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Minimum degree and disjoint cycles in generalized claw-free graphs

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ABSTRACT

For $s \geq 3$ a graph is $K_{1,s}$-free if it does not contain an induced subgraph isomorphic to $K_{1,s}$. Cycles in $K_{1,3}$-free graphs, called claw-free graphs, have been well studied. In this paper we extend results on disjoint cycles in claw-free graphs satisfying certain minimum degree conditions to $K_{1,s}$-free graphs, normally called generalized claw-free graphs. In particular, we prove that if G is $K_{1,s}$-free of sufficiently large order $n = 3k$ with $\delta(G) \geq n/2 + c$ for some constant $c = c(s)$, then G contains k disjoint triangles. Analogous results with the complete graph K_3 replaced by a complete graph K_m for $m \geq 3$ will be proved. Also, the existence of 2-factors for $K_{1,s}$-free graphs with minimum degree conditions will be shown.

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1. Introduction

In this paper we consider only graphs without loops or multiple edges. We let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The *order* of G, usually denoted by n, is $|V(G)|$ and the *size* of G is $|E(G)|$. For any vertex v in G, let $N(v)$ denote the set of vertices adjacent to v and $N[v] = N(v) \cup v$. The *degree* $d(v)$ of a vertex v is $|N(v)|$, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of a vertex in G, respectively. If $U \subset V(G)$, we will use $G[U]$ to denote the subgraph of G induced by the vertices in U and let $E(U_1, U_2)$ denote the set of edges with one end in U_1 and one end in U_2.

Let G and H be graphs. We say that G is *H-free* if H is not an induced subgraph of G. In this paper, we are interested in determining the number of disjoint cycles possible in a $K_{1,s}$-free graph which satisfies certain minimum degree conditions.

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Disjoint cycles in claw-free graphs have been studied in a variety of papers. For example Chen, Faudree, Gould, and Saito investigated the range of the number of cycles in a 2-factor of a 2-connected claw-free graph G of order n with minimum degree $(n-2)/3$ in [1].

Theorem 1. *If G is a 2-connected claw-free graph with $\delta(G) \geq \frac{n-2}{3}$, then G contains a 2-factor with exactly k cycles for $1 \leq k \leq \frac{n-24}{3}$. Furthermore, this result is sharp in the sense that if we lower $\delta(G)$ we cannot obtain the full range of values for k.*

Chen, Markus and Schelp studied independent cycles on the basis of edge density [2].

Theorem 2. *Let $k \geq 1$ and G be a $K_{1,s}$-free graph of order n and size q.*

(1) *If $s = 3$ and $q \geq \frac{1}{2}(3k-1)(3k-4)+1$, then G contains k vertex disjoint cycles.*
(2) *If $s \geq 4$ and $q \geq n16sk^2$, then G contains k disjoint cycles.*

The objective of this paper is to generalize the results for claw-free graphs proved in [3] to $K_{1,s}$-free graphs for $s \geq 4$, and in particular to give analogues for the following three results.

Theorem 3. *Let k be a positive integer. If G is a claw-free graph of order*

$$n \geq 2k^4 - 2k^2 + k$$

with $\delta(G) \geq n/k$, then G contains a 2-factor with $k-1$ components. Further, this value of $\delta(G)$ is best possible.

Theorem 4. *If G is a claw-free graph of order n with $\delta(G) \geq n/3$, then G contains a 2-factor with k disjoint cycles, for $2 \leq k \leq \lfloor n/3 - 2 \rfloor$.*

Theorem 5. *If G is a claw-free graph of sufficiently large order $n = 3k$ with $\delta(G) \geq n/2$, then G contains k disjoint triangles.*

We will need the following results in the proof of the main theorems. The next result, of Komlos, Sarkozy, and Szemeredi [4], verifies a conjecture of Seymour. A consequence of this result is that if G is a graph of sufficiently large order $n = r(k+1)$ with $\delta(G) \geq kn/(k+1)$, then G contains r vertex disjoint copies of K_{k+1}.

Theorem 6. *If $k \geq 1$ and G is a graph of sufficiently large order n with $\delta(G) \geq kn/(k+1)$, then G contains the kth power of a Hamiltonian cycle.*

Ramsey numbers will be used in expressing the bounds on the number of vertex disjoint cycles and vertex disjoint complete graphs in a $K_{1,s}$-free graph with varied minimum degrees. We will denote the Ramsey number $r(K_k, K_m)$ by the shorter notation $r(k, m)$.

Theorem 7 (*Li, Rousseau and Zang [5]*)**.** *The Ramsey number*

$$r(K_k, K_n) \leq (1 + o(1)) \frac{n^{k-1}}{(\log n)^{k-2}}.$$

2. Disjoint complete graphs

The objective is to determine the number of possible disjoint complete graphs K_m for $m \geq 3$ in a $K_{1,s}$-free graph with minimum degree at least n/k for some $k \geq 2$. The graph of Fig. 1 consists of k copies of the graph $K_{n/k}$ with an edge between two copies forming them into a ring. This graph has minimum degree $n/k - 1$. If $n/k = (t+1)m - 1$, then $n = ktm + k(m-1)$, but this graph will contain at most kt disjoint copies of a K_m. However, the order of the graph will accommodate as many as $kt + \lfloor \frac{k(m-1)}{m} \rfloor$ disjoint copies of a K_m. This implies that if G is a $K_{1,s}$-free graph of order n and minimum degree at least n/k, then the maximum number of vertex disjoint copies of a K_m in G that will always exist will be at most $n/m - c$ for some constant $c = c(s, k)$. It will be shown that this does, in fact, always occur.

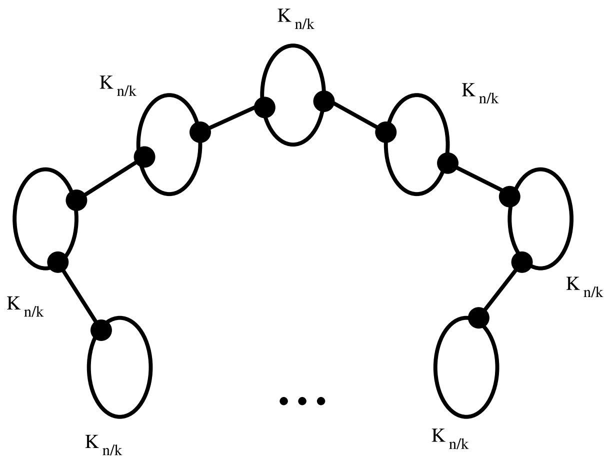

Fig. 1. $K_{1,s}$-free graph G_1 of order $n = ktm + k(m-1)$, $\delta \geq n/k$, but only kt disjoint K_m.

We begin with a look at disjoint triangles.

Theorem 8. *For $s \geq 4$ and $r = r(3, s)$, let G be a $K_{1,s}$-free graph of order n. If G has minimum degree δ, then G contains at least $F_3(n) = (\frac{3(\delta-s+1)}{3\delta+r-s-2})\frac{n}{3}$ disjoint triangles.*

Proof. Select a disjoint cycle system T composed of the maximum number, say t, of triangles. Let $H = G - V(T)$ be the subgraph of G that remains after removing T. No vertex of H can have degree s relative to H, since H is $K_{1,s}$-free and contains no triangles. Thus for each $h \in V(H)$, $d_T(h) \geq \delta - s + 1$.

Consider a triangle $L \in T$ with vertices $\{x, y, z\}$ and let $\{a, b, c\}$ be the degrees of these vertices with respect to H respectively. We can assume with no loss of generality that $a \geq b \geq c$. We will show that $a + b + c \leq r + 2s - 5$. Assume not. If $a \geq r$, then $|N_H(x)| \geq r$, and since G is $K_{1,s}$-free, there is a triangle in H, a contradiction. If $a < r$, then $b \geq s - 1$. Since G is $K_{1,s}$-free, there is an edge in the neighborhood $N_{H\cup\{z\}}(y)$, and so there is a triangle L_1 with vertices y and two vertices of $N_{H\cup\{z\}}(y)$. Since $\lceil (r + 2s - 4)/3 \rceil \geq s + 1$, there is an edge in the neighborhood $N_{H\cup\{z\}}(x)$ that is disjoint from the vertices in L_1. This implies that there is a triangle L_2 with vertices x and two vertices of $N_{H\cup\{z\}}(z)$ that are disjoint from L_1. This contradicts the maximality of T. Thus, we can conclude that the vertices of each triangle in T collectively have at most $r + 2s - 5$ adjacencies in H.

The previous observation implies that $|E(T, H)| \leq t(r + 2s - 5)$, and so

$$(n - 3t)(\delta - s + 1) \leq |E(T, H)| \leq t(r + 2s - 5).$$

Thus,

$$(\delta - s + 1)n \leq (r - s - 2 + 3\delta)t;$$

hence,

$$t \geq \left(\frac{3(\delta - s + 1)}{3\delta + r - s - 2}\right)\frac{n}{3}. \quad \square$$

Consider the case when $\delta \geq n/k$ for $k \geq 2$. Thus,

$$t \geq \left(\frac{3(n/k - s + 1)}{3n/k + r - s - 2}\right)\frac{n}{3},$$

and so

$$t \geq \left(\frac{3n + k(r - s - 2)}{3n + k(r - s - 2)} - \frac{k(r + 2s - 5)}{3n + k(r - s - 2)}\right)\frac{n}{3}.$$

Therefore,

$$t \geq \frac{n}{3} - \left\lceil \frac{(r+2s-5)k}{9} \right\rceil .$$

Corollary 1. *Let $s \geq 4$, $k \geq 2$, and $r = r(3, s)$. If G is a $K_{1,s}$-free graph of sufficiently large order n with minimum degree $\delta(G) \geq n/k$ then G contains at least $\frac{n}{3} - \lceil \frac{(r+2s-5)k}{9} \rceil$ disjoint triangles.*

Thus, for fixed s and k and n sufficiently large, a $K_{1,s}$-free graph with minimum degree n/k has $n/3 - c$ vertex disjoint triangles for some constant $c = c(s, k)$. More specifically, if $s = 4$, then $r = r(3, 4) = 9$, and so we have the following bounds.

Corollary 2. *If G is a $K_{1,4}$-free graph of order n with minimum degree $\delta(G) \geq n/3$ then G contains at least $n/3 - 4$ disjoint triangles, and if the minimum degree $\delta(G) \geq n/2$ then G contains at least $n/3 - 3$ disjoint triangles.*

In $K_{1,s}$-free graphs, strong minimal degree conditions also imply the existence of many vertex disjoint copies of complete graphs K_m for $m \geq 4$. The following result, which is the analogue of Theorem 8, is an example of this.

Theorem 9. *For $s \geq 4$ and $m \geq 4$ let G be a $K_{1,s}$-free graph of order n. If G has minimum degree δ, then G contains at least $F_m(n) = (\frac{\delta - r(s,m-1)+1}{\delta - r(s,m-1)+r(s,m)})\frac{n}{m}$ disjoint copies of a complete graph K_m.*

Proof. Select a disjoint system D composed of the maximum number, say d, of complete graphs K_m. Let $H = G - V(D)$ be the subgraph of G that remains after removing D. No vertex of H can have degree $r(s, m-1)$ relative to H, since H is $K_{1,s}$-free and does contain a copy of K_m. Thus for each $h \in H$, $d_D(h) \geq \delta - r(s, m-1) + 1$.

If a vertex in D has as many as $r(s, m)$ adjacencies in H, then there would be a K_m in H, a contradiction. Thus, the number of edges between a $K_m \in D$ and H will be no more than $m(r(s, m) - 1)$.

The previous observations imply that

$$(n - dm)(\delta - r(s, m-1) + 1) \leq |E(D, H)| \leq dm(r(s, m) - 1).$$

Thus,

$$(\delta - r(s, m-1) + 1)n \leq dm((r(s, m) - 1) + \delta - r(s, m-1) + 1);$$

hence,

$$d \geq \left(\frac{\delta - r(s, m-1) + 1}{\delta - r(s, m-1) + r(s, m)} \right) \frac{n}{m}. \quad \square$$

Consider the case when $\delta \geq n/k$ for $k \geq 2$. Then, in general from Theorem 9,

$$d \geq \left(\frac{\delta - r(s, m-1) + 1}{\delta - r(s, m-1) + r(s, m)} \right) \frac{n}{m},$$

and thus,

$$d \geq \left(\frac{\delta - r(s, m-1) + r(s, m)}{\delta - r(s, m-1) + r(s, n)} - \frac{r(s, m) - 1}{\delta - r(s, m-1) + r(s, m)} \right) \frac{n}{m},$$

or equivalently

$$d \geq \frac{n}{m} - \left(\frac{r(s, m)}{\delta - r(s, m-1) + r(s, m)} \right) \frac{n}{m}.$$

Therefore, when $\delta = n/k$,

$$d \geq \frac{n}{m} - \left(\frac{nr(s, m)k}{mn - kmr(s, m-1) + kmr(s, m)} \right),$$

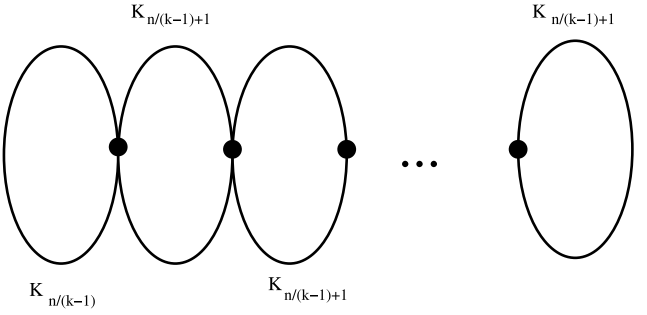

Fig. 2. G_2 composed of $k-1$ blocks with no 2-factor with $k-2$ cycles.

which implies

$$d \geq \frac{n}{m} - \left\lceil \left(\frac{r(s,m)k}{m} \right) \right\rceil,$$

since $r(s,m) - r(s,m-1)$ is a positive integer. □

This results in the following corollary.

Corollary 3. *For $s \geq 4$ and $k \geq 2$ let G be a $K_{1,s}$-free graph of order n. If G has minimum degree n/k, then G contains at least $\frac{n}{m} - c$ vertex disjoint copies of K_m for some $c = c(m,k,s)$. More specifically, G has at least $\frac{n}{m} - \lceil (\frac{r(s,m)k}{m}) \rceil$ vertex disjoint copies of K_m.*

For example, a graph G of sufficiently large order n with minimum degree $n/4$ will have at least $n/4 - 18$ disjoint copies of a K_4, since $r(4,4) = 18$.

3. Disjoint cycles

The objective of this section is to determine the number of possible cycles in a 2-factor in a $K_{1,s}$-free graph with minimum degree at least n/k for some $k \geq 2$. Consider the graph G_2 formed by taking one copy of $K_{n/(k-1)}$ and identifying a vertex with a vertex in a copy of $H_2 = K_{n/(k-1)+1}$. Now identify a new copy of H_2 with a different vertex of the last copy, and repeat this process until we have a "path" of subgraphs with $k-1$ blocks (see Fig. 2). The graph G_2 is $K_{1,s}$-free and has order n, and $\delta(G_2) = n/(k-1) - 1$. Also, $n/(k-1) - 1 \geq n/k$ whenever $n \geq (k-1)k$, and G_2 clearly has a 2-factor with $k-1$ components, but no 2-factor with $k-2$ cycles.

To verify that a $K_{1,s}$-free graph G of order n with $\delta(G) \geq n/k$ has a 2-factor with $k-1$ components, we will need the following lemma on the independence number of such a graph.

Lemma 1. *If G is a $K_{1,s}$-free graph with $\delta(G) \geq n/k$ for $k \geq 2$, then the independence number $\alpha(G) \leq (s-1)k - 1$.*

Proof. Choose an independent set S with $\alpha = \alpha(G)$ vertices. Let $H = G - S$ be the remaining subgraph of order $n - \alpha$. Any vertex of H has degree at most $s-1$ in S as G is $K_{1,s}$-free. Further, each vertex of S has all its neighbors in H. If $E = E(S,H)$ is the set of edges between S and H, then

$$\alpha \left(\frac{n}{k} \right) \leq |E| \leq (s-1)(n-\alpha)$$

and so

$$\alpha \leq \frac{(s-1)kn}{n+(s-1)k} = k(s-1) \left(\frac{n}{n+(s-1)k} \right) < (s-1)k;$$

hence,

$$\alpha(G) \leq (s-1)k - 1. \quad \square$$

Theorem 10. *Let k be a positive integer, and $s \geq 4$. If G is a $K_{1,s}$-free graph of sufficiently large order n with $\delta(G) \geq n/k$, then G contains a 2-factor with $k-1$ components. Further, this value of $\delta(G)$ is best possible, in that $\delta(G) \geq n/(k+1)$ is not sufficient.*

Proof. Suppose we select a vertex disjoint set system $\mathcal{C}$ with $k-1$ cycles $C_1, C_2, \ldots, C_{k-1}$, where $|\cup_{i=1}^{k-1} V(C_i)|$ is as large as possible. We know that such a set exists from Corollary 1. Let $H = G - \cup_{i=1}^{k-1} V(C_i)$.

Observe that with any one cycle C_i, a vertex $h \in V(H)$ has at most $(s-1)k-1$ adjacencies, for otherwise there would exist an independent set (predecessors of adjacencies along with h) of order at least $(s-1)k$, a contradiction to Lemma 1. Thus, $\delta(H) \geq n/k - (k-1)((s-1)k-1)$.

But the bound on $\delta(H)$ implies that H contains a cycle of length at least $\delta(H)+1$. Thus, as $\mathcal{C}$ is as large as possible, each cycle C_i $(1 \leq i \leq k-1)$ contains at least $\delta(H)+1 \geq n/k - c'$ vertices for some constant $c' = c'(k, s)$. This, in turn, implies that $V(H) \leq n/k + c$ for some constant $c = c(s, k)$. Hence, for n sufficiently large, H is dense and, in fact, H is hamiltonian connected, since $2(n/k - c')$ is significantly larger than $n/k + c$.

Claim 1. *No cycle in $\mathcal{C}$ has two independent edges to H.*

Suppose this were not the case; say, C_b has edges w_ih_i and w_jh_j with $w_i, w_j \in V(C_\ell)$ and $h_i, h_j \in V(H)$. Without loss of generality we can assume that $w_i, w_{i+1}, \ldots, w_j$ contains more than half of the vertices of C_b. Therefore, the cycle

$$(w_i, w_{i+1}, \ldots, w_j, h_j, P, h_i, w_i),$$

where P is a hamiltonian path connecting h_i and h_j in H, is a cycle longer than C_b, contradicting our choice of $\mathcal{C}$.

Claim 2. *No two cycles of $\mathcal{C}$ have three independent edges between them.*

Suppose instead that C_a and C_b had three independent edges between them. Without loss of generality say that a_1b_1, a_2b_2 and a_3b_3 are these edges with $a_i \in C_a$ and $b_i \in C_b$, $i = 1, 2, 3$. Also, without loss of generality, suppose that the segment (a_1, a_2) contains less than $|C_a|/3$ vertices and (b_1, b_2) contains less than $|C_b/2|$ vertices. Then, a new cycle

$$C_a' = (a_2, a_2^+, \ldots, a_1, b_1, b_1^-, \ldots, b_2, a_2)$$

replaces C_a and H replaces C_b to form a new system with more vertices than $\mathcal{C}$, a contradiction.

By Claims 1 and 2 we see that some cycles may have a vertex of large degree to H, but then no other vertices of that cycle have any adjacencies in H.

Observe that each vertex of H has edges to $\mathcal{C}$. If this were not true, and $d_{\mathcal{C}}(h) = 0$ for some $h \in V(H)$, then since $d(h) \geq n/k$, this implies that $|H| \geq n/k + 1$. Since every cycle in $\mathcal{C}$ is at least as large as H, this gives the contradiction that $n \geq k(n/k+1) = n+k$. By the same reasoning, no vertex of H has only one edge to $\mathcal{C}$, because if this were the case then we would have $|H| \geq n/k - 1 + 1 = n/k$ and, hence, $|H| = n/k = |C_i|$ for $i = 1, 2, \ldots, k-1$. But then every vertex of every cycle has edges to other cycles, which is in contradiction to one of the claims 1 or 2.

The previous observations imply that each of the cycles C_i and H induce dense subgraphs of order approximately n/k. That is, with the exception of a function of $c^* = c^*(k, s)$ vertices in each cycle, the vertices have degree at least $n/k - c_1$ for some $c_1 = c_1(k, s)$. Since each cycle is only of order at most $n/k + c_2$ for some $c_2 = c_2(k, s)$, these dense subgraphs will have strong hamiltonian properties. For example, even after a small number of vertices are removed, a cycle will span the rest of the dense subgraph.

Now suppose that $H = \{C_0, C_1, \ldots, C_q\}$ are the cycles with edges to other cycles. If we consider these cycles as the vertices of a graph, then among these $q+1$ cycles there are at least $q+1$ independent edges, and a cycle of cycles can be formed.

Say that $\{C_{i_1}, C_{i_2}, \ldots, C_{i_t}, C_{i_1}\}$ are the "vertices" of this cycle. Then, starting in C_{i_1} we may traverse all but a function of k and s vertices before we cross to C_{i_2}. In C_{i_2} we traverse all but a function of k and s vertices before we cross to C_{i_3}, where we traverse a minimum number of vertices (some function k

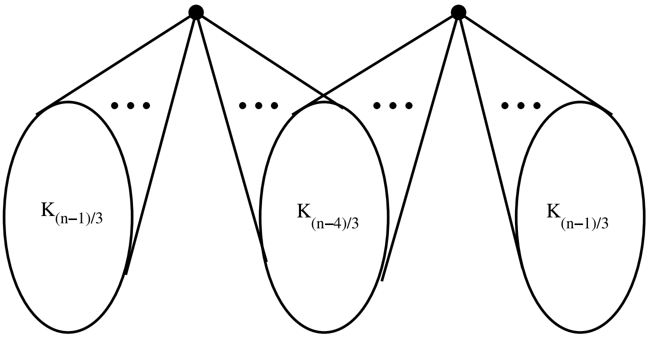

Fig. 3. Claw-free graph G_3, with no 2-factor consisting of two cycles.

and s) before we cross to C_{i_4}. Continuing in this manner we return to C_{i_1}, completing a cycle. Now on the other subgraphs corresponding to this cycle we form new cycles using a maximum number of the remaining dense subgraphs. Thus, at most a function of k and s vertices has been lost from any of the original cycles.

We now form $\mathcal{C}'$ to include all these new cycles, as well as H if it is not a part of these cycles, and all the unchanged cycles from $\mathcal{C}$. This is a system of $k-1$ cycles that includes all but a function of k and s vertices of G, contradicting our choice of $\mathcal{C}$ and completing the proof. □

The graph G_2 in the case $k = 3$ shows that $\delta(G) \geq n/2$ is needed to obtain a Hamiltonian cycle in a $K_{1,s}$-free graph of order n. The graph G_3 of Fig. 4 has order n and $\delta(G_3) = \frac{n-1}{3}$, but clearly cannot be covered by two cycles. Thus $\delta(G) \geq n/3$ is required to have a 2-factor with just two cycles (see Fig. 3).

Theorem 11. *If G is a $K_{1,s}$-free graph of order n with $\delta(G) \geq n/3$, then G contains a 2-factor with k disjoint cycles for $2 \leq k \leq \lfloor n/3 - \frac{r(3,s)+2s-5}{3} \rfloor$.*

Proof. When $k = 2$, the result holds by Theorem 10. Suppose we select a disjoint cycle system $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ for each $t \geq 3$ in the range. We know that such a system exists by Corollary 1. Assume that $\mathcal{C}$ is chosen to contain the maximum number of vertices, and let $H = G - \mathcal{C}$.

Observe that if $d_H(h) > n/(t+1)$ for all $h \in V(H)$, then H contains a cycle of length greater than $n/(t+1)$ and, hence, each cycle in $\mathcal{C}$ has length greater than $n/(t+1)$, or we could find a system larger than $\mathcal{C}$. This implies $|V(G)| = n > (t+1)(n/(t+1)) = n$, a contradiction. Therefore, for each $t \geq 3$ there exists a vertex $h \in V(H)$ such that $d_{\mathcal{C}}(h) \geq n/3 - n/(t+1)$. We also have by Lemma 1 that $\alpha(G) \leq 3s - 4$.

Previous arguments imply that there is a vertex $x \in V(H)$ such that $d_{\mathcal{C}}(x) \geq cn$ for some constant c. Observe that x has at most $3s - 5$ adjacencies to any cycle of $\mathcal{C}$, since more adjacencies would imply an independent set with at least $3s - 3$ vertices using predecessors of the adjacencies of x and x. Therefore, x is adjacent to a function of n different cycles of $\mathcal{C}$, say q. Hence $q \geq cn/(3s-5)$.

Let $X = \{x_1, x_2, \ldots, x_q\}$ be the adjacencies of x in these q cycles. Since $\alpha(X) \leq 3s - 4$, there is a subset $X_1 \subset X$ that induces a complete graph and $|X_1| \geq q^{1/3s}$. Let X_1^+ be the predecessor of the vertices of X_1 on the respective cycles. There is a subset $X_2 \subset X_1^+$ that induces a complete graph. This can be repeated with the successors of the adjacencies of x to form a subset $X_3 \subset X_2$ with at least two vertices. This implies that there are vertices $y_1, y_2 \in X$ in cycles C' and C'' respectively such that $y_1y_2 \in E(G)$, $y_1^+y_2^+ \in E(G)$, and $y_1^-y_2^- \in E(G)$. The two cycles C' and C'' can be replaced by the cycle (x, y_1, y_2, x) and the cycle formed from $C' - \{y_1\}$ and $C'' - \{y_2\}$ using the edges $y_1^+y_2^+$ and $y_1^-y_2^-$. This contradicts the maximality of the cycle system $\mathcal{C}$, and completes the proof of Theorem 4. □

4. Complete graph factors

In [3] it was shown that in a claw-free graph of order $n = 3k$, $\delta(G) \geq n/2$ is sufficient to imply that there are k vertex disjoint triangles (Theorem 5). The minimum degree condition $\delta(G) \geq n/2$ is not sufficient if the triangle K_3 is replaced by the a complete graph K_m for $m \geq 4$ with n divisible by m.

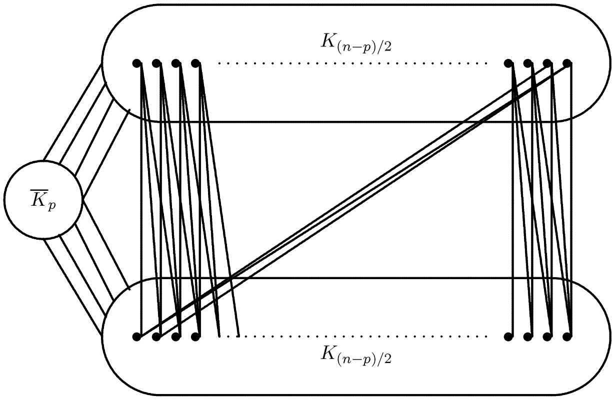

Fig. 4. G_4.

For a fixed integer p with $n-p$ divisible by 2, consider the graph $\overline{K_p} + (K_{(n-p)/2} \cup K_{(n-p)/2})$. Let $X = \{x_1, x_2, \ldots, x_{(n-p)/2}\}$ and $Y = \{y_1, y_2, \ldots, y_{(n-p)/2}\}$ be the vertices of the two complete graphs. For $m \geq 4$ and for each i with $1 \leq i \leq (n-p)/2$ add the edges $x_iy_i, x_iy_{i+1}, \ldots, x_iy_{m-3}$ with the indices taken modulo $(n-p)/2$. Denote this graph by G_4 (see Fig. 4). There is no K_m in G_4 with vertices in both X and Y, and so all copies of a K_m will have all of its vertices in either X or Y or $m-1$ vertices in either X or Y and one vertex in $\overline{K_p}$. Therefore, if n is divisible by m, and there are n/m vertex disjoint copies of a K_m, then $p = p_1 + p_2$ such that $(n-p)/2 - p_i(m-1)$ is divisible by m for $i = 1, 2$. This implies that $p(m-2)$ is divisible by m. Hence, if p is chosen such that $p(m-2)$ is not divisible by m and $p < s$, then G_4 does not contain n/m vertex disjoint copies of a K_m. However, $\delta(G_4) \geq (n+p-8+2m)/2 > n/2$ for $m \geq 4$ and $p \geq 1$. Thus, a minimum degree condition of $\delta(G) \geq n/2 + c$ where $c = c(m, s)$ will be needed to imply the existence of n/m vertex disjoint copies of a K_m.

Our goal in this section is to prove the following result.

Theorem 12. *Let $m \geq 4$ and $s \geq 3$. If G is a $K_{1,s}$-free graph of sufficiently large order $n = km$, then there is a $c = c(s, m)$ such that if $\delta(G) \geq n/2 + c$, G contains k disjoint copies of K_m.*

Proof of Theorem. By Lemma 1, $\alpha(G) \leq 2s-3$. Since G does not contain $2s-3$ independent vertices, Ramsey theory implies that G contains a large clique; in fact, G contains a $K_{n^{\frac{1}{2s-2}}}$. Select such a clique and denote it by A. Let $B \subseteq G - A$ be those vertices of $G - A$ whose degree to A is at most $r^* = m(r(m, 2s-2) - 1)$. Let $C = G - (A \cup B)$.

Observe that

$$|E(A, C)| \geq |A|(n/2 + c - |A|) - r^*|B|.$$

Thus,

$$|C| \geq \frac{|A|(n/2 + c - |A|) - r^*|B|}{|A|},$$

since each vertex in C has at most A adjacencies in A. However, since $|A| \geq n^{\frac{1}{2s-2}}$, and c and r^* are constants and not a function of n,

$$|C| \geq n/2 - o(n).$$

Let

$$B_2 = \{b \in B \mid d_C(b) \geq mr(m, 2s-2)\},$$

and let $B_1 = B - B_2$. Note that each vertex in B_1 has at most $2(m-1)r(m, 2s-2)$ adjacencies in $A \cup C$ and so if B_1 is nonempty,

$$|B_1| \geq n/2 - 2mr(m, 2s-2) \approx n/2.$$

Now we consider the partition $V(G) = B_1 \cup D$, where $D = A \cup B_2 \cup C$. Note that we have both $|B_1| \approx n/2$ and $|D| \approx n/2$. If $|B_1| \equiv 0 \mod m$, then let $B'_1 = B_1$. If $|B_1| \geq n/2$, then every vertex of D must have at least c adjacencies to B_1. Hence, as G is $K_{1,s}$-free and $c = c(s, m)$ is large, we may find a K_m containing $m - 1$ vertices of B_1 and one vertex of D. Remove this copy of a K_m. Continue to do this until we get a subgraph B'_1 of B_1 such that

$$|B'_1| \equiv 0 \mod m.$$

If $|B_1| < n/2$, then each vertex of B_1 has at least c adjacencies to D. As before, we can find a copy of K_m containing one vertex of B_1 and $m - 1$ vertices of D. Remove this K_m and continue this until we get a subgraph B'_1 of B_1 such that

$$|B'_1| \equiv 0 \mod m.$$

Now, since B'_1 is very dense and has order a multiple of m, and n is sufficiently large, we may apply Theorem 6 to B'_1 to obtain an independent set of disjoint copies of K_m that covers all of B'_1.

We can find a copy of K_m in the vertices of B_2 as long as there are at least $r(m, 2s - 2)$ vertices remaining in B_2. Each of the remaining vertices after the deletion of the K_m have at least $mr(m, 2s-2)$ adjacencies in C, so each of these remaining vertices can be placed in a K_m using $m - 1$ vertices in C.

We can find a copy of K_m in the vertices of C as long as there are at least $r(m, 2s - 2)$ vertices remaining in C. Each of the remaining vertices after the deletion of the K_m have at least $(m-1)r(m, 2s-2)$ adjacencies in A, so each of these remaining vertices can be placed in a K_m using $m - 1$ vertices in A. Since A is a complete graph, the remaining vertices of A can be partitioned into disjoint copies of complete graphs K_m. □

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