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Saturation Numbers for Nearly Complete Graphs

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Abstract An upper bound on the saturation number for graphs as well as associated extremal graphs was given by (Kászonyi and Tuza in J. Graph Theory, 10:203–210, 1986). A minor improvement of that result, which was implied in their paper, will be stated. Using this result, a series of exact saturation numbers and associated extremal graphs will be proved for the nearly complete graphs $K_t - E(L)$, where L is a graph of order at most 4.

Keywords Saturation numbers · Extremal graphs · Complete graphs

1 Introduction

We will deal only with finite graphs without loops or multiple edges. Notation will be standard, and generally follow the notation of Chartrand and Lesniak [1]. We let K_p denote the complete graph on p vertices, C_p the cycle on p vertices, and P_p the path on p vertices. We will also use the more compact notation that if H is a subgraph of G, then $G - E(H)$ will be denote by just $G - H$. Thus, for example $K_t - P_4$ will denote the graph obtained from the complete graph K_t by deleting the three edges in a path with four vertices.

Given a graph F, we say that the graph G is F*-free* if G has no subgraph isomorphic to F. A graph G is F*-saturated* if G is F-free, but $G + e$ does contain a copy of F for every edge $e \in E(\overline{G})$, where $\overline{G}$ denotes the complement of G. Specifically we are interested in the following:

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$$sat(n, F) := \min\{|E(G)| \ : \ |V(G)| = n \text{ and } G \text{ is } F\text{-saturated}\},$$
$$Sat(n, F) := \{G \ : \ |V(G)| = n, \ |E(G)| = sat(n, F), \ \text{ and } G \text{ is } F\text{-saturated}\}.$$
$$SAT(n, F) := \{G \ : \ |V(G)| = n, \ \text{ and } G \text{ is } F\text{-saturated}\}.$$

The *saturation number* of F on n vertices is $sat(n, F)$, and $Sat(n, F)$ is the set of graphs on n vertices that are F-saturated with $sat(n, F)$ edges.

In 1986 Kászonyi and Tuza [5] proved the best known general upper bound for $sat(n, F)$ for any fixed graph F, and in fact for a family of graphs. This result verifies that $sat(n, F)$ is linear in n for any graph F. Let $\alpha(F)$ be the independence number of F, and define the following two parameters:

$$u = u(F) = |V(F)| - \alpha(F) - 1,$$

and

$$d = d(F) = \min\{|E(F')| \ : \ F' \text{ is induced by } S \cup x\},$$

where S is a maximal independent set and $x \in V(F) - S$. Thus, d is the minimum degree of a vertex in F relative to a maximal independent set of F.

Consider the graph $G = K_u + H$, where H is a nearly $(d-1)$-regular graph of order $n - u$ [where nearly $(d-1)$-regular means every vertex has degree $d-1$ except for possibly one vertex, which has degree $d-2$]. The addition of any edge to G will generate a vertex of degree d in H. As a result the graph G contains an F-saturated graph, and so some subgraph of G is a candidate to be in $Sat(n, F)$. This leads to the following result of Kászonyi and Tuza.

Theorem 1 [5] $sat(n, F) \leq un + \lfloor (d-1)(n-u)/2 \rfloor - \binom{u+1}{2}$.

However in [5] the saturation number of the star $K_{1,d}$ was determined and the result was the following.

Theorem 2 *For* $n \geq d + \lfloor d/2 \rfloor$,

$$sat(n, K_{1,d}) = ((d-1)/2)n - \frac{1}{2}\lfloor d^2/4 \rfloor.$$

The graph $K_{\lfloor (d+1)/2 \rfloor} \cup A$, where A is a nearly $(d-1)$-regular graph of order $n - \lfloor (d+1)/2 \rfloor$, is a graph in $SAT(n, K_{1,d})$. If the graph $G = K_u + H$ in the proof of Theorem 1 is replaced by the graph $G' = K_u + H'$, where H' is the graph in $Sat(n-u, K_{1,d})$ associated with Theorem 2, then this provides an improved upper bound for the saturation number of a graph in terms of the parameters u and d. This gives the following result, which lowers the bound of Theorem 1 by $\frac{1}{2}\lfloor d^2/4 \rfloor$.

Theorem 3 $sat(n, F) \leq un + \lfloor (d-1)(n-u)/2 \rfloor - \binom{u+1}{2} - \frac{1}{2}\lfloor d^2/4 \rfloor$.

In Sect. 2 the impact of the Kászonyi and Tuza results (Theorems 1 and 2) will be exhibited. A series of known results on the saturation number of a graph for which the bound in Theorem 3 is a sharp bound will be discussed. In Sect. 3, new results dealing with nearly complete graphs of the form $K_t - L$, where L is a graph of order at most 4, will be stated using the bound of Theorem 3 or graphs derived from the Kászonyi and Tuza examples as the sharp upper bound.

2 Known Results

The star $K_{1,t}$ has the following parameters: $\alpha(K_{1,t}) = t$, $u = u(K_{1,t}) = 0$, and $d = d(K_{1,t}) = t$, which implies $sat(n, K_{1,t}) \leq (t-1)/n - \frac{1}{2}\lfloor t^2/4 \rfloor$. Theorem 2 verifies that this bound is sharp.

The complete graph K_t has the following parameters: $\alpha(K_t) = 1$, $u = u(K_t) = t-2$, and $d = d(K_t) = 1$. Thus, $K_{t-2} + \overline{K}_{n-t+2}$ is K_t-saturated, and $sat(n, K_t) \leq (t-2)n - \binom{t-2}{2} = (t-2)(n-t+2) + \binom{t-2}{2}$ by Theorem 3. The following theorem of Erdős, Hajnal, and Moon shows that this bound is sharp.

Theorem 4 [3] *If* $2 \leq t \leq n$, *then*

$$sat(n, K_t) = (t-2)(n-t+2) + \binom{t-2}{2}$$

and $Sat(n, K_t)$ *contains only one graph,* $K_{t-2} + \overline{K}_{n-t+2}$.

The graph $K_2 + \overline{K}_t$ is the book with t pages, and will be denoted by B_t. It is easy to see that B_t has the following parameters: $\alpha(B_t) = t$, $u = u(B_t) = 1$, and $d = d(B_t) = t$. Thus, Theorem 3 implies that $sat(n, B_t) \leq n + \lfloor (t-1)(n-1)/2 \rfloor - 1 - \frac{1}{2}\lfloor t^2/4 \rfloor = \lfloor (t+1)(n-1)/2 \rfloor - \frac{1}{2}\lfloor t^2/4 \rfloor$. The following result of Chen et al. [2] verifies that this bound is sharp.

Theorem 5 [2] *For* $t \geq 2$ *and* $n \geq t^3 + t$,

$$sat(n, B_t) = \lfloor (t+1)(n-1)/2 \rfloor - \frac{1}{2}\lfloor t^2/4 \rfloor.$$

Generalized books $B_{b,t} = K_b + \overline{K}_t$ were considered in [2]. It is easy to see that $B_{b,t}$ has the following paramenters: $\alpha(B_{b,t}) = t$, $u = u(B_{b,t}) = b-1$, and $d = d(B_t) = t$. Thus, Theorem 3 implies that $sat(n, B_{b,t}) \leq (b-1)n + \lfloor (t-1)(n-b+1)/2 \rfloor - \binom{b}{2} - \frac{1}{2}\lfloor t^2/4 \rfloor = (\lfloor (t+2b-3)(n-b+1)/2 \rfloor + (b)(b-1))/2 - \frac{1}{2}\lfloor t^2/4 \rfloor$. The following result in [2] verifies that this bound is sharp.

Theorem 6 [2] *For* $t \geq 2$, $b \geq 3$ *and* $n \geq 4(t+2b)^b$,

$$sat(n, B_{b,t}) = (\lfloor (t+2b-3)(n-b+1)/2 \rfloor + b(b-1))/2 - \frac{1}{2}\lfloor t^2/4 \rfloor.$$

The graph $K_{1,t} + e$ is the graph obtained from a star with t edges by adding one edge between two vertices of degree 1. This is a graph with $\alpha(K_{1,t} + e) = t-1$, $u = u(K_{1,t} + e) = 1$, and $d = d(K_{1,t} + e) = 1$. Thus, Theorem 3 implies that $sat(n, K_{1,t} + e) \leq n - 1$, and the $(K_{1,t} + e)$-saturated graph with this number of edges is the star $K_{1,n-1}$. This bound was shown to be sharp by the following result of Faudree et al. [4].

Theorem 7 [4] *For $t \geq 2$ and $n \geq t+1$,*

$$sat(n, K_{1,t}+e) = n-1,$$

and $Sat(n, K_{1,t}+e) = \{K_{1,n-1}\}$.

The previous result can be generalized to give a class of graphs for which the Kászonyi and Tuza [5] bound is sharp. Let H_s be an arbitrary graph of order s, and let $G_{s,t}$ be the graph obtained from $H_s + \overline{K_t}$ by adding one edge in the graph induced by $\overline{K}_t$.

Theorem 8 *For $s \geq 1$, $t \geq 2s+2$, and n sufficiently large,*

$$sat(n, G_{s,t}) = sn - \binom{s+1}{2}.$$

Proof By Theorem 1, the number of edges in the graph $K_s + \overline{K}_{n-s}$, which is $sn - \binom{s+1}{2}$, gives an upper bound for $sat(n, G_{s,t})$. Thus, $sat(n, G_{s,t}) \leq sn - \binom{s+1}{2}$.

Let $G \in Sat(n, G_{s,t})$. If $\delta = \delta(G) \geq 2s$, then G has sn edges. Thus we can assume that $\delta(G) < 2s$. First consider the case when $\delta(G) \leq s-1$. Choose a vertex v such that $d(v) = \delta(G)$. If there is a vertex $w \in G$ such that $vw \in \overline{G}$ and $d_G(w) < t-1$, then consider $G + vw$, which must contain a copy of $G_{s,t}$ containing the edge vw. However, this is impossible, since there is no edge in $G_{s,t}$ with one endvertex of degree $\leq s$ and the other strictly $< t$. Hence, we can assume that each vertex of G not adjacent to v has degree at least $t-1$. Therefore,

$$\begin{aligned}|G| &\geq (\delta(\delta+1) + (n-\delta-1)(t-1))/2 \\ &\geq (t-1)n/2 - (\delta+1)(t-1-\delta)/2 \geq sn - \binom{s+1}{2}.\end{aligned}$$

Hence, we can assume that $s \leq \delta(G) \leq 2s-1$.

We have $d(v) = \delta(G) = s + r$, where $0 \leq r < s$. Partition the vertices of $G - v$ into three sets $N = N_G(v)$, A, and B, where A is the set of vertices nonadjacent to v of degree at most $t-2$, and B is the set of remaining vertices of G. Thus, each vertex of B has degree at least $t-1$. Let $a = |A|$ and so $|B| = n-s-r-1-a$. The addition of any edge vw for $w \in A$, must generate a copy of $G_{s,t}$ containing vw, and so w must have at least s adjacencies in N. Therefore,

$$\begin{aligned}2|G| &\geq (s+r) + a(2s+r) + (n-s-r-1-a)(t-1) \\ &\geq 2sn - (s+r+1)(2s) + ar.\end{aligned}$$

Thus, if $ar \geq 2sr$, then $|G| \geq sn - \binom{s+1}{2}$, so we can assume $a \leq 2s-1$. Thus, there are at least $n-4s$ vertices of G of degree at least $t-1 \geq 2s+1$. Thus, $|G| \geq (2s+1)(n-4s) \geq 2sn$ for n sufficiently large. This completes the proof of Theorem 8. $\square$

It is clear that there are many known saturation results in which the upper bound given by Kászonyi and Tuza [5] is sharp. New results will be given in the next section with this same property.

3 Nearly Complete Graphs

The objective is to determine the saturation number of the graphs $K_t - H$, where H is a graph of order at most 4. Thus, the possibilities for H are K_2, P_3, K_3, P_4, $K_{1,3}$, C_4, $K_4 - K_{1,2}$, $K_4 - K_2$, K_4, and $2K_2$. In many, and in fact most, of the cases the upper bound given by Theorem 3 is sharp, and in the remaining cases the structure of a minimal saturated graph is suggested by Theorem 3. Of these ten possibilities, three of them are already known, namely K_2, K_3, and K_4, since these are all generalized books (see Theorem 6), and it has already been indicated that these fit the Kászonyi and Tuza upper bound. This is stated in the following summary result.

Theorem 9 *For $n \geq t$,*

(i) $sat(n, K_t - K_2) = \lfloor (2t-5)(n-t+3)/2 \rfloor + \binom{t-3}{2}$.

(ii) $sat(n, K_t - K_3) = \lfloor (2t-6)(n-t+4)/2 \rfloor + \binom{t-4}{2} - 1$.

(iii) $sat(n, K_t - K_4) = \lfloor (2t-7)(n-t+5)/2 \rfloor + \binom{t-5}{2} - 3$.

For the graph $K_t - P_3$ we have the following parameters: $\alpha(K_t - P_3) = 2$, $u(K_t - P_3) = t-3$, and $d(K_t - P_3) = 1$. This would indicate that the graph $K_{t-3} + \overline{K}_{n-t+3} \in Sat(n, K_t - P_3)$. This graph gives the upper bound for the following result which will be proved in Sect. 4.

Theorem 10 *For $t \geq 4$ and $n \geq 7t - 24$,*

$$sat(n, K_t - P_3) = (t-3)(n - t/2 + 1).$$

For the graph $K_t - P_4$ we have the following parameters: $\alpha(K_t - P_4) = 2$, $u(K_t - P_4) = t-3$, and $d(K_t - P_4) = 1$. This would indicate that the graph $K_{t-3} + \overline{K}_{n-t+3} \in Sat(n, K_t - P_4)$. Although this graph is $(K_t - P_4)$-saturated, it is not minimal. However, The graph $K_{t-4} + ((n-t+4)/2)K_2 \in Sat(n, K_t - P_4)$ when $n - t$ is even, and $K_{t-4} + (K_3 \cup ((n-t+1)/2)K_2) \in Sat(n, K_t - P_4)$ when $n - t$ is odd. This graph gives the upper bound for the following result which will be proved in Sect. 4.

Theorem 11 *For $t \geq 5$ and $n \geq 7t - 18$,*

$$sat(n, K_t - P_4) = \lfloor (2t-7)(n-t+4)/2 \rfloor + \binom{t-4}{2} + \theta(n, t),$$

where $\theta(n, t) = 2$ if $n - t$ is odd, and 0 otherwise.

For the graph $K_t - K_{1,3}$ we have the following parameters: $\alpha(K_t - K_{1,3}) = 2$, $u(K_t - K_{1,3}) = t - 3$, and $d(K_t - K_{1,3}) = 1$. The graph $K_{t-3} + \overline{K}_{n-t+3} \in$

$SAT(n, K_t - K_{1,3})$; however, there is a smaller graph in $Sat(n, K_t - K_{1,3})$. The addition of any edge in $R = K_{t-5} + ((\lfloor (n-t+5)/4 \rfloor)K_4 \cup K_{n-t+5-4\lfloor (n-t+5)/4 \rfloor})$ will induce a $K_5 - K_{1,3}$ disjoint from the vertices in the K_{t-5}. It will be shown that $R \in Sat(n, K_t - K_{1,3})$, and the number of edges in R is $((2t-7)n - t^2 + 6t - \theta(n,t))/2$, where $\theta(n,t) = 5, 8, 9, 8$ respectively when $n - t \equiv 3, 2, 1, 0 \bmod 4$. This graph gives the upper bound for the following result which will be proved in Sect. 4.

Theorem 12 *For $t \geq 5$ and $n \geq 10t - 16$,*

$$sat(n, K_t - K_{1,3}) = ((2t-7)n - t^2 + 6t - \theta(n,t))/2,$$

where $\theta(n,t) = 5, 8, 9, 8$ respectively when $n - t \equiv 3, 2, 1, 0$ mod 4.

For the graph $K_t - C_4$ we have the following parameters: $\alpha(K_t - C_4) = 2$, $u(K_t - C_4) = t - 3$, and $d(K_t - C_4) = 1$. This would indicate that the graph $K_{t-3} + \overline{K}_{n-t+3} \in Sat(n, K_t - C_4)$. Although this graph is $(K_t - C_4)$-saturated, it is not minimal. However, The graph $K_{t-4} + (K_3 \cup \overline{K}_{n-t+1}) \in SAT(n, K_t - C_4)$. This graph gives the upper bound for the following result which will be proved in Sect. 4.

Theorem 13 *For $t \geq 5$ and $n \geq 7t - 25$,*

$$sat(n, K_t - C_4) = (t-4)(n-t+4) + \binom{t-4}{2} + 3.$$

For the graph $K_t - (K_4 - K_{1,2})$ we have the following parameters: $\alpha(K_t - (K_4 - K_{1,2})) = 3$, $u(K_t - (K_4 - K_{1,2})) = t - 4$, and $d(K_t - (K_4 - K_{1,2})) = 2$. This would indicate that the graph $K_{t-4} + \lfloor (n-t+4)/2 \rfloor K_2 \in Sat(n, K_t - (K_4 - K_{1,2}))$. This graph gives the upper bound for the following result which will be proved in Sect. 4.

Theorem 14 *For $t \geq 5$ and $n \geq 9t - 36$,*

$$sat(n, K_t - (K_4 - K_{1,2})) = \lfloor (2t-7)(n-t+4)/2 \rfloor + \binom{t-4}{2}.$$

For the graph $K_t - (K_4 - K_2)$ we have the following parameters: $\alpha(K_t - (K_4 - K_2)) = 3$, $u(K_t - (K_4 - K_2)) = t - 4$, and $d(K_t - (K_4 - K_2)) = 1$. This would indicate that the graph $K_{t-4} + \overline{K}_{n-t+4} \in Sat(n, K_t - (K_4 - K_2))$. This is the upper bound for the following result which will be proved in Sect. 4.

Theorem 15 *For $t \geq 5$ and $n \geq 7t - 31$,*

$$sat(n, K_t - (K_4 - K_2)) = (t-4)(n - t/2 + 3/2).$$

An interesting case involves the nearly complete graphs $K_t - 2K_2$. For this graph we have the following parameters: $\alpha(K_t - (2K_2)) = 2$, $u(K_t - (2K_2)) = t - 3$, and $d(K_t - (2K_2)) = 2$. This would indicate that the graph $K_{t-3} + (\lfloor (n-t+3)/2 \rfloor)K_2 \in Sat(n, K_t - (2K_2))$. Although this graph is $(K_t - (2K_2))$-saturated, it is not minimal. In the case when n is even, let D be the graph obtained from $\overline{K}_{t-3} \cup \lfloor (n-2t+6)/2 \rfloor K_2$

by making all of the vertices of the $\overline{K}_{t-3}$ adjacent to one of the remaining vertices of D. In the case when n is odd, let D be the graph obtained from $\overline{K}_{t-2} \cup \lfloor (n-2t+5)/2 \rfloor K_2$ by making all of the vertices of the $\overline{K}_{t-2}$ adjacent to one of the remaining vertices of D. Consider the graph R, which is obtained from the graph $K_{t-3} + D$ by deleting a perfect matching between K_{t-3} and the $\overline{K}_{t-3}$ in D when n is even, and by deleting a matching with $t-3$ edges between K_{t-3} and the $\overline{K}_{t-2}$ along with one additional edge such that each vertex in the $\overline{K}_{t-2}$ is the endvertex of a missing edge, when n is odd. The graph $R \in Sat(n, K_t - 2K_2)$, and gives the upper bound for the following result which will be proved in Sect. 4.

Theorem 16 *For $t \geq 5$ and $n \geq 8t - 25$,*

$$sat(n, K_t - 2K_2) = \lfloor ((2t-5)n - t(t-3) - 1)/2 \rfloor.$$

4 Proofs

The proofs of the theorems stated in Sect. 3 will be given. The structure of all of the proofs are the same, so to reduce repetition, we will outline this general structure prior to giving the individual proofs. We will also introduce the notation that results, and not repeat it in each case. Let G be a graph in $SAT(n, K_t - S)$ for some subgraph S of K_t. Let x be a vertex of G of minimal degree $\delta = \delta(G)$, $B = N_G(x)$, and A the remaining vertices of G. Thus, $|B| = \delta$, and $|A| = n - \delta - 1$. An edge xy for some $y \in A$ will be added, which will produce a subgraph H of G with H isomorphic to $K_t - S$ and containing xy. The graph H will contain some $s \geq 1$ vertices of A, which we will denote by $\{y, y_1, y_2, \ldots, y_{s-1}\}$, and B will contain the remaining $t - s - 1$ vertices of H. Note that, $xy_i \notin G$ for $1 \leq i < s$.

In most cases the copy of H in $G + xy$ will have y adjacent to a least $t - s - 1$ vertices of A, which form a complete graph. This implies that a count on the sum of the degrees of the vertices in G would give the following inequality:

$$2|E(G)| \geq 2\delta + (t-s-1)(n-\delta-1) + 2\binom{t-s-1}{2} + (n-\delta-1)\delta,$$

where the 4 terms give lower bounds on the number of edges incident to x doubled, edges between A and B, edges in B doubled, and sum of degrees of the vertices in A respectively. This count will be supplemented in some cases, such as the additional edges in B that may not be in a copy of H when the edge xy is added.

Proof of Theorem 10 It is direct and straightforward to check that $R = K_{t-3} + \overline{K}_{n-t+3} \in SAT(n, K_t - P_3)$ and has $(t-3)(n - t/2 + 1)$ edges.

Let $G \in SAT(n, K_t - P_3)$. Since each pair of vertices in $K_t - P_3$ has at least $t - 3$ common adjacencies, each pair of non-adjacent vertices of G must have at least $t - 4$ common adjacencies and $\delta \geq t - 4$. Add an edge xy to G, where x has minimum degree and $y \in A$ to produce H, a copy of $K_t - P_3$ in G.

Case 1: $\delta > t-2$.

Each vertex of A must have at least $t-4$ adjacencies in B. Also, the addition of the edge xy will result in H, a copy of $K_t - P_3$ in G, and so there will be at least $\binom{t-4}{2}$ edges in B. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2\delta + (t-4)(n-\delta-1) + 2\binom{t-4}{2} + (n-\delta-1)\delta \\ &= 2\delta + (\delta+t-4)(n-\delta-1) + (t-4)(t-5).\end{aligned}$$

Since, $\delta \geq t-1$, $\delta + t - 4 > 2t-6$, and so for $n \geq 7t-24$ it is straightforward to determine that G has as many as $|E(R)|$ edges.

Case 2: $\delta = t-4$.

The graph H will contain three vertices $\{y, y_1, y_2\}$ of A, all vertices of B and x. The copy of $P_3 \in \overline{H}$ will contain x and the two vertices $\{y_1, y_2\}$ of A in H. Thus, B forms a complete graph, y is adjacent to all of the vertices of B and at least two vertices of A. Since y is a typical vertex of A, all vertices of A have the same properties as y. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2(t-4) + (t-4)(n-t+3) + 2\binom{t-4}{2} + (n-t+3)(t-2) \\ &= (2t-6)n - t^2 + 5t - 6.\end{aligned}$$

Since, $(2t-6)n - t^2 + 5t - 6 \geq 2((t-3)(n-t/2+1)$, this implies $\delta \geq t-3$.

Case 3: $\delta = t-3$.

The graph H will contain either two or three vertices of A. In the case when there are two vertices y and y_1 in A, then y_1 will be the center of the P_3 in $\overline{H}$, and y will be adjacent to all of the vertices of B, which is complete. If there are three vertices in A, then x will be the center of the P_3 in $\overline{H}$, and y will be adjacent to the other two vertices y_1, y_2 of A in H and at least $t-4$ vertices of B forming a complete graph. Also, each vertex of B must have degree at least $t-4$ in B. Let i be the number of vertices $y \in A$ associated with an H having two vertices in A. This results in the following bound on the number of edges in G:

$$\begin{aligned}2|E(G)| \geq\ & 2(t-3) + i(t-3) + (n-t+2-i)(t-4) + 2\binom{t-3}{2} \\ & + i(t-3) + (n-t+2-i)(t-2) \\ =\ & (2t-6)n - t^2 + 5t - 6.\end{aligned}$$

This implies $\delta \geq t-2$.

Case 4: $\delta = t-2$.

The graph H will contain one, two, or three vertices of A. This implies that each vertex of A will have at least $t-4$ adjacencies in B and B will have a complete subgraph with $t-4$ vertices. Since each vertex of B has degree at least $t-4$ in B, there are at least $\binom{t-2}{2} - 2$ edges in B. If that is precisely the number of edges in B, then the graph H associated with each vertex $y \in A$ will contain the same $t-4$

vertices of B. Therefore two of the vertices of B will have additional adjacencies in A not counted in the $t-4$ adjacencies of each vertex of A. This gives the following lower bound for the number of edges in G:

$$2|E(G)| \geq 2(t-2) + (t-4)(n-t+1) + 2\left(\binom{t-2}{2} - 2\right) + (n-t+1)(t-2)$$
$$+2 = (2t-6)n - (t-2)(t-3).$$

Thus, G has at least as many edges as R, which completes the proof of Case 4 and Theorem 10. □

Proof of Theorem 11 It can easily be checked that the graph $R = K_{t-4} + ((\lfloor(n-t+4)/2\rfloor)K_2)$ is in $SAT(n, K_t - P_4)$ when $n-t$ is even, and $K_{t-4} + (K_3 \cup ((n-t+1)/2)K_2) \in SAT(n, K_t - P_4)$ when $n-t$ is odd. The number of edges in R is $\lfloor((2t-7)n - (t-4)(t-2))/2\rfloor + \theta(n,t)$, where $\theta(n,t) = 2$ if $n-t$ is odd, and 0 otherwise

Let $G \in SAT(n, K_t - P_4)$. Since each pair of vertices in $K_t - P_4$ has at least $t-4$ common adjacencies, each pair of non-adjacent vertices of G must have at least $t-5$ common adjacencies and $\delta \geq t-5$.

Case 1: $\delta \geq t-2$.

The copy H of $K_t - P_4$ in G will have at most 3 vertices of A in H. Thus, y will have at least $t-4$ adjacencies in B which form a complete subgraph of B.

$$2|E(G)| \geq 2\delta + (t-4)(n-\delta-1) + 2\binom{t-4}{2} + (n-\delta-1)\delta$$
$$= 2\delta + (\delta+t-4)(n-\delta+1) + (t-4)(t-5).$$

Since, $\delta \geq t-2$, $\delta+t-4 > 2t-7$, and for $n \geq 7t-18$ it is straightforward to determine G has more than $|E(R)|$ edges.

Case 2: $\delta = t-5$.

The graph H must contain four vertices $\{y, y_1, y_2, y_3\}$ of A with x nonadjacent to $\{y_1, y_2, y_3\}$. However, this gives a contradiction, since this implies $K_{1,3} \in \overline{H}$. Thus, we can assume that $\delta \geq t-4$.

Case 3: $\delta = t-4$.

The graph H contains three vertices of A, which are y, y_1, y_2, and all of the vertices of B. Thus the P_4 in $\overline{H}$ will contain (y_1, x, y_2) along with some other vertex in $A \cup B$. Since each vertex of A is adjacent to all of the vertices of B, the $P_4 \in \overline{H}$ will be without loss of generality (y, y_1, x, y_2), and $yy_2, y_1y_2 \in E(G)$. Thus, y will be adjacent to at least one vertex of A of degree 2, and B will be a complete graph. The property that every vertex of A is adjacent to a vertex of degree 2 implies that each component of A will contain a cycle, and so A has at least $|A|$ edges. Thus,

$$|E(G)| \geq (t-4)(n-t+4) + \binom{t-4}{2} + (n-t+3)$$
$$= (t-3)(n-t+4) + (t-4)(t-5)/2 - 1.$$

Since, $2((t-3)(n-t+4)) + (t-4)(t-5) - 2 > (2t-7)(n-t+4) + (t-4)(t-5) + 2\theta(n,t)$, this implies $\delta \geq t-3$.
Case 4: $\delta = t-3$.

The graph H will contain two or three vertices of A, and the vertex y will have at least $t-4$ adjacencies in B. If for some y there are only two vertices y, y_1 in $H \cap A$, then B is either complete or missing at most one edge. At least one of y or y_1 will have $t-3$ adjacencies in B. Also, if B is not complete, then y has degree at least $t-2$ with $t-3$ adjacencies in B. If $n-t$ is odd, then the sum of the degrees of A relative to A will exceed $|A|$ or some vertex other than y or y_1 will have $t-3$ adjacencies in B, since the vertices of A cannot all have degree precisely one relative to A. Thus, in this case when for some y, $|H \cap A| = 2$,

$$\begin{aligned}2|E(G)| &\geq 2(t-3) + (t-4)(n-t+2) + 2\left(\binom{t-3}{2} - 1\right)\\ &\quad +2 + (n-t+2)(t-3) + \theta'(n,t)\\ &= (2t-7)(n-t+4) + (t-4)(t-5) + \theta'(n,t),\end{aligned}$$

where $\theta'(n,t) \geq 2$. Therefore, since $\lceil (2t-7)(n-t+4) + (t-4)(t-5)\rceil + \theta'(n,t) \geq \lfloor (2t-7)(n-t+4) + (t-4)(t-5)\rfloor + \theta(n,t)$, the number of edges in G is at least as large as the number of edges in R.

We now deal with the case where each copy of H has three vertices in A, which are $\{y, y_1, y_2\}$. Then $y_1y_2 \in E(G)$, and with no loss of generality we can assume that $yy_1 \in E(G)$. Thus, each vertex of $y \in A$ has $t-4$ adjacencies in B that form a complete graph, and is adjacent to a vertex of degree 2 in A. Thus, every component of G in A has a cycle, and so G has at least $|A|$ edges in A. Thus, we have

$$\begin{aligned}|E(G)| &\geq (t-3) + (t-4)(n-t+2) + \binom{t-4}{2} + (n-t+2)\\ &= (t-3)(n-t+4) + (t-4)(t-5)/2 + (t-3).\end{aligned}$$

Hence, in this case, since $n \geq 7t-18$, G has more than $\lfloor (2t-7)(n-t+4)/2\rfloor + \binom{t-4}{2} + \theta(n,t)$ edges. This completes the proof of Case 4 and Theorem 11. $\square$

Proof of Theorem 12 Consider the graph $R = K_{t-5} + ((\lfloor (n-t+5)/4\rfloor)K_4 \cup K_{n-t+5-4\lfloor (n-t+5)/4\rfloor})$. The addition of any edge in R will induce a $K_5 - K_{1,3}$ disjoint from the vertices in the K_{t-5}, and so it is easy to check that $R \in SAT(n, K_t - K_{1,3})$. The number of edges in R is $((2t-7)n - t^2 + 6t - \theta(n,t))/2$, where $\theta(n,t) = 5, 8, 9, 8$ respectively when $n - t \equiv 3, 2, 1, 0 \bmod 4$.

Let $G \in SAT(n, K_t - K_{1,3})$. Since each pair of vertices in $K_t - K_{1,3}$ has at least $t-4$ common adjacencies, each pair of non-adjacent vertices of G must have at least $t-5$ common adjacencies and $\delta \geq t-5$.
Case 1: $\delta \geq t-1$.

The copy H of $K_t - K_{1,3}$ in G will have at most 4 vertices of A in H. Thus, y will have at least $t-5$ adjacencies in B, which form a complete graph. Thus,

$$2|E(G)| \geq 2\delta + (t-5)(n-\delta-1) + 2\binom{t-5}{2} + (n-\delta-1)\delta$$
$$= 2\delta + (\delta+t-5)(n-\delta-1) + (t-5)(t-6).$$

Since, $\delta \geq t-1$, $\delta+t-5 > 2t-7$, and so for $n \geq 10k-16$ it is straightforward to verify that G has more than $|E(R)|$ edges.
Case 2: $\delta = t-5$.

The graph H must contain four vertices $\{y, y_1, y_2, y_3\}$ of A with x nonadjacent to $\{y_1, y_2, y_3\}$. Thus, y is adjacent to $\{y_1, y_2, y_3\}$ and all of the $t-5$ vertices of B. In fact, the vertices $\{y, y_1, y_2, y_3\} \cup B$ form a complete subgraph of H. Thus, each vertex of A is in a K_4. Therefore, it can be checked directly that the sum of the degrees of the vertices of A relative to A is at least $3(n-t+4)+\beta(n,t)$ where $\beta(n,t) = 3, 4, 3, 0$ respectively when $n-t \equiv 3, 2, 1, 0 \bmod 4$. Therefore,

$$2|E(G)| \geq 2(t-5) + (t-5)(n-t+4) + 2\left(\binom{t-5}{2}\right) + (n-t+4)(t-2)$$
$$+\beta(n,t) = (2t-7)n - t^2 + 6t - 8 + \beta(n,t).$$

Thus, G has as many edges as R, so we can assume that $\delta \geq t-4$.
Case 3: $\delta = t-4$.

The graph H contains either three vertices of A, which are y, y_1, y_2, or four vertices of A, which are $\{y, y_1, y_2, y_3\}$. In the first category y is adjacent to both $\{y_1, y_2\}$ and all of the vertices of B, which is complete. Thus, the vertices of $B \cup \{x\} \cup \{y, y_1, y_2\}$ induce a copy of $K_t - K_{1,3}$ without the edge xy, so this category cannot occur. In the second category y is adjacent to the vertices $\{y_1, y_2, y_3\}$ and at least $t-5$ vertices of B, which form a complete subgraph. Thus, each $y \in A$ will be in $K_4 \subseteq A$. Hence, we have that the sum of the degrees of the vertices in A relative to A is at least $3(n-t+3)+\beta(n,t)$ where $\beta(n,t) = 4, 3, 0, 3$ respectively when $n-t \equiv 3, 2, 1, 0 \bmod 4$. We can assume that each vertex in B has at least $t-5$ adjacencies in B. If not, consider the a vertex $z \in B$ that does not have $t-5$ adjacencies in B. If z is not adjacent to a vertex in $y' \in A$, then since y' and z must have at least $t-5$ common adjacencies, they have at least one common adjacency in A. This implies that either there would be at least $2t$ edges between z and vertices in A not involved in the count of edges between A and B from the copies of H, or a large number of the vertices of A (at least $(|A|-2t)/2t$) would be adjacent to a neighbor of z in A. In the calculation on the lower bound on the number of edges in $E(G)$ we can assume that each vertex in B has degree at least $t-5$, since the bound would be much larger for $n \geq 10t-16$ by the observation just made.

$$2|E(G)| \geq 2(t-4) + (n-t+3)(t-5) + 2\binom{t-4}{2} + (t-2)(n-t+3) + \beta(n,t)$$
$$= (2t-7)n - t^2 + 6t - 9 + \beta(n,t).$$

Clearly, since $\beta(n,t) - 9 \geq -\theta(n,t)$, G has at least as many edges as R. Thus, we can assume that $\delta \geq t-3$.

Case 4: $\delta = t - 3$.

The graph H will contain two, three, or four vertices of A. In the first category y will be adjacent to $t-3$ vertices of B, which is a complete graph. In the second category, y will be adjacent to at least $t-4$ vertices of B which form a complete graph and will be in a K_3 in A. However, this case cannot occur, since just as in the previous case, this implies G contains a copy of $K_t - K_{1,3}$. In the last category, y will be adjacent to at least $t-5$ vertices of B which forms a complete graph, and will be in a K_4 of A. As observed, before, each vertex of B can be assumed to have degree at least $t-5$ relative to B. Thus, if i, k are the number of vertices of A in the first and third categories respectively, then a lower bound on the number of edges in G is given by the following:

$$2|E(G)| \geq 2(t-3) + i(t-3) + k(t-5) + 2\left(\binom{t-3}{2} - 2\right) + 4 + i(t-3) + k(t-2) = (2t-7)n - t^2 + 6t - 8 + i.$$

Using an analysis essentially identical to that of the previous case it can be shown that there are additional edges that imply that G has as many edges as R.

Case 5: $\delta = t - 2$.

The graph H will contain one, two, three, or four vertices of A. In the first category y will be adjacent to $t-5$ vertices of B, which form a complete graph. In the second category y will be adjacent to at least $t-3$ vertices of B, which form a complete graph. In the third category, y will be adjacent to $\{y_1, y_2\}$ and at least $t-4$ vertices of B which form a complete graph and will be in a K_3 in A. However, as observed in the previous two cases, this category cannot occur, since $K_t - K_{1,3}$ is a subgraph of G. In the last category, y will be adjacent to $\{y_1, y_2, y_3\}$ at least $t-5$ vertices of B, which form a complete graph, and also y will be in a K_4 of A. As observed, before, each vertex of B has degree at least $t-2$ and can be assumed to have degree at least $t-5$ relative to B. Thus, if i, j, ℓ are the number of vertices of A in the first, second, and fourth categories respectively, then a lower bound on the number of edges in G is given by the following:

$$2|E(G)| \geq 2(t-2) + i(t-5) + j(t-3) + \ell(t-5) + 2\left(\binom{t-2}{2} - 4\right) + 8 + (n-t+1)(t-2) = (2t-7)n - t^2 + 6t - 5 + 2j.$$

This implies G has more edges than R, so this completes the proof of Case 5 and Theorem 12. □

Proof of Theorem 13 Consider the graph $R = K_{t-4} + (K_3 \cup \overline{K}_{n-t+1})$. The addition of any edge from $\overline{R}$ will give a $2K_2$ disjoint from the $K_{t-4} \in R$, and so $R \in SAT(n, K_t - C_4)$ and has $(t-4)n - (t^2 - 7t + 6)/2$ edges.

Let $G \in SAT(n, K_t - C_4)$. Since each pair of vertices in $K_t - C_4$ has at least $t-4$ common adjacencies, each pair of non-adjacent vertices of G must have at least $t-5$ common adjacencies which form a complete graph, and $\delta \geq t-5$.

Case 1: $\delta \geq t-2$.

Each vertex of A must have at least $t-5$ adjacencies in B. Therefore,

$$2|E(G)| \geq 2\delta + (t-5)(n-\delta-1) + 2\binom{t-5}{2} + (n-\delta-1)\delta$$
$$= (\delta+t-5)(n-\delta-1) + 2\delta + (t-5)(t-6).$$

Since, $\delta \geq t-2$, $\delta+t-5 > 2t-8$, and since $n \geq 7t-25$, it is straightforward to verify that G has more than $|E(R)|$ edges.

Case 2: $\delta = t-5$.

The graph H will contain four vertices $\{y, y_1, y_2, y_3\}$ of A, all vertices of B and x. The vertex x is not adjacent to any of the vertices $\{y_1, y_2, y_3\}$, so $K_{1,3} \in \overline{H}$, a contradiction. Thus, $\delta \geq t-4$.

Case 3: $\delta = t-4$.

The graph H will contain three vertices of A, which will be $\{y, y_1, y_2\}$. Thus, y will be adjacent to all of the vertices of B, which will be complete, and $y_1y_2 \in E(G)$. This implies each vertex of A is adjacent to all of the vertices of B, and there will be an edge in A disjoint from the vertex. Since, two edge-disjoint edges in A implies the existence of a $K_t - C_4 \in G$, we can assume that there are three edges in A that form a triangle. Thus, we have the following edge count:

$$2|E(G)| \geq 2(t-4) + (t-4)(n-t+3) + 2\binom{t-4}{2} + (n-t+3)(t-4) + 6$$
$$= 2(t-4)(n-t+4) + 2\binom{t-4}{2} + 6.$$

This implies $|E(G)| \geq |E(R)|$, and so $\delta \geq t-3$.

Case 4: $\delta = t-3$.

The graph H will contain two or three vertices of A. In both the two and three vertex case in $H \cap A$, the vertex y will be adjacent to at least $t-4$ vertices of B that form a complete graph. As a result, a lower bound on the number of edges in G is the following:

$$2|E(G)| \geq 2(t-3) + (n-t+4)(t-4) + 2\binom{t-4}{2} + (n-t+4)(t-3)$$
$$\geq (2t-7)n - t^2 + 8t - 14.$$

Since, $2t-7 > 2t-8$, the graph G has more edges than R for $n \geq 7t-25$. This completes the proof of Case 4 and Theorem 13. □

Proof of Theorem 14 It is straightforward to verify that $R = K_{t-4} + \lfloor (n-t+4)/2 \rfloor K_2 \in SAT(n, K_t - (K_4 - K_{1,2}))$ and has $\lfloor (2t-7)(n-t+4)/2 \rfloor + \binom{t-4}{2} = \lfloor ((2t-7)n - t^2 + 6t - 8)/2 \rfloor$ edges.

Let $G \in SAT(n, K_t - (K_4 - K_{1,2})$. Since each pair of vertices in $K_t - (K_4 - K_{1,2})$ has at least $t-4$ common adjacencies, each pair of vertices of non-adjacent vertices of G must have at least $t-5$ common adjacencies and $\delta \geq t-5$. Thus, $|B| = \delta$ and $|A| = n - \delta - 1$.

Case 1: $\delta \geq t-1$.

Each vertex of A must have at least $t-5$ adjacencies in B, and these vertices will form a complete graph in B. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2\delta + (t-5)(n-\delta-1) + 2\binom{t-5}{2} + (n-\delta-1)\delta \\ &= (\delta+t-5)(n-\delta-1) + 2\delta + (t-5)(t-6).\end{aligned}$$

Since, $\delta \geq t-1$, $\delta + t - 5 > 2t-7$, and since $n \geq 9t-36$, it is straightforward to verify that G has more than $|E(R)|$ edges.

Case 2: $\delta = t-5$.

The graph H will contain four vertices $\{y, y_1, y_2, y_3\}$ of A, all vertices of B, and x. The copy of $K_4 - K_{1,2} \in \overline{H}$ will contain x and all of the vertices $\{y_1, y_2, y_3\}$ of A in H. Thus, B forms a complete graph, y is adjacent to at all of the vertices of B and all of the vertices $\{y_1, y_2, y_3\}$ of A. All vertices of A have the same properties as y. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2(t-5) + (t-5)(n-t+4) + 2\binom{t-5}{2} + (n-t+4)(t-2) \\ &= (2t-7)n - (t-4)(t-2).\end{aligned}$$

Thus, G has as many edges as R.

Case 3: $\delta = t-4$.

The graph H will contain either three or four vertices of A. In the case when there are four vertices $\{y, y_1, y_2, y_3\}$ in A, y will be adjacent to at least $t-5$ vertices of B, which form a complete graph, and all of the vertices $\{y_1, y_2, y_3\}$. Also, each vertex of B must have degree at least $t-5$ relative to B. If all of the vertices of A have this property, then there is the following bound on the number of edges in G:

$$\begin{aligned}2|E(G)| &\geq 2(t-4) + (n-t+3)(t-5) + 2\binom{t-4}{2} + (n-t+3)(t-2) \\ &= (2t-7)n - (t-3)(t-3).\end{aligned}$$

Since $\lceil ((2t-7)n - t^2 + 6t - 9)/2 \rceil \geq \lfloor ((2t-7)n - t^2 + 6t - 8)/2 \rfloor$, G has as many edges as R.

Thus, we assume that this does not occur for each vertex $y \in A$. In the case when there are three vertices $\{y, y_1, y_2\}$ in A, y will be adjacent to the $t-4$ vertices of B, which form a complete graph, and at least one vertex in A. Assume there are i vertices in the first category and j vertices in the second category. We know that $j \geq 1$ and $i + j = n - t + 3$. Thus, we have the following:

$$\begin{aligned}2|E(G)| &= i(t-5) + j(t-4) + 2(t-4) + 2\binom{t-4}{2} + i(t-2) + j(t-3)\\ &= (2t-7)n - (t-3)(t-3).\end{aligned}$$

Since $\lceil((2t-7)n - t^2 + 6t - 9)/2\rceil = \lfloor((2t-7)n - t^2 + 6t - 8)/2\rfloor$, we can assume that $\delta \geq t-3$.

Case 4: $\delta = t-3$.

The graph H will contain four, three, or two vertices of A. In the case of four vertices, the vertex y will have degree at least $t-2$, since it will be adjacent to at least 3 vertices of A, and will be adjacent to a complete graph with $t-5$ vertices in B. In the second case y will be adjacent to a complete graph of order $t-4$ in B. In the last case, y will be adjacent to $t-4$ vertices of B, which will be a complete graph. Let i, j, and k be the number of vertices of A in each of the categories. We sill first consider the case when all of the vertices are in the first category. This gives the following lower bound for the number of edges in G:

$$\begin{aligned}2|E(G)| \geq 2(t-3) + (n-t+2)(t-5) + 2\binom{t-5}{2}\\ +(n-t+2)(t-2) = (2t-7)n - t^2 + 2t + 10.\end{aligned}$$

However, this count assumes that all of the edges in B are in a complete graph of order $t-5$, and there are two additional vertices in B not considered. Consider the case when a vertex $z \in B$ does not have at least $t-5$ adjacencies in B. Then for each $y \in A$, either $yz \in E(G)$ or y and z must have a common adjacency in A. This implies that either z is adjacent to at least $|A|/2$ of the vertices of A or at least $|A|/2$ of the vertices of A are adjacent to neighborhood of z in A. This implies that there are at least $(n-t+2)/2$ additional edges in G, which certainly exceeds the number of additional edges obtained by assuming that each vertex in B has at least $t-5$ adjacencies in B. Thus, we will assume that each vertex in B has at least $t-5$ adjacencies in B. This implies that G has an additional sum of degree count of at least $4(t-5)+4 = 4t-16$. Thus, $2|E(G)| \geq (2t-7)n - t^2 + 6t - 6$, and so G has more edges than R. Hence, we can assume some of the vertices in A are in category two or three. This will give the following count, again assuming that each vertex in B has degree at least $t-5$ relative to B for the reason given above.

$$\begin{aligned}2|E(G)| \geq 2(t-3) + i(t-5) + j(t-4) + k(t-4) + 2\binom{t-4}{2} + i(t-2)\\ +(j+k)(t-3) + 2(t-5) + 2 = (2t-7)n - t^2 + 6t - 8.\end{aligned}$$

This implies G has as many edges as R, so we can assume that $\delta \geq t-2$.

Case 5: $\delta = t-2$.

The graph H will contain four, three, two, or one vertices of A. In the case of 4 vertices in A, the vertex y will be adjacent to at least 3 vertices of A, and will be adjacent to a complete graph with $t-5$ vertices in B. In the case of three vertices in A, y will be adjacent to a complete graph of order $t-4$ in B. In the case of two vertices in A, y will be adjacent to at least $t-4$ vertices of B, which will be a complete graph. In the case of one vertex in A, then y could be adjacent to as few as $t-5$ vertices of B, but B will contain at least $\binom{t-2}{2}-4$ edges and each vertex of B will have degree at least $t-2$. If this case occurs for at least one vertex of A, then there is the following bound on the number of edges in G:

$$2|E(G)| \geq 2(t-2)+(n-t+1)(t-5)+2\left(\binom{t-2}{2}-4\right)+6$$
$$+(n-t+1)(t-2)=(2t-7)n-t^2+6t-7.$$

We now consider the remaining cases collectively, and let i, j, and k be the number of vertices in each of the first three categories respectively. This gives the following lower bound for the number of edges in G:

$$2|E(G)| \geq 2(t-2)+i(t-5)+(j+k)(t-4)+2\left(\binom{t-5}{2}-6\right)+6$$
$$+(n-t+1)(t-2)=(2t-7)n-t^2+6t-11+j+k.$$

Thus, if $j+k \geq 2$, then G has as many edges as R, so we can assume that $j+k \leq 1$. If all of the vertices of A are in the first category with the exception of at most one, then at least two of the vertices of A will have degree at least $t-1$ or the number of edges in B will be at least $\binom{t-5}{2}-4$. Thus, in all of the cases G has more edges than R, which completes the proof of Case 5 and Theorem 14. □

Proof of Theorem 15 It is easy to check that $R=K_{t-4}+\overline{K}_{n-t+4} \in SAT(n, K_t-(K_4-K_2))$ and has $(t-4)n-\binom{t-3}{2}$ edges.

Let $G \in SAT(n, K_t-(K_4-K_2))$. Since each pair of vertices in $K_t-(K_4-K_2)$ has at least $t-4$ common adjacencies, each pair of vertices of non-adjacent vertices of G must have at least $t-5$ common adjacencies and $\delta \geq t-5$.

Case 1: $\delta \geq t-2$.

Each vertex of A must have at least $t-5$ adjacencies in B, and these vertices form a complete graph in B. Therefore,

$$2|E(G)| \geq 2\delta+(t-5)(n-\delta-1)+2\binom{t-5}{2}+(n-\delta-1)\delta$$
$$=(\delta+t-5)(n-\delta-1)+2\delta+(t-5)(t-6).$$

Since, $\delta \geq t-2$, $\delta+t-5>2t-7$, and since $n \geq 7t-31$, G can be shown to have more than $|E(R)|$ edges.

Case 2: $\delta = t-5$.

The graph H will contain four vertices $\{y, y_1, y_2, y_3\}$ of A, all vertices of B and x. The copy of $K_4-K_2 \in \overline{H}$ will contain x and the three vertices $\{y_1, y_2, y_3\}$ of A

in H. Thus, B forms a complete graph, y is adjacent to all of the vertices of B and at least three vertices of A. All vertices of A have the same properties as y. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2(t-5)+(t-5)(n-t+4)+2\binom{t-5}{2}+(n-t+4)(t-2)\\&=(2t-7)n-(t-4)(t-2).\end{aligned}$$

Since, $(2t-7) > (2t-8)$, for $n \geq 7t-31$, G has more edges than R.
Case 3: $\delta = t-4$.

The graph H will contain either four or three vertices of A. In the case when there are four vertices $\{y, y_1, y_2, y_3\}$ in A, y will be adjacent to at least $t-5$ vertices of B, which form a complete graph, and all of the vertices $\{y_1, y_2, y_3\}$. If all of the vertices of A have this property, then just as in Case 2, G will have more edges than R. In the case when there are three vertices $\{y, y_1, y_2\}$ in A, y will be adjacent to at least $t-4$ vertices of B, which form a complete graph. Assume there are i vertices in the first category and j vertices in the second category. We know that $j \geq 1$ and $i+j=n-t+3$. Thus, we have the following:

$$\begin{aligned}2|E(G)| &\geq i(t-5)+j(t-4)+2(t-4)+2\binom{t-4}{2}+i(t-2)+j(t-4)\\&=(2t-8)n-(t-3)(t-4)+i \geq 2|E(R)|.\end{aligned}$$

This implies $\delta \geq t-3$.
Case 4: $\delta = t-3$.

The graph H will contain four, three, or two vertices of A. In the case of 4 vertices, the vertex y will have degree at least $t-2$ and will be adjacent to a complete graph with $t-5$ vertices in B. In the other two cases y will be adjacent to a complete graph of order $t-4$ in B. Let i, j, and k be the number of vertices of A in each of the categories. This gives the following lower bound for the number of edges in G:

$$\begin{aligned}2|E(G)| &\geq i(t-5)+j(t-4)+k(t-4)+2\binom{t-5}{2}+2(t-3)\\&\quad+i(t-2)+(j+k)(t-3)\\&=(n-t+2)(2t-7)+2\binom{t-5}{2}+2(t-3).\end{aligned}$$

Since $n \geq 7t-31$, this implies that G has more edges than R, which completes the proof of Case 4 and Theorem 15. □

Proof of Theorem 16 As was described earlier, let D be the graph obtained from $\overline{K}_{t-3} \cup \lfloor (n-2t+6)/2 \rfloor K_2$ when n is even and $\overline{K}_{t-2} \cup \lfloor (n-2t+5)/2 \rfloor K_2$ when n is odd by making all of the vertices of the $\overline{K}_{t-3}$ for even n and $\overline{K}_{t-2}$ for odd n adjacent to a fixed vertex of the remaining vertices of D. Then, R is the graph obtained from $K_{t-3}+D$ by deleting a perfect matching between K_{t-3} and $\overline{K}_{t-3}$ when n is even, and by deleting a deleting a matching between with $t-3$ edges between K_{t-3} and the $\overline{K}_{t-2}$ along with one additional edge such that each vertex in the K_{t-2} is the end vertex of a missing edge. We will show that the graph $R \in Sat(n, K_t - 2K_2)$, and it is easily checked that $|E(R)| = \lfloor((2t-5)n - t(t-3) - 1)/2\rfloor$ edges.

Let $G \in SAT(n, K_t - 2K_2)$. Since the vertices in $K_t - 2K_2$ have degree $t-1$ or $t-2$, $\delta \geq t-3$ and each pair of nonadjacent vertices of G will have at least $t-5$ common adjacencies. Also, $|B| = \delta$ and $|A| = n - \delta - 1$.

Case 1: $\delta \geq t$.

The graph H will have either one or two vertices of A, the vertex y has at least $t-4$ adjacencies in B, and there will be at least a complete graph K_{t-4} in the neighborhood of y in B. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2\delta + (t-4)(n-\delta-1) + 2\binom{t-4}{2} + (n-\delta-1)\delta \\ &= (\delta+t-4)(n-\delta-1) + 2\delta + (t-4)(t-5).\end{aligned}$$

Since, $\delta \geq t$, $\delta + t - 4 > 2t - 5$, and so for $n \geq 8t - 25$, it can be shown that G has more than $|E(R)|$ edges.

Case 2: $\delta = t-1$.

The graph H will have either one or two vertices of A. Thus, the vertex y has at least $t-4$ adjacencies in B, and $H \cap B$ will contain at least a nearly complete graph K_{t-3} with at most one missing edge. Also, each vertex of B has degree at least $t-5$ relative to B. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2(t-1) + (t-4)(n-t) + 2\left(\binom{t-1}{2} - 7\right) + 8 + (n-t)(t-1) \\ &= (2t-5)n - t^2 + 4t - 6.\end{aligned}$$

Since, $(2t-5)n - t^2 + 4t - 6 \geq (2t-5)n - t^2 + 3t - 1$, for $t \geq 5$, we can assume that $\delta \leq t-2$.

Case 3: $\delta = t-2$.

The graph H will contain either one or two vertices of A. In the case of just one vertex y, then y will be adjacent to at least $t-3$ vertices of B, and B will be nearly complete missing a matching with at most 2 edges. Let j be the number of vertices of A with this property. If H contains two vertices $\{y, y_1\}$, then y will be adjacent to at least $t-4$ and y_1 will be adjacent to at least $t-3$ vertices that form a complete graph with at most one edge missing. If the vertex y is adjacent to precisely $t-4$ vertices of B, then y is adjacent to at least two vertices of A, say $\{y_1, z\}$. Let A' be those vertices with degree precisely 2 in B, i' be the number of such vertices, and i be the number of such vertices with degree at least $t-3$ in B.

If $y \in A'$, then consider any other vertex $y' \in A' - \{y, z\}$. The addition of the edge yy' will result in a $K_t - 2K_2$ containing yy'. It is straightforward to check that for this to occur, y' will have to be adjacent to at least one of y_1 or z. Thus, the sum of the degrees of y_1, z in B will be increased by at least $i' - 2$ as a result. This gives the following count on the number of edges in G:

$$
\begin{aligned}
2|E(G)| &\geq 2(t-2)+2\left(\binom{t-2}{2}-2\right)+i(t-4)+i'(t-2)\\
&\quad +j(t-3)+(n-t+2)(t-2)+i+(i'-4)\\
&=(2t-5)n-t^2+6t-16.
\end{aligned}
$$

Since, $(2t-5)n-t^2+6t-16 \geq 2((t-3)n-t^2+3t-1$, for $t \geq 5$, this implies $\delta \leq t-3$.

Case 4: $\delta = t-3$.

The graph H will contain precisely two vertices of A, which we denote by y and y_1. The vertex y will be adjacent to at least $t-4$ vertices of B and y_1 will be adjacent to all of the vertices of B. If $d_B(y) = t-4$, then B will be a complete graph. In general B will be complete except for possibly one edge. Let A' be the set of all vertices of A with degree $t-4$ relative to B, and let s be the number of vertices. If y has $t-3$ adjacencies in B, then y has $t-2$ adjacencies relative to $(A-A') \cup B$.

If $A' = \emptyset$, then the following count results:

$$
\begin{aligned}
2|E(G)| &\geq 2(t-3)+2\left(\binom{t-3}{2}-1\right)+(n-t+2)(t-3)+(n-t+2)(t-2)\\
&=(2t-5)n-t^2+4t-6.
\end{aligned}
$$

Since $\lceil((2t-5)n-t^2+4t-6)/2\rceil \geq \lfloor(2t-5)n-t^2+3t-1\rfloor$ for $t \geq 5$, we can assume that $A \neq \emptyset$. If each vertex of A' has degree at least $t-2$, then since each vertex of A' contributes an additional edge count to vertices in $A-A'$, and the vertices of $A-A'$ have degree at least $t-2$, the following bound exists:

$$
\begin{aligned}
2|E(G)| &\geq 2(t-3)+2\left(\binom{t-3}{2}\right)+s(t-4)+(n-t+2-s)(t-3)\\
&\quad +(n-t+2)(t-2)+s=(2t-5)n-t^2+4t-4.
\end{aligned}
$$

Thus, $|E(G)| \geq |E(R)|$ in this case.

We are left with the case when some of the vertices of A' have degree $t-3$. Note, that if $u, v \in A'$ with $uv \in E(G)$, then each of u and v have degree at least $t-2$. Also, if $u, v \in A'$ are nonadjacent and have the same non-adjacency in B, then the addition of the edge uv results in a copy of $K_t - 2K_2$, and so each of u and v will have two additional adjacencies in A and have degree at least $t-2$. Let A'' be the vertices of A' of degree $t-3$, and let $s' = |A''|$. Each pair of vertices in A'' are nonadjacent, and each will have a distinct nonadjacency in B. The addition of an edge between a pair of vertices in A'' results in a $K_t - 2K_2$, so the vertices must have a common adjacency in $A-A'$. This implies there is one vertex in $A-A'$ adjacent to all of the vertices of A''. Therefore,

$$\begin{aligned}2|E(G)| &\geq 2(t-3) + 2\left(\binom{t-3}{2}\right) + s(t-4) + (n-t+2-s)(t-3)\\ &\quad +(n-t+2-s')(t-2) + s\\ &= (2t-5)n - t^2 + 4t - 4 - s'.\end{aligned}$$

Since $s' \leq t-3$, $\lceil((2t-5)n-t^2+4t-4-s')/2\rceil \geq \lceil((2t-5)n-t^2+3t-1)/2\rceil \geq \lfloor((2t-5)n-t^2+3t-1)/2\rfloor$, this completes the proof of Case 3 and Theorem 16. □

5 Concluding Remarks and Questions

The upper bound of the Kászonyi and Tuza result for the saturation number applied to nearly complete graphs gives either an exact value or a very good approximation. This leaves the obvious question:

Question 1 *Is there a universal lower bound for the saturation number sat* $(n, K_t - F)$ *where the order of* F *is small compared to* t*? Can the bound be expressed using the parameters used in the Kászonyi and Tuza result.*

A more specialized class of graphs which could be easier to deal with is $K_t - T_s$, where T_s is a tree with $s < t$ or s much smaller than t.

Question 2 *What is the saturation for number sat*$(n, K_t - T_s)$*, where* $s < t$*. In particular what is true for trees such as paths, cycles, brooms, etc.?*

Several examples have been given of graphs for which the Kászonyi and Tuza result gives the precise value of the saturation number. It would be interesting to know if there other classes of graphs for which this occurs.

Question 3 *What are classes of graphs for which the upper bound expressed by the Kászonyi and Tuza result is the precise saturation number of the graph.*

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