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A note on powers of Hamilton cycles in generalized claw-free graphs

Ralph J. Faudree[a,*], Ronald J. Gould[b]

[a] University of Memphis, Memphis, TN 38152, United States
[b] Emory University, Atlanta, GA 30322, United States

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ABSTRACT

Seymour conjectured for a fixed integer $k \geq 2$ that if G is a graph of order n with $\delta(G) \geq kn/(k+1)$, then G contains the kth power C_n^k of a Hamiltonian cycle C_n of G, and this minimum degree condition is sharp. Earlier the $k = 2$ case was conjectured by Pósa. This was verified by Komlós et al. [4]. For $s \geq 3$, a graph is $K_{1,s}$-free if it does not contain an induced subgraph isomorphic to $K_{1,s}$. Such graphs will be referred to as *generalized claw-free graphs*. Minimum degree conditions that imply that a generalized claw-free graph G of sufficiently large order n contains a kth power of a Hamiltonian cycle will be proved. More specifically, it will be shown that for any $\epsilon > 0$ and for n sufficiently large, any $K_{1,s}$-free graph of order n with $\delta(G) \geq (1/2 + \epsilon)n$ contains a C_n^k.

1. Introduction

In this paper we consider only graphs without loops or multiple edges. We let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The *order* of G, usually denoted by n, is $|V(G)|$ and the *size* of G is $|E(G)|$. For any vertex v in G, let $N(v)$ denote the set of vertices adjacent to v and $N[v] = N(v) \cup v$. The *degree* $d(v)$ of a vertex v is $|N(v)|$, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of a vertex in G, respectively. Given subgraphs H_1 and H_2, $E(H_1, H_2)$ will denote the edges between H_1 and H_2. The notation will generally follow that in Chartrand and Lesniak [1].

Let G and H be graphs. We say that G is *H-free* if H is not an induced subgraph of G. More specifically, we are interested in $K_{1,s}$-free graphs for $s \geq 3$, which we will call *generalized claw-free graphs*. We are interested in determining the minimum degree $\delta(G)$ in a $K_{1,s}$-free graph G of order n which implies that the kth power C_n^k of a Hamiltonian cycle is present in G.

Seymour [7] conjectured for a fixed integer $k \geq 2$ that if G is a graph of order n with $\delta(G) \geq kn/(k+1)$, then G contains the kth power C_n^k of a Hamiltonian cycle C_n of G, and this minimum degree condition is sharp. The special case $k = 2$ was conjectured earlier by Pósa [6]. This was verified by Komlós et al. [4].

Theorem 1 ([4]). *For a fixed integer $k \geq 2$, any graph G of sufficiently large order n with $\delta(G) \geq kn/(k+1)$ contains a C_n^k. Also, the minimum degree condition is sharp.*

The following result for generalized claw-free graphs will be proved.

Theorem 2. *Let $k \geq 2$ and $s \geq 3$ be fixed integers. For any given $\epsilon > 0$ there is a constant $c = c(k, s, \epsilon)$ such that if G is a $K_{1,s}$-free graph of order $n \geq c$ with $\delta(G) \geq (1/2 + \epsilon)n$, then G contains a C_n^k.*

* Corresponding author.
E-mail address: rfaudree@memphis.edu (R.J. Faudree).

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2. Examples

The complete t-partite graph with partite sets of order $n_1, n_2, \ldots, n_t$ will be denoted by $K_{n_1,n_2,\ldots,n_t}$. For a fixed positive integer $k \geq 2$ and n divisible by $k+1$, the slightly unbalanced complete multipartite graph $G = K_{n/(k+1)+1,n/(k+1)-1,n/(k+1),\ldots,n/(k+1)}$ does not contain a C_n^k and $\delta(G) = kn/(k+1) - 1$. This verifies that the result of Komlós et al. [4] is sharp.

A graph being generalized claw-free places additional restrictions on the graph, and so possibly a lower minimum degree condition will imply the existence of powers of a Hamiltonian cycle. For example, the complete multipartite graph has many induced generalized claws.

For a fixed $k \geq 2$ consider the graph $G = K_{2k-1} + (K_{(n-2k+1)/2} \cup K_{(n-2k+1)/2})$ for n odd. The graph G is $K_{1,3}$-free (claw-free) and $\delta(G) = (n+2k-3)/2$. There is no C_n^k in G, since the vertex cut that separates two (nonadjacent) vertices of a C_n^k contains at least $2k$ vertices. Thus, at least $\delta(G) \geq n/2 + c$ will be needed to imply the existence of a power of a Hamiltonian cycle.

3. Proof

Before giving the proof of Theorem 2, some notation and critical results must be presented. In a series of two papers [2,3], results on cycles and factorizations in claw-free graphs and in generalized claw-free graphs with minimum degree conditions were proved. In each case a minimum degree condition of approximately $n/2$ in a graph of order n is sufficient to give a factorization into complete graphs. If a graph G of order n contains the kth power C_n^k of a Hamiltonian cycle, it certainly contains a factorization of complete graphs K_{k+1} if n is divisible by $k+1$.

Theorem 3 *([2]). If G is a claw-free graph of sufficiently large order $n = 3k$ with $\delta(G) \geq n/2$, then G contains k disjoint triangles.*

Theorem 4 *([3]). Let $m \geq 4$ and $s \geq 3$. If G is a $K_{1,s}$-free graph of sufficiently large order $n = rm$, then there is a $c = c(s, m)$ such that if $\delta(G) \geq n/2 + c$, then G contains r disjoint copies of K_m.*

Given a graph H, the extremal number $ext(n, H)$ is the maximal number of edges in a graph of order n that does not contain H as a subgraph. The following result of Kővari et al. gives a bound on the extremal number $ext(n, K_{p,q})$ for the complete bipartite graph $K_{p,q}$.

Theorem 5 *([5]). Let $p \leq q$ be positive integers. Then, there exists a $c' = c'(p, q)$ such that*

$$ext(n, K_{p,q}) \leq c'(p, q)n^{2-1/p}.$$

Proof of Theorem 2. Select an integer $m \geq 6k$ and m sufficiently large. We will first consider the case where n is divisible by m. By Theorem 4, there are $r = n/m$ vertex disjoint copies of K_m in G if $n \geq c = c(s, k, m)$. Label these r copies of K_m as $H_1, H_2, \ldots, H_r$. □

Claim. *For each H_i, there are at least $\lceil r/2 \rceil$ different H_j with $j \neq i$ such that $|E(H_i, H_j)| > c''(2k, 2k)m^{2-1/2k} = c'(2k, 2k)(2m)^{2-1/2k}$. Therefore, by Theorem 5 there is a complete bipartite graph $K_{2k,2k}$ between the vertices of H_i and H_j.*

Proof of Claim. Without loss of generality consider the graph H_1, and assume that the claim is not true. We can assume that between H_1 and each of the H_j for $2 \leq j \leq d$ with $d \leq \lceil r/2 \rceil$ there are at least $c'(2k, 2k)m^{2-1/2k}$ edges, but this is not true for those H_j with $j > d$. This implies

$$m(1/2 + \epsilon)n - m^2 < |E(H_1, G - H_1)| \leq (d-1)m^2 + (r-d)c'(2k, 2k)m^{2-1/2k},$$

since there are at most m^2 edges between H_1 and H_j for $j \leq d$ and at most $c'(2k, 2k)m^{2-1/2k}$ edges for the remaining H_j for $j > d$. However, this implies

$$\frac{\left(\frac{1}{2}\right)\left(\frac{n}{m}\right)}{1 - \frac{c'}{m^{1/2k}}} + \frac{\left(\epsilon - \frac{c'}{m^{1/2k}}\right)\left(\frac{n}{m}\right)}{1 - \frac{c'}{m^{1/2k}}} < d.$$

Thus, for m sufficiently large and $n = mr$, clearly $d > \lceil r/2 \rceil = \lceil n/2m \rceil$. □

Now, form a new graph F in which the vertices of the graph F are the graphs $H_i (1 \leq i \leq r)$, and there is an edge between an H_i and an H_j if there are more than $c'(2k, 2k)m^{2-1/2k}$ edges in G between H_i and H_j. Thus, the graph F has $r = n/m$ vertices, and by the claim, $\delta(F) \geq r/2$. Thus, H is a Hamiltonian graph by Dirac's Theorem.

The complete graphs $\{H_i : (1 \leq i \leq r)\}$ can be placed in cycle order, say $(H_1, H_2, \ldots, H_r, H_1)$, such that there is a complete bipartite graph $K_{2k,2k}$ between consecutive complete graphs H_j and H_{j+1}. Thus, between consecutive complete graphs H_j and H_{j+1}, vertex disjoint complete bipartite graphs $K_{k,k}$ can be selected. Therefore, a Hamiltonian cycle C_n can be chosen in G by using the order of the graphs $(H_1, H_2, \ldots, H_r, H_1)$ and an arbitrary ordering of the vertices in each H_i except that the first k vertices are part of the $K_{k,k}$ with H_{i-1} and the last k vertices are part of the $K_{k,k}$ with H_{i+1}. This results in a kth power of a Hamiltonian cycle C_n^k.

The previous calculations were done under the assumption that m divides n. If this is not true, then it is easily seen that one of the complete graphs H_i can be selected to have $m + t$ vertices for some $1 \leq t < m$, and the same argument applies to the collection of $\{H_i \ : \ (1 \leq i \leq r)\}$. this follows since only the constant in the bound on the extremal result for bipartite graphs would change with the change in the size of one H_i. This completes the proof of Theorem 2. □

4. Questions

The most natural open question is the following:

Question 1. *What is the sharp minimum degree condition that implies that a $K_{1,s}$-free graph of order n contains the kth power C_n^k of a Hamiltonian cycle?*

It would be of interest to determine whether the weaker question could be answered.

Question 2. *Is there a minimum degree condition of the form $\delta(G) \geq n/2 + o(n)$, or more specifically a condition of the form $\delta(G) \geq n/2 + c$, that implies that a $K_{1,s}$-free graph of order n contains the kth power C_n^k of a Hamiltonian cycle?*

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