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FORBIDDEN SUBGRAPHS AND HAMILTONIAN PROPERTIES OF GRAPHS

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Various sufficient conditions are given, in terms of forbidden subgraphs, that imply a graph is either homogeneously traceable, hamiltonian or pancyclic.

We consider only finite undirected graphs without loops or multiple edges. Notation or terms not defined here can be found in [1]. Let G be a graph and let $S \subseteq V(G)$. The *subgraph* $\langle S \rangle$ *induced by* S is the graph with vertex set S and whose edge set consists of those edges of G incident with two vertices of S. The distance $d(u, v)$ between vertices u and v in a connected graph G is the minimum number of edges in a $u-v$ path. The *diameter* of a graph G is $\max_{u,v \in V(G)} d(u, v)$. A graph is *hamiltonian* (*traceable*) if it has a cycle (path) containing all its vertices. A *pancyclic* graph of order p contains a cycle of length l for each l $(3 \leq l \leq p)$. A graph is *panconnected* if, for each pair u, v of distinct vertices, there is a $u-v$ path of length l for each l $(d(u, v) \leq l \leq p-1)$. A graph G is *homogeneously traceable*, if, for each vertex v in G, there exists a hamiltonian path with initial vertex v. Homogeneously traceable nonhamiltonian graphs exist for all orders p, except $3 \leq p \leq 8$ (see [2]).

The following implications are well-known and the reverse implications fail to hold:

$$\text{panconnected} \Rightarrow \text{pancyclic} \Rightarrow \text{hamiltonian} \Rightarrow \text{homogeneously traceable.}$$

Let Z_i be that graph obtained by identifying a vertex of K_3 and an end-vertex of P_{i+1}. Note also that Z_{i+1} is that graph obtained by subdividing a bridge of Z_i.

Theorem A [4]. *If G is a 2-connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_1, then G is hamiltonian.*

We note that the proof of Theorem A actually shows that either G is a cycle or

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G is pancyclic. We now show that slightly more general conditions yield the same result.

Theorem 1. *If G is a 2-connected graph that contains no induced subgraphs isomorphic to $K_{1,3}$ or Z_2, then G is a cycle or pancyclic.*

Proof. The result is trivial if $|V(G)| \leq 4$. So suppose G satisfies the hypothesis, is not a cycle, and has order at least 5. Let C: $v_0 v_1 \cdots v_{k-1} v_0$ $(k \geq 2)$ be an arbitrary cycle of length k. We show that if $V(G) \neq V(C)$, then a $(k+1)$-cycle can be found.

Since G is not a cycle and contains no induced $K_{1,3}$, then it must contain a 3-cycle. Since G is not connected there exists $x_0 \in V(G) - V(C)$ such that x_0 is adjacent to a vertex of C. Without loss of generality we may assume x_0 is adjacent to v_1 (or else relabel C). There is no induced $K_{1,3}$ in G so the edge $v_0 v_2$ is in G or a $(k+1)$-cycle would be produced. Since G is 2-connected, there exists an $x_0 - v_2$ path P, not containing v_1. Consider the subpath P': $x_0 x_1 \cdots x_l v_j$ where v_j is the first vertex of C on P.

Case 1. Suppose $v_j = v_2$ (or by symmetry $v_j = v_0$). The graph $\langle\{v_0, v_1, v_2, x_0, x_1\}\rangle \cong Z_2$ unless one of the edges $x_1 v_0$, $x_1 v_1$ or $x_1 v_2$ is present. Consider $x_1 v_2$ (and note a similar argument holds for $x_1 v_0$). If $v_{k-1} = v_2$, then $x_0 x_1 v_2 v_1 x_0$ is a cycle of length 4. If $v_{k-1} \neq v_2$, then $\langle\{v_1, v_2, v_3, x_l\}\rangle$ implies that $v_1 v_3 \in E(G)$. If $v_{k-1} = v_3$, then $v_0 v_1 x_0 x_1 v_2 v_0$ is a 5-cycle. If $v_{k-1} \neq v_3$, we consider $\langle\{v_0, v_1, v_3, x_0\}\rangle$. If $x_0 v_3$ is an edge of G, then $v_0 v_2 x_1 x_0 v_3 v_4 \cdots v_{k-1} v_0$ is a $(k+1)$-cycle. Hence we see that $v_0 v_3$ is an edge of G. The graph $\langle\{v_0, v_2, v_3, x_1\ x_0\}\rangle \cong Z_2$ unless one of the edges $x_0 v_3$, $x_1 v_0$ or $x_1 v_3$ is in G. But the inclusion of any of these edges produces a $(k+1)$-cycle.

If $x_1 v_1 \in E(G)$, we may consider the path P'': $x_1 x_2 \cdots x_l v_2$ and repeat the above argument. Eventually, either the previous possibility results or a $(k+1)$-cycle is formed.

Case 2. Suppose $3 \leq j \leq k-1$. Since G contains no induced Z_2, it is easily seen that for some i $(0 \leq i \leq l)$, $v_1 x_i$ and $x_i v_j$ are edges of G. Observe that $v_2 \neq v_{j-1}$ and $v_{k-1} \neq v_j$ or a $(k+1)$-cycle can be found. Further, $v_{j-1} v_{j+1} \in E(G)$ since G contains no induced $K_{1,3}$.

Consider $\langle\{v_0, v_1, v_2, x_i, v_j\}\rangle$. It follows that either $v_j v_0 \in E(G)$ in which case

$$v_0 v_{k-1} \cdots v_{j+1} v_{j-1} \cdots v_1 x_i v_j v_0$$

is a $(k+1)$-cycle or $v_j v_2 \in E(G)$, and

$$v_0 v_{k-1} \cdots v_{j+1} v_{j-1} \cdots v_2 v_j x_i v_1 v_0$$

is a $(k+1)$-cycle. Thus $v_j v_1 \in E(G)$ and $\langle\{v_1, x_i, v_j, v_2, v_3\}\rangle \cong Z_2$ unless at least one of $x_i v_2$, $x_i v_3$, $v_j v_2$, $v_j v_3$ or $v_1 v_3$ is an edge of G. The first three yield immediate $(k+1)$-cycles. If $v_j v_3 \in E(G)$, then

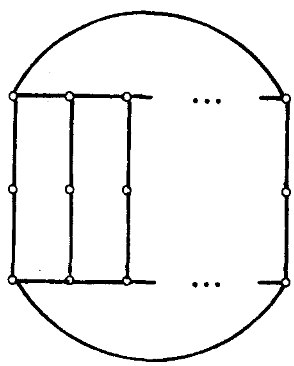
$$v_0 v_{k-1} \cdots v_{j+1} v_{j-1} \cdots v_3 v_j x_i v_1 v_2 v_0$$

is a $(k+1)$-cycle.

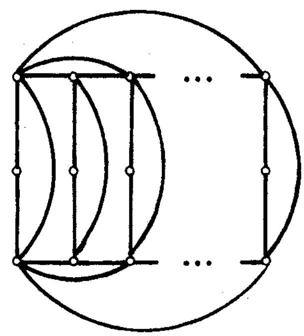
Finally, if $v_1v_3 \in E(G)$, then $\langle\{v_0, v_1, v_2, v_j, v_{j-1}\}\rangle \cong Z_2$ unless one of v_0v_j, v_2v_j, v_0v_{j-1}, v_2v_{j-1} or v_1v_{j-1} is in G. But each edge produces a $(k+1)$-cycle and the result follows. □

Remark 1. We note that the hypothesis of Theorem 1 does not imply that the graph is panconnected. Consider K_n ($n \geq 3$) with vertices $x_1, x_2, \ldots, x_n$. Let G be that graph obtained by subdividing the edge x_1x_2 and name the new vertex x. This graph does not contain an induced $K_{1,3}$ or Z_2 yet there is no $x-x_1$ path of length 2.

Furthermore, we cannot omit either of the induced subgraphs from the hypothesis. Fig. 1(a) shows a nonhamiltonian graph with no induced Z_2. It is constructed by taking two copies of C_{2n+1} ($n > 1$) and joining corresponding vertices in each copy by a path of length 2. Fig. 1(b) shows a nonhamiltonian graph with no induced $K_{1,3}$. It is constructed by taking two copies of K_{2n+1} ($n > 1$) and joining corresponding vertices in each copy by an edge and a path of length 2.



(a) Contains no Z_2.



(b) Contains no $K_{1,3}$.

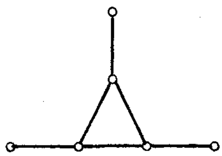
Fig. 1.

The following was shown in [3].

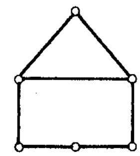
Theorem B. *If G is a 2-connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or F (see Fig.* 2(a)), *then G is hamiltonian.*

We note that the conditions of this theorem are not enough to imply that the graph be pancyclic; for example the graph of Fig. 2(b) is 2-connected, contains no induced subgraph isomorphic to $K_{1,3}$ or F and is not pancyclic. However it is easily shown that the hypothesis of Theorem 1 implies the hypothesis of Theorem B. The proof is routine and not included.

Proposition 2. *If G is 2-connected and contains no induced subgraph isomorphic to $K_{1,3}$ or Z_2, then G contains no induced subgraphs isomorphic to F.*



(a) The graph F.



(b) A nonpancyclic graph.

Fig. 2.

The graph in Fig. 1(b) shows that Z_2 cannot be replaced by Z_3 in Theorem 1. In Theorem 3 we modify the set of forbidden subgraphs to include Z_3, but the conclusion is weaker than that of Theorem 1. Let B be that graph obtained by identifying a vertex in two distinct copies of K_3.

Theorem 3. *If G is a 2-connected graph that contains no induced subgraph isomorphic to $K_{1,3}$, B, or Z_3, then G is hamiltonian.*

Proof. Suppose G satisfies the hypothesis and is not hamiltonian. Choose a cycle C: $v_0v_1 \cdots v_kv_0$ ($k>2$) of maximum length in G. Let $x_0 \in V(G)-V(C)$ such that x_0 is adjacent to a vertex of C. Without loss of generality suppose x_0 is adjacent to v_1. Since $K_{1,3}$ is not an induced subgraph of G, $v_0v_2 \in E(G)$ or a longer cycle would be present. Since G is 2-connected, there must exist an x_0-v_2 path P that does not contain v_1. Suppose v_j is the first vertex of P on C. It is clear that $v_{j-1}v_{j+1} \in E(G)$ and that $j \neq 0, 2, 3, 4, k-1$, or k, for otherwise C would not be of maximum length. Since G contains no induced Z_3, we can find a v_1-v_j path P', that is a subpath of P, and is disjoint from $V(C)-\{v_1, v_j\}$. Further, the length of P' is at most 3.

Suppose the length of P' is 3, that is, suppose P' is the path $v_1x_0x_1v_j$. By considering $\langle\{v_0, v_1, v_2, x_0, x_1, v_j\}\rangle$, one can readily establish that v_1v_j is an edge of G. The subgraph $\langle\{v_1, v_2, x_0, v_j\}\rangle \cong K_{1,3}$ unless v_2v_j is an edge of G. But now the cycle

$$v_0v_1x_0x_1v_jv_2v_3 \cdots v_{j-1}v_{j+1}v_kv_0$$

has length longer than C which is a contradiction.

Therefore we may assume that the length of P' is 2, that is, suppose P' is the path $v_1x_0v_j$. We first consider the graph $\langle\{v_0, v_1, v_2, x_0, v_j, v_{j+1}\}\rangle$. We need only consider whether v_0v_j, v_1v_j, v_2v_j, v_0v_{j+1}, v_1v_{j+1} or v_2v_{j+1} are edges of G. If any one of v_0v_j, v_2v_j, v_1v_{j+1}, or v_2v_{j+1} is an edge of G, a cycle longer than C is immediately produced. If $v_1v_j \in E(G)$, then $\langle\{v_0, v_1, v_2, x_0, v_j\}\rangle \cong B$, unless G contains an edge previously considered. Thus v_0v_{j+1} is an edge of G and we next consider $\langle\{v_0, v_j, v_{j+1}, v_{j+2}\}\rangle$. If v_jv_{j+2} is an edge of G, then either $\langle\{v_{j-1}, x_0, v_j, v_{j+2}\}\rangle \cong K_{1,3}$ or a longer cycle exists. The edge v_0v_j has already been handled; hence we conclude that v_0v_{j+2} is an edge of G. Now the graph $\langle\{v_0, v_{j+2}, v_{j+1}, v_j, v_{j-1}\}\rangle \cong B$

unless an edge already analyzed exists in G. Thus, all cases produce a cycle longer than C, and hence we conclude that G is hamiltonian □

Remark 2. The hypothesis of Theorem 3 does not imply that of Theorem B (see Fig. 3).

We also note that the hypothesis of Theorem 3 does not imply the graph is pancyclic (see Fig. 2(b)).

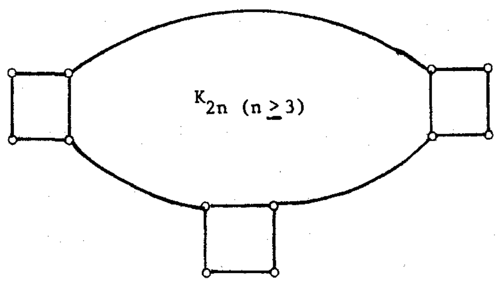


Fig. 3. A graph with no induced $K_{1,3}$, Z_3 or B that contains an induced F.

Theorem C [5]. *If G is a 2-connected graph of diameter at most 2 that contains no induced subgraph isomorphic to $K_{1,3}$, then G is hamiltonian.*

We note that 2-connected nontraceable graphs of diameter 3 with no induced $K_{1,3}$ can be found (see Fig. 4). However, we may modify the set of forbidden subgraphs to produce homogeneously traceable graphs of diameter at most 3.

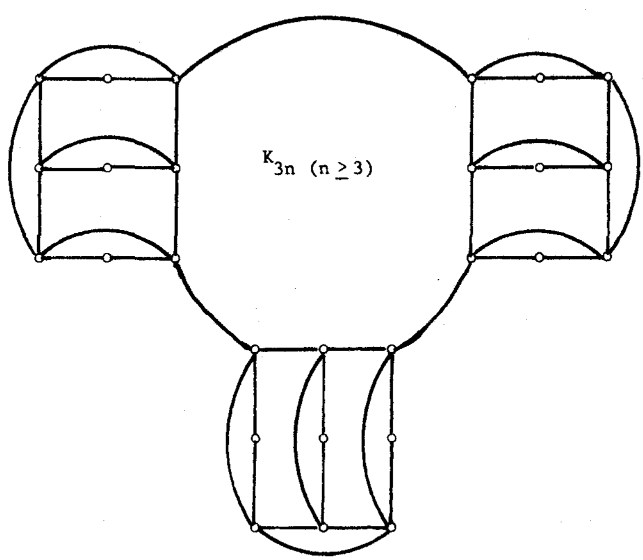


Fig. 4. A graph with diameter 3 and no induced $K_{1,3}$ that is not traceable.

Theorem 4. *If G is a 2-connected graph of diameter at most 3 and G contains no induced subgraph isomorphic to $K_{1,3}$ or B, then G is homogeneously traceable.*

Proof. By Theorem C we need only consider graphs of diameter 3. So suppose G satisfies the hypothesis of the Theorem but is not homogeneously traceable. Thus there exists a vertex v_0 that is not the initial vertex of a hamiltonian path in G. Let $P: v_0v_1 \cdots v_n$ be a longest path with initial vertex v_0. Thus there exists $x \in V(G)-V(P)$ such that x is adjacent to a vertex v_i of P. Furthermore, we may assume $i<n$ or a longer path would be evident and we may assume $0<i$ or else v_0 would be a cutvertex. Observe that $v_{i-1}v_{i+1} \in E(G)$. Hence $i<n-1$. Since $2 \leq d(x, v_n)<3$ we consider two cases.

Case 1. Suppose $d(x, v_n)=2$. Since v_n must be adjacent only to vertices of P, the vertex intermediate to x and v_n on the distance path must lie on P. If xv_0v_n is the path of length 2, then $\langle\{x, v_0, v_1, v_n\}\rangle \not\cong K_{1,3}$ implies that v_1v_n is an edge of G. It follows that the path $v_0xv_iv_{i+1} \cdots v_nv_1v_2 \cdots v_{i-1}$ is longer than P and has initial vertex v_0. Thus we may assume xv_iv_n $(0<i<n)$ is the distance path, or we would merely change our choice of v_i above. Now $v_0v_1 \cdots v_{i-1}v_{i+1}v_{i+2} \cdots v_nv_ix$ is a path longer than P.

Case 2. Suppose $d(x, v_n)=3$. Let $xv_iv_jv_n$ be the $x-v_n$ distance path. Clearly $v_i, v_j \in P$.

Subcase 2a. Suppose $j=0$. Since $\langle\{v_0, v_1, v_i, v_n\}\rangle \not\cong K_{1,3}$ at least one of v_1v_i, v_iv_n or v_1v_n is an edge of G. Except for v_1v_n longer paths are easily established. So suppose $v_1v_n \in E(G)$ and consider $\langle\{v_0, v_i, v_{i+1}, x\}\rangle$. The edges xv_{i+1} and xv_0 yield longer paths easily; while if $v_0v_{i+1} \in E(G)$ the path $v_0v_{i+1}v_{i+2} \cdots v_nv_1v_2 \cdots v_{i-1}v_ix$ is a longer path.

Subcase 2b. Suppose $0<j<i-1$. By considering $\langle\{v_j, v_{j+1}, v_i, v_n\}\rangle$ and $\langle\{v_j, v_{j-1}, v_i, v_n\}\rangle$ we must have v_iv_{j-1} and $v_{j+1}v_n$ as edges of G. Since $\langle\{v_j, v_{j-1}, v_i, v_{j+1}, v_n\}\rangle \not\cong B$ at least one of $v_{j-1}v_{j+1}$, $v_{j-1}v_n$, v_iv_{j+1} and v_iv_n is an edge of G. However, a longer path results in each case.

Subcase 2c. Suppose $j=i-1$. The path $v_0v_1 \cdots v_{i-1}v_nv_{n-1} \cdots v_ix$ is a longer path.

Subcase 2d. Suppose $j=i+1 \neq n-1$ $(j=n-1$ is Subcase 2f). Since $\langle\{v_i, v_{i+1}, v_{i+2}, v_n\}\rangle \not\cong K_{1,3}$ we can conclude that $v_nv_{i+2} \in E(G)$. But since $\langle\{v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_n\}\rangle \not\cong B$ at least one of $v_{i-1}v_{i+2}$, $v_{i-1}v_n$, v_iv_{i+2} or v_iv_n is in G. Again a longer path is easily established in each case.

Subcase 2e. Suppose $i+1<j<n-1$. The argument for this case is analogous to that of Subcase 2b.

Subcase 2f. Suppose $j=n-1$. With this being the final possibility we may assume for each longest path P of length n with initial vertex v_0 and final vertex v_n and any vertex x not on P, that $d(x, v_n)=3$ and $xv_iv_{n-1}v_n$ is a distance path where $1 \leq i \leq n-2$. Since G is 2-connected v_n is adjacent to some vertex of P, say v_k. Clearly $k \neq i-1$ or i. If $0<k<i-1$ and since $\langle\{v_k, v_n, v_{k+1}, v_{k-1}\}\rangle \not\cong K_{1,3}$, then v_nv_{k-1}, v_nv_{k+1} or $v_{k-1}v_{k+1}$ is in G. Longer paths are immediate for the first two; so

suppose $v_{k+1}v_{k-1}$ is an edge of G. Then

$$Q: v_0v_1 \cdots v_{k-1}v_{k+1} \cdots v_nv_k$$

is a longest path with initial vertex v_0. Further, $x \notin V(Q)$ so the distance from x to v_n is 2. But this contradicts the fact that $d(x, v_n) = 3$.

A similar argument applies when $i < k < n-1$.

Hence we may additionally assume that the last vertex of every longest path with initial vertex v_0 is adjacent to v_0. Consider

$$Q: v_0v_1 \cdots v_{i-1}v_{i+1} \cdots v_{n-1}v_ix.$$

Now Q is a longest path with initial vertex v_0 and end vertex x so that $xv_0 \in E(G)$. Then $\langle\{v_0, x, v_1, v_n\}\rangle \cong K_{1,3}$ unless one of xv_1, xv_n or v_1v_n is an edge of G. In any case a new path longer than Q (or P) is apparent. □

Since homogeneously traceable nonhamiltonian graphs have no vertices adjacent to two or more vertices of degree 2, the following is immediate.

Corollary 5. *If G is a 2-connected graph of diameter at most 3 that contains no induced subgraph isomorphic to $K_{1,3}$ or B and G contains a vertex adjacent to exactly two vertices of degree 2, then G is hamiltonian.*

Remark 3. The supposition that the diameter be 3 or less in Theorem 4 cannot be weakened. The family of graphs shown in Fig. 5 is constructed by taking a copy of K_m ($m \geq 3$) and a copy of K_n ($n \geq 3$) and joining vertices with the paths shown. These graphs are not homogeneously traceable as x is not the initial vertex of a hamiltonian path. However, these graphs have diameter 4 and contain no induced $K_{1,3}$ or B.

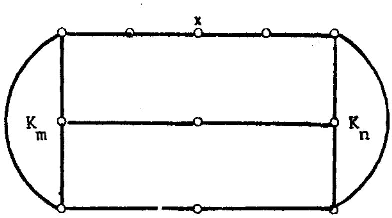


Fig. 5.

We also note that 2-connectedness is necessary in Theorems 1, 3 and 4. The graph F of Fig. 2(a) contains none of the forbidden subgraphs of Theorems 1, 3 and 4, but is not traceable.

Conclusion

We feel that 'forbidden subgraphs' offer an interesting approach to 'hamiltonian' problems. We would like to point out some possible directions.

The graph $K_{1,3}$ plays a major role in the results of this paper as well as those in [3–7]. Can results be found that do not use $K_{1,3}$? Perhaps $K_{1,n}$ $(n>3)$ can be of some help.

In [7], Oberly and Sumner and [6] Kanetkar and Rao combine forbidden subgraphs with local connectivity. Can forbidden subgraphs be combined with other degree restrictions to yield new results?

A simple alteration to Fig. 1(b) shows that the diameter restriction of Theorem 4 cannot be dropped. Can a reasonable set of forbidden subgraphs be found that eliminates the need for this restriction?

Acknowledgement

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