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Pancyclicity of 4-connected {claw, generalized bull}-free graphs

Michael Ferrara [a,*,1], Silke Gehrke [b], Ronald Gould [b], Colton Magnant [c], Jeffrey Powell [d]

[a] Department of Mathematical and Statistical Sciences, University of Colorado Denver, Denver, CO 80217, United States
[b] Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, United States
[c] Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, United States
[d] Mathematics and Computer Science, Samford University, Birmingham, AL 35229, United States

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ABSTRACT

A graph G is pancyclic if it contains cycles of each length ℓ, $3 \leq \ell \leq |V(G)|$. The generalized bull $B(i, j)$ is obtained by associating one endpoint of each of the paths P_{i+1} and P_{j+1} with distinct vertices of a triangle. Gould, Łuczak and Pfender (2004) [4] showed that if G is a 3-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 4$ then G is pancyclic. In this paper, we prove that every 4-connected, claw-free, $B(i, j)$-free graph with $i + j = 6$ is pancyclic. As the line graph of the Petersen graph is $B(i, j)$-free for any $i+j = 7$ and is not pancyclic, this result is best possible.

1. Introduction

All graphs in this paper are simple. A graph G is *hamiltonian* if it contains a spanning cycle, and is *pancyclic* if it contains cycles of each length ℓ, $3 \leq \ell \leq |V(G)|$. We consider all cycles to have an implicit clockwise orientation. With this in mind, given a cycle C and a vertex x on C, we let x^+ denote the successor of x under this orientation and let x^- denote the predecessor. We define x^{+i} recursively with $x^{+1} = x^+$ and $x^{+(i+1)} = (x^{+i})^+$ for $i > 1$ and define x^{-i} analogously. For any other vertex y on C, we let xCy denote the path from x to y on C in the clockwise direction of the orientation and xC^-y denote the path from x to y on C in the counterclockwise direction. When convenient, we will also let $C(x, y)$ denote $V(x^+Cy^-)$, that is, the set of vertices lying between x and y on C when traversed in the clockwise direction. We will use the term *arc* to describe these paths on a cycle. Given a subgraph H of G and a vertex $v \in G - H$, by a $v - H$ path we mean a path P with endpoints v and $w \in H$ such that $P \cap H = \{w\}$. For a set of vertices A in G and a subgraph H of G, we let $N_G(A)$ denote the neighborhood of A in G and $N_H(A)$ denote the neighborhood of A in H. When $A = \{x\}$, we write $N_G(x)$ and $N_H(x)$, respectively. Furthermore let $d_G(x) = |N_G(x)|$ and $d_H(x) = |N_H(x)|$.

Given a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$*-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F} = \{K_{1,3}\}$, then G is said to be *claw-free*. The *net*, N, is the graph obtained by attaching a pendant vertex to each vertex in a triangle. The *generalized net* $N(i, j, k)$ is obtained by associating one endpoint of each of the paths P_{i+1}, P_{j+1} and P_{k+1} with distinct vertices of a triangle. We refer to the generalized net $N(i, j, 0)$ as the *generalized bull*, and denote this by $B(i, j)$.

The following well-known conjecture of Matthews and Sumner [8] has provided the impetus for a great deal of research into the hamiltonicity of claw-free graphs.

* Corresponding author.
E-mail addresses: michael.ferrara@ucdenver.edu (M. Ferrara), gehrkesilke@gmail.com (S. Gehrke), rg@mathcs.emory.edu (R. Gould), cmagnant@georgiasouthern.edu (C. Magnant), jspowel1@samford.edu (J. Powell).
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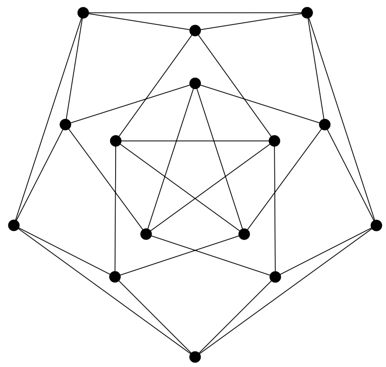

Fig. 1. The line graph of the Petersen graph.

Conjecture 1.1 (*The Matthews–Sumner Conjecture*)**.** *If G is a 4-connected claw-free graph, then G is hamiltonian.*

In [10] Ryjáček demonstrated that this was equivalent to a conjecture of Thomassen [14] that every 4-connected line graph is hamiltonian. Also in [10], Ryjáček showed that every 7-connected, claw-free graph is hamiltonian. More recently, in [6], Kaiser and Vrána showed that every 5-connected claw-free graph G with minimum degree at least six is hamiltonian, which currently represents the best general progress towards affirming Conjecture 1.1. As the general conjecture has proven difficult, a number of authors have considered the hamiltonicity of $\{K_{1,3}, G'\}$-free graphs for various choices of G'. These include proofs that every 4-connected $\{K_{1,3}, H\}$-free graph is hamiltonian when H is the hourglass [1] or a chain of three triangles [9], as well as results that any 3-connected $\{K_{1,3}, P_{11}\}$-free [15] graph is hamiltonian.

In this paper, we are not only interested in the hamiltonicity of highly connected claw-free graphs, but also in their pancyclicity. Significantly fewer results of this type can be found in the literature, in part because it has been shown in many cases [11,12] that closure techniques such as those in [10] do not apply to pancyclicity.

In [13], Shepherd showed the following, which extended a well-known result of Duffus, Gould and Jacobson [2].

Theorem 1.2. *Every 3-connected, $\{K_{1,3}, N\}$-free graph is pancyclic.*

Gould, Łuczak and Pfender [4] obtained the following characterization of forbidden pairs of subgraphs that imply pancyclicity in 3-connected graphs. Here Ł denotes the graph obtained by connecting two disjoint triangles with a single edge.

Theorem 1.3. *Let X and Y be connected graphs on at least three vertices such that neither X nor Y are P_3 and Y is not $K_{1,3}$. Then the following statements are equivalent:*

1. *Every 3-connected $\{X, Y\}$-free graph G is pancyclic.*
2. *$X = K_{1,3}$ and Y is a subgraph of one of the graphs from the family*

$$\mathcal{F} = \{P_7, Ł, B(4, 0), B(3, 1), B(2, 2), N(2, 1, 1)\}.$$

The Matthews–Sumner conjecture and Theorem 1.3 together inspire the following general question.

Problem 1. Characterize those pairs of graphs (X, Y) such that every 4-connected, (X, Y)-free graph is pancyclic.

In [3], the following was shown.

Theorem 1.4. *Every 4-connected, claw-free, P_{10}-free graph is either pancyclic or is the line graph of the Petersen graph. Consequently, every 4-connected, claw-free, P_9-free graph is pancyclic.*

The line graph of the Petersen graph is 4-connected and contains no cycle of length 4 (see Fig. 1).

The main result of this paper is the following, which represents new progress towards Problem 1.

Theorem 1.5. *Every 4-connected $\{K_{1,3}, B(i, j)\}$-free graph, where $i + j = 6$, is pancyclic.*

As the line graph of the Petersen graph is $B(i, j)$-free for all $i + j = 7$, this result is best possible in the sense that the condition on $i + j$ could not be increased.

2. Proof of Theorem 1.5

Before we proceed, we introduce some additional notation. For the remainder of the paper, we will let $\langle w + xyz\rangle$ denote a $K_{1,3}$ in G, induced or otherwise, with center vertex w and pendant vertices x, y and z. Also, we let $N(xyz; x_1 \cdots x_i, y_1 \cdots y_j, z_1 \cdots z_k)$ denote a copy of $N(i, j, k)$ with central triangle xyz and appended paths $xx_1 \cdots x_i$, $yy_1 \cdots y_j$, and $zz_1 \cdots z_k$. A copy of the bull $B(i, j)$ is denoted $B(xyz; x_1 \cdots x_i, y_1 \cdots y_j)$ where xyz is the central triangle with appended paths $xx_1 \cdots x_i$ and $yy_1 \cdots y_j$.

The following two results allow us to establish the hamiltonicity of the graphs under consideration.

Theorem 2.1 (*Hu and Lin [5]*). *If G is a 3-connected, $\{K_{1,3}, N(5,2,2)\}$- or $\{K_{1,3}, N(4,3,2)\}$-free graph, then G is hamiltonian.*

Theorem 2.2 (*Lai et al. [7]*). *If G is a 3-connected, $\{K_{1,3}, N(8,0,0)\}$-free graph, then G is hamiltonian.*

By these results, we immediately get the following corollary which provides hamiltonicity of all graphs considered in this paper.

Corollary 2.3. *If G is a 3-connected, $\{K_{1,3}, B(6,0)\}$-, $\{K_{1,3}, B(5,1)\}$-, $\{K_{1,3}, B(4,2)\}$-or $\{K_{1,3}, B(3,3)\}$-free graph, then G is hamiltonian.*

Our strategy for the proof of Theorem 1.5 is to show that for $t \geq 4$ the presence of a t-cycle in our graph implies the existence of a $(t-1)$-cycle. In the absence of such a cycle, we show that the graph contains either an induced $K_{1,3}$ or each of $B(6,0)$, $B(5,1)$, $B(4,2)$ and $B(3,3)$. Given a cycle C, an edge $xy \notin C$ with $x, y \in V(C)$ is called a *chord* of C, and x and y are called *chordal vertices* of C. A *hop* is a chord xy of C where there is exactly one vertex between x and y on C.

Lemma 2.4. *Let G be a 4-connected $K_{1,3}$-free graph containing a cycle C of length $t \geq 4$. If C has a chord or if there is a vertex $w \in G \setminus C$ with at least 4 neighbors on C, then G contains another cycle C' of length $t-1$.*

Proof. Given a cycle C, a path P with endpoints x and y such that $V(P) \cap V(C) = \{x, y\}$ *shortens* xCy if $|V(P)| < |xCy|$. In this case we say that P is a *shortening path* that *covers* the arc xCy. Note that a chord of C is certainly a shortening path, but other paths may be as well. Let X denote the set of vertices on C that are not incident to a chord of C, and call any vertex in $V(C) - X$ a *chordal* vertex of C.

Let C be a cycle as given in the statement of the lemma and note that we may assume C has no hops. We would now like to show that there exist a pair of (not necessarily disjoint) shortening paths of C, each of length at most two, that shorten disjoint arcs of C. Recall that either C has a chord, or there is some vertex $w \in G - C$ such that $d_C(w) \geq 4$. Assume the latter, and note that since G is claw-free and has no hops, each vertex with a neighbor x on C must also be adjacent to either x^+ or x^-. The assumption that $d_C(w) \geq 4$ implies that there must be two pairs of vertices in $N_C(w)$ that are consecutive on C. Let w_1, w_1^+, w_2 and w_2^+ denote these vertices, and note that w_2 is neither w_1^{+2} nor w_1^{+3} and similarly that w_1 is neither w_2^{+2} nor w_2^{+3} as any of these possibilities results in a cycle of length $t-1$ in G. Thus $w_1 w w_2^+$ and $w_1^+ w w_2$ comprise the desired shortening paths.

If there is no vertex outside C with four neighbors, then by the conditions of the lemma, C must have at least one chord. Among all chords of C, choose the chord xy so that $|xCy|$ is a minimum. We will show that we can either find a cycle of length $t-1$ or that there are in fact two vertex-disjoint, non-crossing chords. Now, to avoid the induced claw $\langle y + y^- x y^+ \rangle$, we must have that $xy^+ \in E(G)$ as the edge xy^- would create a chord with $|xCy^-| < |xCy|$. Similarly, to avoid the induced claw $\langle x + x^- y x^+ \rangle$, we have $x^- y \in E(G)$. To avoid the induced claw $\langle y + y^- x^- y^+ \rangle$, either $x^- y^- \in E(G)$ or $x^- y^+ \in E(G)$ since C has no hops. If $x^- y^- \in E(G)$, then the cycle $x^- y^- C^- x y^+ C x^-$ is the desired cycle of length $t-1$. If $x^- y^- \notin E(G)$ and $x^- y^+ \in E(G)$, then the chords xy and $x^- y^+$ are the desired vertex-disjoint, non-crossing chords. Note that we can consider these chords as shortening paths that cover disjoint arcs of C.

We now select two shortening paths P_L and P_R of length at most two which cover disjoint arcs of C. Let x_L and y_L (respectively x_R and y_R) denote the endpoints of P_L (resp. P_R). In particular, assume that x_R, y_R, x_L and y_L appear in that order when C is traversed in the clockwise direction where $x_R y_R \notin E(G)$ and $x_L y_L \notin E(G)$. We select P_L and P_R such that $|x_L C y_L \cap X| + |x_R C y_R \cap X|$ is minimum and, subject to this, such that $|x_L C y_L| + |x_R C y_R|$ is minimum. As each chord of C is a shortening path, this implies that there is no chord of C with both endpoints in $x_R C y_R$, with the possible exception of $x_R y_R$, and we may draw a similar conclusion about $x_L C y_L$. Finally, without loss of generality suppose that $x_R C y_R$ contains at least as many vertices of X as $x_L C y_L$.

Now, let C_L denote the cycle $y_L C x_L P_L y_L$, that is, the shortening of C obtained via P_L. Recall that every chordal vertex in $x_L C y_L$ must have a neighbor in $y_L^+ C x_L^-$. Thus, as G is claw-free and C has no hops, any chordal vertex x in $x_L C y_L$ must be adjacent to some vertices y and y^+ in $y_L C x_L$. Thus, it is possible to increase the length of C_L by one by inserting x between y and y^+. Inserting all chordal vertices from $x_L C y_L$ into C_L allows the creation of cycles of lengths $|C_L|$ to $t - |x_L C y_L \cap X|$. If no vertices in $x_L C y_L$ are also in X, then this allows us to construct a $t-1$ cycle in G. Thus, we may assume that $x_L C y_L \cap X$ is nonempty, and recall that since $x_R C y_R$ contains at least as many vertices of X as $x_L C y_L$, $|C_L \cap X| \geq |x_L C y_L \cap X|$.

We now proceed to extend C_L using vertices in $G - C$. Since G has minimum degree at least four, each vertex in X has at least two neighbors in $G - C$. We also claim that by the minimality conditions placed on P_L and P_R, every vertex of $G - C$ can be adjacent to at most three vertices in $x_R C y_R$ as otherwise there would be a shortening path with one of $x_R C y_R \cap X$ or $x_R C y_R$ having smaller cardinality. Further, suppose $v \in G - C$ has three neighbors in X covered by $x_R C y_R$. Either these three neighbors are consecutive or there is a shortening path that contradicts the minimality of P_R. Furthermore, v has no other neighbors in C since otherwise G contains an induced claw or v has four consecutive neighbors on C.

Let $X \cap x_R C y_R = \{x_1, x_2, \ldots, x_l\}$ for some l which by assumption satisfies $l \geq |x_L C y_L \cap X|$. Note that each vertex x_i has at least two neighbors in $G - C$ and each of these neighbors is adjacent to either x_i^- or x_i^+. We claim that one vertex from $G - C$ can be inserted into C_L for each vertex of $X \cap x_R C y_R$ (which allows us to find cycles of all lengths from $t - |x_L C y_L|$

up to our desired length of $t-1$). Since each x_i has at least two neighbors in $G-C$ that could be inserted, the only way that it is not possible to insert distinct vertices for each x_i is if there are consecutive vertices x^-, x and x^+ on C such that $N_{G-C}(x^-, x, x^+) = \{u, v\}$ for some u, v in $G-C$. Since G is claw-free, we immediately have that uv is an edge in G, and that u and v have no other neighbors on C. Now, assume that without loss of generality u has some neighbor $u' \neq v$ in $G-C$. As C is hop-free, the claw $\langle u + u'x^-x^+\rangle$ implies that $u'x^-$ or $u'v^+$ is an edge in G, which contradicts our assumption that $N_{G-C}(x^-, x, x^+) = \{u, v\}$. Consequently, the set $\{x^-, x, x^+\}$ is a cut of size three in G, which contradicts our assumption that G is 4-connected. This completes the proof. □

From this result we immediately get the following corollary.

Corollary 2.5. *If G is 4-connected and $\{B, K_{1,3}\}$-free where B is one of $B(6,0)$, $B(5,1)$, $B(4,2)$ or $B(3,3)$ then G is pancyclic provided all cycles of length at least four contain chords.*

We now present some results which will allow us to focus strictly on finding short cycles in order to prove that G is pancyclic.

The first lemma takes advantage of the fact that, via Corollary 2.5, G must contain induced cycles. We omit the proof as it is standard.

Lemma 2.6. *Let $C = C_t$ be an induced cycle in a $K_{1,3}$-free graph G with $t \geq 9$. If there exists a vertex $w \in G-C$ with exactly two neighbors on C then G contains an induced $B(6,0)$, $B(5,1)$, $B(4,2)$ and $B(3,3)$.*

The following lemma allows us to find a shorter cycle when a vertex has three or more neighbors on an induced cycle.

Lemma 2.7. *Let $C = C_t$ for $t \geq 6$ be an induced cycle in a 4-connected $K_{1,3}$-free graph G and suppose that all vertices $v \in G-C$ with $d_C(v) \geq 1$ have $d_C(v) \geq 3$. Then G contains a cycle of length $t-1$.*

Proof. Assume that G does not contain a cycle of length $t-1$, and choose a vertex $w \in G-C$ with $d_C(w) \geq 1$. By assumption w must have three neighbors on C and since G is $K_{1,3}$-free and G has no $(t-1)$ cycle, these neighbors must all be consecutive on C. Let $v_1v_2\cdots v_t$ denote the vertices of C in order, and let V_i denote the set of vertices in $G-C$ which are adjacent to $\{v_{i-1}, v_i, v_{i+1}\}$ where these indices are taken modulo t. For $v, w \in V_i$, the claw $\langle v_{i-1} + v_{i-2}vw\rangle$ for $v, w \in V_i$ implies that the sets V_i must all be complete.

Claim 1. *For $w_i \in V_i$, $N(w_i) \subseteq \{v_{i-1}, v_i, v_{i+1}\} \cup V_{i-1} \cup V_i \cup V_{i+1}$.*

Proof. For a contradiction, suppose $z \in N(w_i)$ and $z \notin \{v_{i-1}, v_i, v_{i+1}\} \cup V_{i-1} \cup V_i \cup V_{i+1}$. Considering the claw $\langle w_i + zv_{i-1}v_{i+1}\rangle$, we must have either zv_{i-1} or zv_{i+1} in G. Without loss of generality, suppose $zv_{i+1} \in E(G)$. By assumption, z must have three consecutive edges to C but since $z \notin V_i \cup V_{i+1}$, we must have $z \in V_{i+2}$. Then the cycle $v_{i-1}w_izv_{i+3}Cv_{i-1}$ is a $(t-1)$-cycle, a contradiction. □

Next we claim that there are at most two sets V_i which are empty and furthermore, if V_i and V_j are both empty with $i < j$, then $j = i+1$. Suppose that the sets V_i and V_j are empty and $j \neq i+1$. By Claim 1 and the fact that C is induced, the set $\{v_i, v_j\}$ forms a 2-cut of G, a contradiction to the assumption that G is 4-connected. Hence, $j = i+1$ and there can be at most two empty sets.

Since $t \geq 6$ and at most two V_i are empty, we may assume without loss of generality that $V_s \neq \emptyset$ for $1 \leq s \leq t-2$. Choose a vertex x_i in V_i for each $1 \leq i \leq t-2$. If $t = 2m$ and m is odd, then $v_tx_1v_2x_3v_4\cdots v_{\frac{t-2}{2}}x_{\frac{t-2}{2}}v_{\frac{t-4}{2}}\cdots x_2v_1v_t$ is a cycle of length $t-1$ in G. If $t = 2m$ and m is even, then $v_tx_1v_2x_3v_4\cdots x_{\frac{t-2}{2}}v_{\frac{t-2}{2}}v_{\frac{t-4}{2}}\cdots x_2v_1v_t$ is a cycle of length $t-1$ in G. Now, if $t = 2m+1$ and m is odd, then $v_1x_2v_3x_4\cdots x_{\frac{t-1}{2}}v_{\frac{t-1}{2}}x_{\frac{t-3}{2}}\cdots v_2x_1v_1$ is a cycle of length $t-1$ in G. Finally, if $t = 2m+1$ and m is even, then $v_1x_2v_3x_4\cdots v_{\frac{t-1}{2}}x_{\frac{t-1}{2}}x_{\frac{t-3}{2}}\cdots v_2x_1v_1$ is a cycle of length $t-1$ in G, completing the proof. □

From these lemmas we get the following corollary.

Corollary 2.8. *If G is a 4-connected $\{K_{1,3}, B\}$-free graph where B is one of $B(6,0)$, $B(5,1)$, $B(4,2)$ or $B(3,3)$, then G is pancyclic as long as it contains cycles of length four, five, six and seven.*

Proof. By Corollary 2.3, G is hamiltonian and since G is 4-connected, no hamiltonian cycle is induced. So, this hamiltonian cycle has a chord, and by Lemma 2.4, G contains a $(n-1)$-cycle. Let C be a t-cycle of G for some $9 \leq t \leq n-1$. If C is not induced, then Lemma 2.4 implies the existence of a $(t-1)$-cycle so suppose C is induced and there exists no $(t-1)$-cycle in G. Then by Lemmas 2.6–2.7, we obtain an induced copy of B, contradiction. Since G is 4-connected and $K_{1,3}$-free, G clearly contains a triangle and the result follows. □

2.1. Proof of Theorem 1.5

We first make some general observations which will be used heavily. Let G be a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph where $i+j = 6$ and suppose G contains no C_t for $4 \leq t \leq 7$. By Theorem 1.4, since the line graph of the Petersen graph contains $B(i,j)$, we may assume there is an induced P_{10} say P, in G, with vertices $p_1, p_2, \ldots, p_{10}$.

We also prove another small fact for use in the first few cases.

Fact 2.9. *If there is a vertex $v \in G - P$ with three consecutive neighbors on P, then G contains C_4, C_5 and C_6.*

Proof. Let v be a vertex in $G-P$ and assume that p_i, p_{i+1} and p_{i+2} are elements of $N_P(v)$. Further, let w be a neighbor of p_{i+1} in $G-P$ that is distinct from v. As G is claw-free, w must also be adjacent to either p_i or p_{i+2} and hence if v is also adjacent to either p_{i+3} or p_{i-1} then we obtain cycles of length four, five and six. Thus, we may assume that no vertex in $G-P$ is adjacent to 4 consecutive vertices on P.

Without loss of generality, suppose $wp_i \in E(G)$. Since G is 4-connected, v must be adjacent to some vertex x that, as outlined above, does not lie on P. To avoid an induced claw centered at v, we must have either $xp_i \in E(G)$ or $xp_{i+2} \in E(G)$. Either case produces all desired cycles unless $x = w$ so we therefore conclude that $vw \in E(G)$.

At this point, $\{p_{i+2}, p_i, w\}$ comprises a 3-cut that separates v and p_{i+1} from the rest of the graph. Since G is 4-connected, there must be another edge from either v or p_{i+1} to a vertex $x \notin \{p_{i+2}, p_i, w\}$. If $xp_{i+1} \in E(G)$ then since P is induced we have that $x \notin P$. Hence either xp_i or xp_{i+2} must be in G to avoid a claw, in either case producing all desired cycles. Similarly if $xv \in E(G)$, we also get that either xp_i or xp_{i+2} is an edge in G, again producing all desired cycles. □

The remainder of the proof of Theorem 1.5 is broken into Lemmas 2.10–2.13, each showing the existence of a small cycle.

Lemma 2.10. *Every* 4*-connected $\{K_{1,3}, B(i,j)\}$-free graph, where $i+j=6$, contains a C_4.*

Proof. Let G be a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph, where $i+j=6$ and $i \geq j$, and suppose that there is no C_4 in G. Note that since G is 4-connected, $K_{1,3}$-free and contains no C_4, G must be 4-regular.

As P is induced, each p_ℓ, $2 \leq \ell \leq 9$, has at least two neighbors in $G-P$. Since G is $K_{1,3}$-free, each of these neighbors must be adjacent to either $p_{\ell-1}$ or $p_{\ell+1}$. To avoid a C_4, for each $1 \leq t \leq 9$ there is a vertex v_t adjacent to both p_t and p_{t+1}. Note that these v_t may not be distinct. Certainly $v_t \neq v_{t+1}$, v_{t+2}, or v_{t+3} as each of these equalities would imply the existence of a C_4.

The remainder of the proof is broken into cases in which each $B(i,j)$ with $i+j=6$ is forbidden.

Case 1. $i=j=3$.

The bull $B = B(p_5p_6v_5; p_4p_3p_2, p_7p_8p_9)$ cannot be induced, and therefore implies that either $v_5 = v_1$ or $v_5 = v_9$, as any other edge in B would result in a C_4. Suppose without loss of generality that $v_5 = v_9$, so that v_5p_9 and v_5p_{10} are edges. As $v_6 \notin \{v_5, v_7, v_8, v_9\}$ the bull $B_1 = B(p_6p_7v_6; p_5p_4p_3, p_8p_9p_{10})$ implies that $v_6 = v_2$. Finally, to avoid a C_4, v_7 is not adjacent to any vertex in $\{p_2, p_3, p_5, p_6, p_9, v_5, v_6\}$. Now, as v_7p_3 and v_7p_5 are not in G, we also know that $v_7p_4 \notin E(G)$. However, this means the bull $B(p_5v_5p_6; p_4p_3p_2, p_9p_8v_7)$ is induced, a contradiction.

Case 2. $i=4$ *and* $j=2$.

As the bull $B(p_5p_6v_5; p_4p_3p_2p_1, p_7p_8)$ cannot be induced and neither v_5p_7 nor v_5p_8 is in $E(G)$, as either edge would create a C_4, we have that v_5p_1 (and possibly v_5p_2) is in $E(G)$. Similarly, $B(p_6p_5v_5; p_7p_8p_9p_{10}, p_4p_3)$ implies that v_5p_{10} (and possibly v_5p_9) is in $E(G)$. However, then $\langle v_5 + p_1, p_5, p_{10}\rangle$ is an induced claw, a contradiction.

Case 3. $i=5$ *and* $j=1$.

Consider the bull $B(p_4p_3v_3; p_5p_6p_7p_8p_9, p_2)$, and note that $v_3p_5, v_3p_6 \notin E(G)$ as either of these would create a C_4. We now consider several possible cases. First, if $v_3 = v_7$, then $B(p_3p_2v_2; p_4p_5p_6p_7p_8, p_1)$ must be induced, as any additional edges would create a C_4 in G, a contradiction. If $v_3 = v_8$, then $v_4p_\ell \notin E(G)$ for all $6 \leq \ell \leq 9$ so that v_4p_{10} must be in $E(G)$ lest the bull $B(p_5p_4v_4; p_6p_7p_8p_9p_{10}, p_3)$ is induced. Now since $B(p_3p_2v_2; p_4p_5p_6p_7p_8, p_1)$ is not induced, $v_2 = v_6$ since all other edges would produce a C_4. Then $B(v_2p_2p_3; p_7p_8p_9p_{10}v_4, p_1)$ is necessarily induced, as all edges within this structure would either produce an induced $K_{1,3}$ or a C_4. Finally, if $v_3 = v_9$, then the bull $B(p_8p_9v_8; p_7p_6p_5p_4p_3, p_{10})$ is necessarily induced, as otherwise we would again contradict the assumption that G is claw-free and does not contain a C_4.

Case 4. $i=6$ *and* $j=0$.

Recall that v_1, v_2, and v_3 are distinct, and note that for $t \leq 3$ the bulls $B_t = B(p_tp_{t+1}v_t; p_{t+2}\cdots p_{t+7})$ imply that v_t is adjacent to one of p_{t+4}, p_{t+5}, p_{t+6}, or p_{t+7}. In particular, we have that $v_1 \in \{v_5, v_6, v_7, v_8\}$, $v_2 \in \{v_6, v_7, v_8, v_9\}$, and also that $v_3 \in \{v_7, v_8, v_9\}$ or $v_3p_{10} \in E(G)$ but v_3p_9 is not. Note that v_1 and v_2 can have no common neighbor on P except for p_2 (such a neighbor would force a C_4), and similarly v_2 and v_3 can have no common neighbor on P except for p_3. With this in mind, there are several possibilities. We will consider cases based on v_3. If $v_3p_{10} \in E(G)$ but $v_3p_9 \notin E(G)$, then either (i) $v_1 = v_5$ and $v_2 = v_7$, (ii) $v_1 = v_5$ and $v_2 = v_8$, or (iii) $v_1 = v_6$ and $v_2 = v_8$. In (i) and (ii), the bull $B(v_1p_1p_2; p_6p_7p_8p_9p_{10}v_3)$ is induced (as otherwise we get a C_4 or an induced claw). In (iii), the bull $B(v_3p_3p_4; p_{10}p_9p_8p_7v_1p_1)$ is similarly induced. Now, if $v_3 = v_9$, then either $v_1 = v_5$ and $v_2 = v_7$, which leads to the induced bull $B(v_1p_1p_2; p_6p_7p_8p_9v_3p_4)$, or $v_1 = v_8$ and $v_2 = v_6$, which leads to the induced bull $B(v_1p_1p_2; p_8p_7p_6p_5p_4v_3)$. The restrictions above on common neighbors between v_i and v_{i+1} for $i \in \{1,2\}$ implies that $v_3 \neq v_8$ unless $v_1 = v_8$ as well. However, this immediately leads to a C_4. Thus, the only remaining possibility is that $v_3 = v_7$. Suppose $v_3 = v_7$. Now, if $v_1 = v_5$ and $v_2 = v_9$, then the bull $B(v_3p_7p_8; p_4p_5v_1p_2v_2p_{10})$ is induced. If $v_1 = v_6$ and $v_2 = v_9$, then the bull $B(p_6v_1p_7; p_5p_4p_3v_2p_9p_8)$ is induced. This final contradiction completes the proof. □

Lemma 2.11. *Every* 4*-connected $\{K_{1,3}, B(i,j)\}$-free graph, where $i+j=6$, contains a C_5.*

Proof. Let G be a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph, where $i+j = 6$ (assume again that $i \geq j$) and suppose there is no C_5 in G. As above consider an induced P_{10}, $P = p_1 \cdots p_{10}$, but note that we cannot ensure the existence of the vertices $v_1, \ldots, v_9$ here, as we are not prohibiting C_4 as a subgraph of G.

Case 1. $i = j = 3$.

We note first that since P is induced, $d_{G-P}(p_5) \geq 2$; let v be one such vertex so that v is also adjacent to either p_4 or p_6. Without loss of generality, suppose vp_4 is an edge of G and observe that by Fact 2.9 neither vp_3 nor vp_6 is an edge in G. Also, the edges vp_i with $i \in \{1, 2, 7, 8\}$ are forbidden as each of these creates a C_5 in G. All other edges between vertices in $B(p_4p_5v; p_3p_2p_1, p_6p_7p_8)$ are forbidden as P is induced, forcing a contradiction.

Case 2. $i = 4$ *and* $j = 2$.

Again let $v \in N_{G-P}(p_5)$, and assume first that $vp_6 \in E(G)$ so that by Fact 2.9, vp_7 and vp_4 are not in $E(G)$. In order to avoid a C_5, we also know that $vp_i \notin E(G)$ for $i \in \{2, 3, 8, 9\}$. Consideration of $B(p_5p_6v; p_4p_3p_2p_1, p_7p_8)$ implies that we must have $vp_1 \in E(G)$ and symmetrically, we must also have $vp_{10} \in E(G)$ but this gives us an induced claw centered at v using p_1, p_5 and p_{10}, a contradiction. Thus, we may assume $vp_4 \in E(G)$.

Now, as $B(p_5p_4v; p_6p_7p_8p_9, p_3p_2)$ is not induced, it follows that vp_9, and hence vp_{10} in G. However, then $B(vp_9p_{10}; p_4p_3p_2p_1, p_8p_7)$ is necessarily induced by Fact 2.9 and the fact that G contains no C_5.

Case 3. $i = 5$ *and* $j = 1$.

Let v and w be vertices in $N_{G-P}(p_3)$, and note that both of v and w are also adjacent to either p_2 or p_4. Suppose first that both vp_2 and wp_2 are edges in G, so that by Fact 2.9, $vp_4 \notin E(G)$ and, to avoid a C_5, we also do not have vp_5 or vp_6 in G. Consequently, the bull $B(p_3p_2v; p_4p_5p_6p_7p_8, p_1)$ implies that vp_8 (and possibly also vp_7) must be an edge of G. Similarly, we have that wp_8 is in $E(G)$ so that $vp_3p_2wp_8v$ is a C_5 in G. The case where wp_4 and vp_4 are in $E(G)$ is handled in a nearly identical fashion.

Thus, assume that vp_2 and wp_4 are in G. As above, we have that vp_8 is an edge in G, and similarly that wp_9 is as well. Then, $vp_3wp_9p_8v$ is a C_5 in G.

Case 4. $i = 6$ *and* $j = 0$.

Let v and w be vertices in $N_{G-P}(p_2)$, and assume first that wp_3 and vp_3 are both in G. Examination of the bull $B(p_3p_2w; p_4p_5p_6p_7p_8p_9)$ implies that w is adjacent to p_7 and p_8, p_8 and p_9, or p_9 and p_{10}, and the bull $B(p_3p_2v; p_4p_5p_6p_7p_8p_9)$ allows us to reach a similar conclusion about v. However, v and w must have either common or consecutive neighbors in the subpath of P from p_7 to p_{10}, and this leads to a C_5 in G, a contradiction. If vp_1 and wp_1 are edges in G, then we reach a similar conclusion and contradiction.

If vp_1 and wp_3 are edges in G, then w is adjacent to p_7 and p_8, p_8 and p_9, or p_9 and p_{10}, and v is adjacent to p_6 and p_7, p_7 and p_8, or p_8 and p_9. This implies that v and w have either common or consecutive neighbors in the subpath of P from p_6 to p_{10} unless v is adjacent to p_6 and p_7 and w is adjacent to p_9 and p_{10}.

We therefore examine the neighbors of p_9 in $G-P$, and similarly conclude that there are vertices v' and w' in $N_{G-P}(p_9)$ such that w' is adjacent to p_8, p_1 and p_2, and v' is adjacent to p_{10}, p_4 and p_5. However, as no vertex in $G-P$ has five neighbors on P, v, w, v' and w' must be distinct vertices so that $wp_3p_2w'p_9w$ is a C_5 in G. □

Lemma 2.12. *Every* 4*-connected* $\{K_{1,3}, B(i,j)\}$*-free graph, where* $i+j = 6$, *contains a* C_6.

Proof. Let G be a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph, where $i+j = 6$ (assume $i \geq j$) and suppose there is no C_6 in G.

Case 1. *Either* $i = j = 3$ *or* $i = 4$ *and* $j = 2$.

Choose $v \in N_{G-P}(p_5)$ so that v must also be adjacent to either p_4 or p_6. We may assume $vp_4 \in E(G)$ as the case where vp_6 is in G is handled in a nearly identical manner. Since neither $B(p_5p_4v; p_6p_7p_8, p_3p_2p_1)$ nor $B(p_5p_4v; p_6p_7p_8p_9, p_3p_2)$ may be induced, we must get that either $vp_2 \in E(G)$ or $vp_7 \in E(G)$ since all other edges would produce a C_6. However, by Fact 2.9, v is adjacent to neither p_3 nor p_6, which implies (as P is induced and G is claw-free) that either v is adjacent to p_1 and p_2, or is adjacent to p_7 and p_8. In both cases, $C_6 \in G$, a contradiction.

Case 2. $i = 5$ *and* $j = 1$.

Let v be a neighbor of p_3 in $G-P$ and suppose that $vp_2 \in E(G)$ (the case where $vp_4 \in E(G)$ is identical). Fact 2.9 and the assumption that G has no C_6 imply that v is not adjacent to any vertex in $\{p_1, p_4, p_5, p_6, p_7\}$. Since $B(p_3p_2v; p_4p_5p_6p_7p_8, p_1)$ is not induced, we must have the vp_8, and hence vp_9 in $E(G)$. Now let $w \neq v$ be another vertex in $N_{G-P}(p_3)$ so that again w must be adjacent to either p_2 or p_4. If $wp_2 \in E(G)$, then by the same argument, $wp_8, wp_9 \in E(G)$ and hence $wp_8p_9vp_2p_3w$ is a C_6 in G. If $wp_4 \in E(G)$, then wp_9 and wp_{10} are edges in G, so that $vp_3wp_{10}p_9p_8v$ is a C_6.

Case 3. $i = 6$ *and* $j = 0$.

Let $v \in N_{G-P}(p_2)$ and assume that $vp_1 \in E(G)$. The case when $vp_3 \in E(G)$ can be handled in a similar manner. Fact 2.9 and the assumption that G contains no C_6 imply that v also cannot be adjacent to any vertex in $\{p_3, p_4, p_5, p_6\}$.

Since the bull $B(p_2p_1v; p_3p_4p_5p_6p_7p_8)$ cannot be induced, we must have vp_8 (and possibly also vp_7) in $E(G)$. Now let $w \neq v$ be another neighbor of p_8 in $G-P$. Then w is also adjacent to either p_9 or p_7. Suppose that $wp_9 \in E(G)$. An argument similar to the above yields that $wp_2 \in G$, implying the existence of the C_6 given by $wp_2p_1vp_8p_9w$. As the case when $wp_7 \in E(G)$ is similar, this completes the proof. □

Lemma 2.13. *Every* 4*-connected* $\{K_{1,3}, B(i,j)\}$*-free graph, where* $i+j = 6$, *contains a* C_7.

Proof. Suppose that G is a 4-connected, claw-free graph that does not contain a C_7. We once again consider an induced P_{10}, $P = p_1 \cdots p_{10}$.

Claim 2. *If a vertex v in $G - P$ is adjacent to vertices $p_\ell, p_{\ell+1}, p_t$ and p_{t+1} with $\ell + 1 < t$, then $7 \leq |\ell - t| \leq 8$.*

Proof. Let v be a vertex in $G-P$ adjacent to p_ℓ, $p_{\ell+1}$, p_t, and p_{t+1} with $\ell < t$, and assume to the contrary that $2 \leq |\ell-t| \leq 6$. If $4 \leq |\ell - t| \leq 7$, then G immediately contains a C_7, so we may suppose that $2 \leq |\ell - t| \leq 3$. If $|\ell - t| = 3$, then since G is 4-connected, there is some vertex $x \neq v$ in $N_{G-P}(p_{\ell+2})$. Since G is claw-free and P is induced, x is either adjacent to $p_{\ell+1}$ or p_t, so that either $vp_\ell p_{\ell+1} x p_{\ell+2} p_t p_{t+1} v$ or $vp_\ell p_{\ell+1} p_{\ell+2} x p_t p_{t+1} v$ is a C_7 in G.

Thus, we may assume that $t = \ell + 2$, namely that v is adjacent to p_ℓ, $p_{\ell+1}$, $p_{\ell+2}$ and $p_{\ell+3}$. Since G is 4-connected, p_ℓ and $p_{\ell+3}$ cannot separate v, $p_{\ell+1}$ and $p_{\ell+2}$ from the remainder of G. We therefore have that there are distinct vertices $u_1, u_2 \in \{v, p_{\ell+1}, p_{\ell+2}\}$ and distinct vertices y_1 and y_2 in $G - P - v$ such that u_1y_1 and u_2y_2 are edges in G. Since G is claw-free, if $u_1 = p_{\ell+1}$, then y_1 is adjacent to either p_ℓ or $p_{\ell+2}$ and if $u_1 = p_{\ell+2}$, then y_1 is adjacent to $p_{\ell+1}$ or $p_{\ell+3}$. Similarly, if $u_1 = v$, then y_1 is adjacent to at least one vertex in each of $\{p_\ell, p_{\ell+2}\}$, $\{p_\ell, p_{\ell+3}\}$, and $\{p_{\ell+1}, p_{\ell+3}\}$. We reach identical conclusions if u_2 is each of $p_{\ell+1}$, $p_{\ell+2}$ or v.

For any choices of u_1 and u_2, these additional edges immediately imply that G contains a C_7, except in the case where, without loss of generality, $u_1 = p_{\ell+1}$, $u_2 = p_{\ell+2}$ and both $y_1p_{\ell+2}$ and $y_2p_{\ell+1}$ are edges in G. However, in this case, the claw $\langle p_{\ell+1} + y_1y_2v\rangle$ implies that either $y_1y_2 \in E(G)$ or, without loss of generality, $y_1v \in E(G)$. If $y_1y_2 \in E(G)$, then $vp_\ell p_{\ell+1} y_1 y_2 p_{\ell+2} p_{\ell+3} v$ is a C_7 in G. If y_1v is an edge in G, then $\langle v + p_\ell y_1 p_{\ell+3}\rangle$ implies that y_1 is either adjacent to p_ℓ or $p_{\ell+3}$. Either possibility implies the existence of a C_7 in G. □

Claim 3. *If there are vertices v and x in G such that v is adjacent to p_ℓ, $p_{\ell+1}$ and $p_{\ell+2}$, and x is adjacent to p_ℓ and $p_{\ell+2}$, then G contains a C_7.*

Proof. By symmetry, we may assume that $\ell > 1$. Claim 2 and the claw $\langle p_\ell + vxp_{\ell-1}\rangle$ then together imply that vx is an edge in G. As G is 4-connected, p_ℓ and $p_{\ell+2}$ cannot separate $\{v, x, p_{\ell+1}\}$ from the remainder of G. Therefore, there are distinct vertices y_1 and y_2 in $G - (P \cup \{v, x\})$ and distinct vertices $u_1, u_2 \in \{x, v, p_{\ell+1}\}$ such that $u_1y_1, u_2y_2 \in E(G)$. Since each of x, v, and $p_{\ell+1}$ are adjacent to p_ℓ and $p_{\ell+2}$, each of y_1 and y_2 is adjacent to at least one of p_ℓ and $p_{\ell+2}$ as well. Subject to these observations, it is straightforward to check that any way the neighbors of y_1 and y_2 are chosen from $\{p_\ell, p_{\ell+1}, p_{\ell+2}, x, v\}$, we obtain a C_7 in G. □

Case 1. $i = 6$ *and* $j = 0$.

By Claim 2, no vertex in $G-P$ has four consecutive neighbors on P. We now claim that there is no vertex v in $G-P$ that is adjacent to p_1, p_2, and p_3. Indeed, assume otherwise, and consider the bull $B(p_3p_2v; p_4p_5p_6p_7p_8p_9)$ which, since G contains no C_7 and v cannot be adjacent to p_4, must be induced unless vp_9 is in G. However, then $\langle v + p_1p_3p_9\rangle$ is necessarily induced, a contradiction.

As P is induced, p_1 has three neighbors in $G-P$, call them v_1, v_2 and v_3. Suppose first that none of v_1, v_2 or v_3 is adjacent to p_2, which implies that $v_1v_2v_3$ must be a triangle in G. Now, consider the bull $B(p_1v_1v_2; p_2p_3p_4p_5p_6p_7)$, which, since neither v_1 nor v_2 is adjacent to p_2, would imply that G contains a C_7 unless (without loss of generality) v_1 is adjacent to p_7. To avoid an induced claw or a C_7 in G, v_1 must also be adjacent to p_8. Now the bull $B(p_1v_2v_3; p_2p_3p_4p_5p_6p_7)$ also implies that (without loss of generality) v_2 is adjacent to p_7 and p_8.

Symmetrically, p_{10} must also have three neighbors in $G-P$, call them x_1, x_2, and x_3. Note that $x_i \neq v_1$ for any i, as then v_1 would be adjacent to p_1, p_6, and p_{10}, forming an induced claw in G. As x_i is similarly not equal to v_2 for any i, we may assume without loss of generality that v_1 and v_2 are not any of x_1, x_2, or x_3. Since G contains no C_7, x_1 and x_2 are immediately not adjacent to p_5. If x_1 (or equivalently x_2) is adjacent to p_6, then $x_1p_6p_7v_1p_8p_9p_{10}x_1$ is a C_7 in G.

Assume that either x_1 or x_2 is adjacent to p_9, say x_1, and consider the bull $B(p_9p_{10}x_1; p_8p_7p_6p_5p_4p_3)$. Recall that no vertex in $G-P$ is adjacent to p_1, p_2, and p_3. Since p_1 and p_{10} behave symmetrically, there is also no vertex in $G-P$ that is adjacent to p_8, p_9, and p_{10}. In particular, as $x_1p_9, x_1p_{10} \in E(G)$, x_1 cannot be adjacent to p_8. As x_1 is also not adjacent to p_6 and G is claw-free, we conclude that $x_1p_7 \notin E(G)$ as well. Finally, $x_1p_4 \notin E(G)$ as it would create the C_7 given by $x_1p_4p_5p_6p_7p_8p_9x_1$. So, we must have $x_1p_3, x_1p_2 \in E(G)$, but this provides a contradiction as we now have the C_7 given by $x_1p_2p_1v_2p_7p_8p_9x_1$.

Thus, we may conclude that neither x_1 nor x_2 is adjacent to p_9, so that the claw $\langle p_{10}+x_1x_2p_9\rangle$ implies that x_1x_2 is an edge in G. We now consider the bull $B(p_{10}x_1x_2; p_9p_8p_7p_6p_5p_4)$ which is induced unless, without loss of generality, x_1 has a neighbor in $\{p_4, \ldots, p_9\}$. By assumption, x_1 is not adjacent to p_9, and either x_1p_5 or x_1p_6 would form a C_7 in G. Since v_1 is adjacent to both p_7 and p_8, the vertex x_1 cannot be adjacent to p_7 and p_8 as this forms a C_7. Therefore, x_1 must be adjacent to p_3 and p_4. However, then the bull $B(x_1x_2p_{10}; p_4p_5p_6p_7v_1p_1)$ is necessarily induced, as every possible edge within this substructure either creates a C_7 or an induced claw.

We may therefore suppose that some vertex in $N_{G-P}(p_1)$, say v_1, is adjacent to p_2. As we have already ruled out the possibility that $v_1p_3 \in E(G)$, the bull $B(p_2p_1v_1; p_3p_4p_5p_6p_7p_8)$ is induced unless v_1 is adjacent to either p_4 and p_5 or to p_8 and p_9. Since p_4 and p_5 would contradict Claim 2, we may assume v_1 is adjacent to p_8 and p_9.

Note then that v_1 is not adjacent to p_{10}, as then the claw $\langle v_1 + p_1p_8p_{10}\rangle$ is induced. By symmetry, there is some neighbor v of p_{10} that is also adjacent to p_9 and also by a symmetric argument, v must be adjacent to p_3 and p_2. However, then $v_1p_1p_2p_3vp_{10}p_9v_1$ is a C_7 in G, the final contradiction that completes this case.

Case 2. $i = 5$ *and* $j = 1$.

Again by Claim 2, no vertex in G is adjacent to four consecutive vertices on P. We next wish to show that there is no vertex v in $G - P$ such that $N_P(v) = \{p_2, p_3, p_4\}$. Assume otherwise, and let v be such a vertex and, since G is 4-connected and v cannot have any other neighbors on P, there is some vertex $x \in N_{G-P}(v)$. The claw $\langle v + xp_2p_4 \rangle$ implies that x must be adjacent to p_2 or p_4.

Suppose first that x is adjacent to p_4 but is not adjacent to p_2 and consider the bull $B(vp_4x; p_2, p_5p_6p_7p_8p_9)$. Now, x cannot be adjacent to any vertex in $\{p_6, p_7, p_8, p_9\}$ by Claim 2 and the assumption that G is claw-free. Since x is not adjacent to p_2, we have that B is induced unless xp_5 is an edge in G. Given that G is 4-connected, p_2 and p_5 cannot separate $\{p_3, p_4, x, v\}$ from the rest of G. Thus, there is some vertex y, distinct from p_2 and p_5, with a neighbor in $\{p_3, p_4, x, v\}$. However, since P is induced and x is not adjacent to p_2, any neighbor of y in this set forces y to be adjacent to consecutive vertices on the C_6 given by $xp_5p_4p_3p_2vx$, forming a C_7 in G. Similarly, if x is adjacent to p_2 but not p_4, the bull $B(p_2vx; p_1, p_4p_5p_6p_7p_8)$ implies that x is either adjacent to p_1 or p_5 and again we can use the connectivity of G to demonstrate the existence of a C_7 in G. Thus we have that x is adjacent to both p_2 and p_4, contradicting Claim 3 and implying that there is no vertex v in $G - P$ that is adjacent to p_2, p_3 and p_4. A nearly identical argument yields that there is no vertex v in $G - P$ that is adjacent to p_3, p_4 and p_5.

Now consider a vertex $w \in G - P$ that is adjacent to p_4, and note that w is adjacent to either p_3 or p_5, but not both. If wp_3 is in $E(G)$, then the bull $B(p_3p_4w; p_2, p_5p_6p_7p_8p_9)$ is induced unless w is adjacent to p_6 and p_7, contradicting Claim 2. If wp_5 is an edge in G, then by Claim 2 and the fact that w is not adjacent to p_3, the bull $B(p_4p_5w; p_3, p_6p_7p_8p_9p_{10})$ is induced unless w is adjacent to p_6. Symmetrically, we may assume that there is some vertex w' in $G-P$ that is adjacent to p_7, p_6 and p_5. As G is 4-connected and $\{p_4, p_7\}$ would separate $\{w, w', p_5, p_6\}$ from the rest of G, one of these four vertices must have a neighbor w'' in $G - P$. As G is claw-free, the vertex w'' is adjacent to one of the following pairs of vertices: p_5 and p_6, w and p_4, w and p_6, w' and p_5, or w' and p_7. In each of these cases, G necessarily contains a C_7 unless w'' is adjacent to p_5 and p_6. However, then the claw $\langle p_5 + ww'w'' \rangle$ implies that one of the edges ww', ww'', or $w'w''$ is in G. Each of these edges implies that G contains a copy of C_7, as desired.

Case 3. $i = 4$ *and* $j = 2$.

This case proceeds in a manner nearly identical to that for $B(5, 1)$, and so we only provide a sketch here in the interest of concision. Using Claim 3, one can show that there is no vertex in $G - P$ adjacent to p_i, p_{i+1} and p_{i+2} for $3 \le i \le 6$. We then consider a vertex v in $G - P$ that is adjacent to p_5, and therefore also to one of either p_4 or p_6. By Claim 2, if v is adjacent to p_4, then $B(p_4p_5v; p_3p_2, p_6p_7p_8p_9)$ is induced, and if v is adjacent to p_6, then $B(p_6p_5v; p_7p_8, p_4p_3p_2p_1)$ is induced. In both cases, we have a contradiction.

Case 4. $i = j = 3$.

Using Claims 2 and 3, along with an argument similar to those in the previous cases, we have that no vertex in $G - P$ is adjacent to p_4, p_5, and p_6, or adjacent to p_5, p_6, and p_7. We therefore consider a vertex v in $N_{G-P}(p_5)$, which is necessarily also adjacent to either p_4 or p_6. If v is adjacent to p_6, then, as v cannot also be adjacent to p_4 or p_7, the bull $B(p_5p_6v; p_4p_3p_2, p_7p_8p_9)$ is necessarily induced.

Thus, we may assume that v is adjacent to p_4 and p_5, and more so that there is no vertex in $G-P$ adjacent to both p_5 and p_6. Considering the bull $B(p_4p_5v; p_3p_2p_1, p_6p_7p_8)$, we conclude that vp_3 is an edge in G, and that v has no additional edges on P. Thus, since $d_G(v) \ge 4$, there is some vertex x in $N_{G-P}(v)$ and as G is claw-free, x is also adjacent to either p_3 or p_5. If x is adjacent to p_5, then since x cannot be adjacent to p_6, the edge xp_4 is also in G. However, then the bull $B(p_4p_5x; p_3p_2p_1, p_6p_7p_8)$ is necessarily induced, as x cannot be adjacent to p_3 by Claim 3.

So, assume $xp_5 \notin E(G)$ and $xp_3 \in E(G)$. Since $d_G(p_5) \ge 4$, there is some vertex $y \ne v$ in $N_{G-P}(p_5)$. As y cannot be adjacent to both p_5 and p_6, we have that yp_4 is an edge of G. However, then the claw $\langle p_5 + p_6yv \rangle$ implies that yv is an edge of G, so that there is some neighbor of v adjacent to p_4 and p_5, a possibility that has been prohibited. This is the final contradiction that completes the proof of the lemma. □

From Lemmas 2.10–2.13 and Corollary 2.8, we immediately obtain Theorem 1.5.

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