kK_{1,t})$. To see this observe that the graph $K_t \cup \overline{K}_{n-k(t+1)+1}$ is easily seen to close to a complete graph with at least $2(t+1)$ vertices. Then, arbitrary edges between the subgraph $(k-1)K_{1,t}$ of H and this complete graph can be added, and after that any edge can be added. Thus, $\mathbf{wsat}(n, kK_{1,t}) \leq (k-1)t + \binom{t}{2}$.

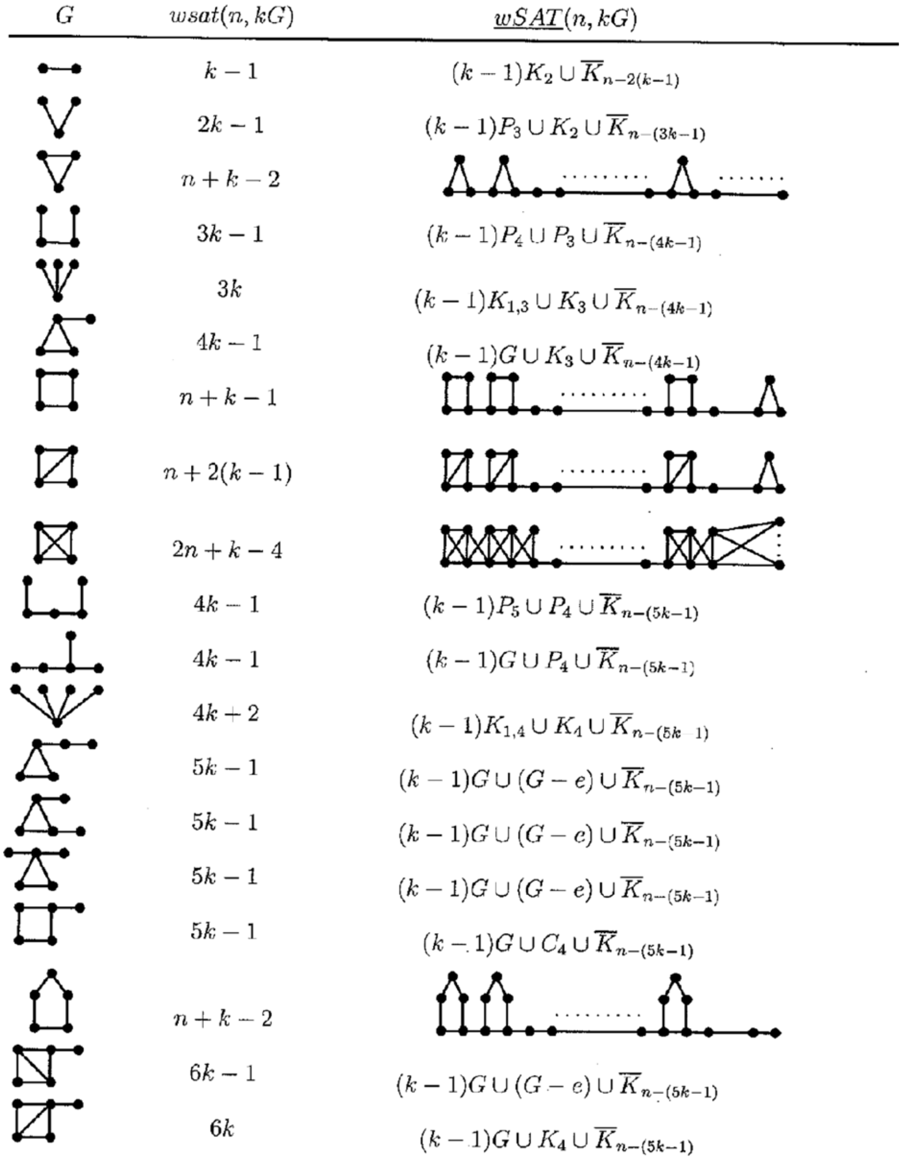

Fig. 1. Small order graphs.

Let H be a graph of order n in $\mathbf{wSAT}(n, kK_{1,k})$, and let $u_1, u_2, \ldots, u_{k-1}$ be the centers of $k-1$ vertex disjoint $K_{1,t}$ stars in H. When the first edge is added to G it must be incident to a vertex v_1 that has degree at least $k-1$ in H with edges disjoint from the $k-1$ disjoint $K_{1,t}$ stars. All edges from $u_1, u_2, \ldots, u_{k-1}$ to the remaining vertices in the graph can also be added at this point. All of the remaining edges that would be incident to v_1 can be added to form H_1. The next edge added to H_1 must be incident to a vertex v_2 that now has degree $k-1$ in H_1, and so v_2 must have degree at least $k-2$ edges in $H - v_1$. Thus the argument can be continued to obtain vertices $v_3, \ldots, v_{k-1}$ such that v_j has degree at least $k-j$ in $H - \{v_1, v_2, \ldots, v_{j-1}\}$. Thus, H has at least $1 + 2 + \cdots + (k-1) = \binom{k}{2}$ edges in addition to the initial $(k-1)t$ edges in the $(k-1)K_{1,t}$. This completes the proof of Theorem 6. □

Theorem 7. *Let $F = T_{p_1} \cup T_{p_2} \cdots \cup T_{p_k}$ be a forest with k trees of order $p_1, p_2, \ldots, p_k$ respectively with $p = \sum_{i=1}^{k} p_i$. If $H \in \underline{\mathbf{wSAT}}(2p, T_{p_k})$, then*

$$\mathbf{wsat}(n, F) \le p - p_k - (k-1) + |E(H)|,$$

and

$$T_{p_1} \cup T_{p_2} \cup \cdots \cup T_{p_{k-1}} \cup H \cup \overline{K}_{n-3p+p_k} \in \mathbf{wSAT}(n, F),$$

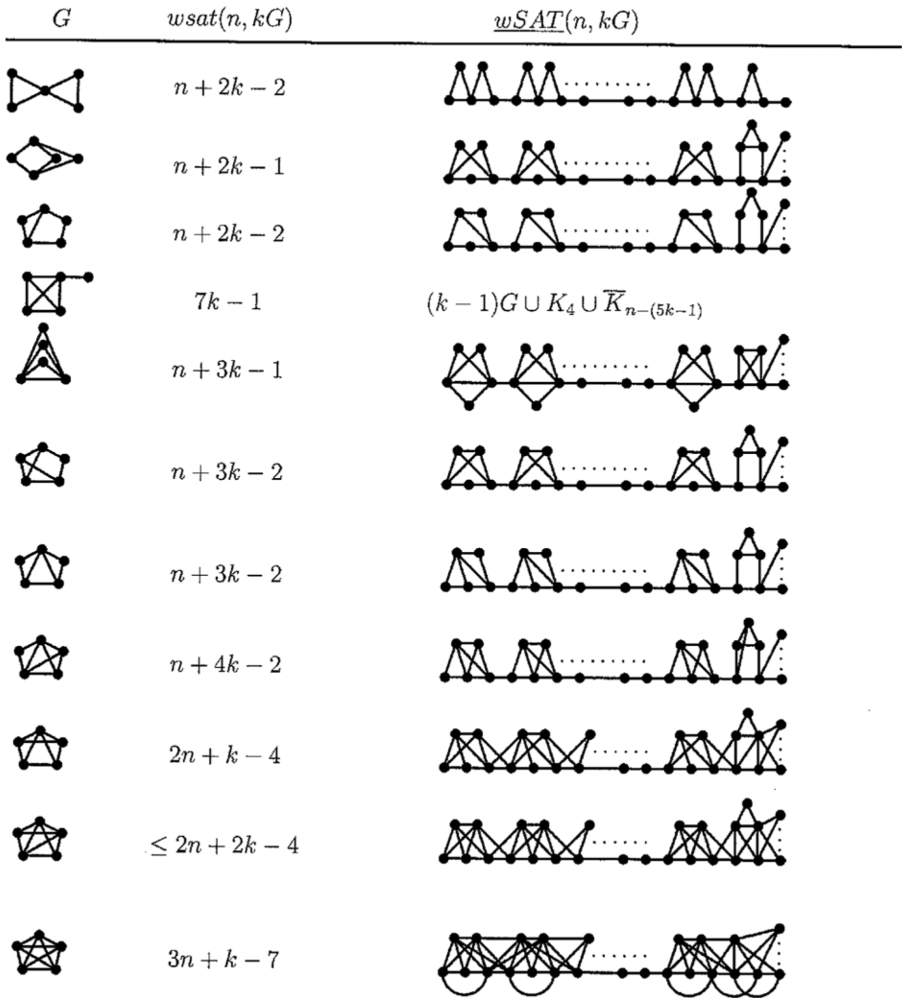

Fig. 1. (*continued*)

for $n \geq 3p$. *If* T_{p_k} *is minimum weakly saturated, then* $\mathbf{wsat}(n, F) = p - k - 1$, *and*

$$T_{p_1} \cup T_{p_2} \cup \cdots \cup (T_{p_k} - e) \cup \overline{K}_{n-p} \in \mathbf{wSAT}(n, F),$$

for e *an edge in* T_{p_k} *and* $n \geq 3p$.

Proof. Since $H \in \mathbf{wsat}(2p, T_{p_k})$, the graph $(T_{p_k} - e) \cup \overline{K}_{n-p}$ will close to a complete graph H'. Then, since each of the trees in F has a vertex of degree 1, any vertex not in H' can be made adjacent to all of the vertices of H'. This will yield a complete graph, and so $\mathbf{wsat}(n, F) \leq p - p_k - (k - 1) + |E(H)|$. In the case when T_{p_k} is minimum weakly saturated, the graph H can be replaced by $T_{p_k} - e$ for any edge $e \in T_{p_k}$, which implies that $\mathbf{wsat}(n, F) \leq p - k - 1$. There is equality, since any graph in $\mathbf{wSAT}(n, F)$ must contain $F - e$ for any edge $e \in F$. This completes the proof of Theorem 7. □

3. Disjoint copies of small order graphs

Using the results of the previous section, the techniques used in that section, and the next theorem (Theorem 8) all the weak saturation numbers of kG for $k \geq 1$ and G a connected graph of order at most 5 can be determined precisely except for $G = K_5 - e$. Fig. 1 gives the weak saturation number of $\mathbf{wsat}(n, G)$ and describes a graph in $\underline{\mathbf{wSAT}}(n, G)$.

Theorem 8. *For* $k \geq 1$ *and n sufficiently large,*

$$\mathbf{wsat}(n, k(K_5 - 2K_2)) = 2n - k + 4.$$

Proof. Let $G = K_5 - 2K_2$, and let B be the balanced bipartite graph with parts B_1 and B_2 of order 2 in which the degree sequence of the vertices in each of the B_i for $i = 1, 2$ is $1, 2$ ($B = P_4$). Thus, B has 3 edges. For any $e \in E(G)$, consider the graph H' which is obtained from $(k-1)G \cup (G-e)$ by adding a copy of the edges of B between consecutive copies of the graphs G in some order and ending with $G - e$. Thus, $k - 1$ copies of B will be added. Let H be the graph obtained from H' by adding an independent set with $n - (5k)$ vertices each with 2 adjacencies in $G - e$ of H. Thus, H has n vertices and $2n+k-4$ edges. It is easily verified that H is weakly G-saturated, since G is self weakly saturated, the bipartite graph between copies of G will cause 2 copies of G to complete, and each of the vertices in the independent set can be adjoined to the complete graph formed from $G - e$. Hence, $\mathbf{wsat}(n, k(K_5 - 2K_2)) \le 2n + k - 4$.

To show a lower bound, consider a graph $H \in \mathbf{wSAT}(n, kG)$. In some order delete edges from a copy of G in H until no G remains. Denote this graph by H'. Thus, at least $k - 1$ edges will be deleted to form H'. Clearly, H' is K_t saturated, and since $\mathbf{wsat}(K_5 - 2K_2) = 2n - 3$ (see [4]), this implies $\mathbf{wsat}(n, k(K_5 - 2K_2)) \ge 2n - 3 + k - 1 = 2n + k - 4$. This completes the proof of Theorem 8. $\square$

The same constructions and calculations made in Theorem 8 for $(K_5 - 2K_2)$ can be made for $(K_5 - K_2)$ using the fact that $\mathbf{wsat}(n, K_5 - K_2) = 2n - 2$ (see [6]). They yield the following inequality: $2n + k - 3 \le \mathbf{wsat}(n, K_5 - K_2) \le 2n + 2k - 4$.

4. Question

In Sections 2 and 3 it was shown for $k \ge 2$ and n sufficiently large that $ws(n, kG) = ws(n, G) + k - 1$ when G is a complete graph, a cycle, or $K_5 - K_2$. This leads to the following natural question.

Question 1. *For $k \ge 2$ and n sufficiently large for which connected graphs G is $ws(n, kG) = ws(n, G) + k - 1$?*

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