Property $P_{d,m}$ and Efficient Design of Reliable Networks

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For $d \geq 1$ and $m \geq 1$, a graph has property $P_{d,m}$ if there exist at least m vertex-disjoint paths of length at most d between each pair of vertices. Property $P_{d,m}$, which has a strong connection to wide diameter, is one way of measuring the reliability of a network. In this article, we first examine the relationship of $P_{d,m}$ to other similar properties and then we prove several results regarding the extremal number for property $P_{d,m}$ (the minimum number of edges needed for a graph to have the property). In particular, we find (i) the extremal number for graphs of certain orders when $d = 2$, (ii) several extremal graphs when $d \geq 3$, (iii) a new lower bound on the extremal number when $d \geq 3$, $m \geq 3$, and (iv) a new upper bound on the extremal number when d, m are even with $d = 4k + 2$ ($k \geq 1$) and $m \geq 4$. © 2012 Wiley Periodicals, Inc. NETWORKS, Vol. 60(3), 167–178 2012

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1. INTRODUCTION

Connectivity is frequently the basis for measuring reliability in networks. Whitney's Theorem, based on Menger's Theorem [14], provides the link between these ideas:

Theorem 1.1 (Whitney's Theorem). [19] *A graph G of order $n \geq 2$ is k-connected ($1 \leq k \leq n-1$) if and only if for each pair u, v of distinct vertices there are at least k internally disjoint u–v paths in G.*

Thus, a graph representing a highly connected network can be seen as reliable since even after several vertex failures, each vertex can still communicate to any of the other working vertices through the remaining paths. Several properties have been introduced to precisely describe this measure of reliability. One such property is property $P_{d,m}$, which has the added benefit of providing many short paths. A graph G has property $P_{d,m}$ if there exist at least m vertex-disjoint paths of length at most d between any pair of vertices. Note that a graph with property $P_{d,m}$ has a very specific type of container between each pair of vertices. For vertices x, y in G, an (x, y)-container is a set of internally disjoint paths between x and y. So a graph with property $P_{d,m}$ has a (x, y)-container with at least m paths of length d or less between every pair of vertices x, y in the graph.

These sort of containers provide the basis for several common parameters in the study of the reliability of networks. In particular, property $P_{d,m}$ provided the basis for the concept of w-diameter, which was introduced by Flandrin and Li [9]. The w-diameter is also more commonly known as the wide diameter, which was the term used by Hsu [10] when he independently introduced the same concept. For a m-connected graph G, the wide diameter of G is essentially the minimum number l for which there exist m internally disjoint paths in G of length at most l between every pair of vertices. Note that, for some number d, an m-connected graph has wide diameter less than or equal to d if and only if the graph satisfies property $P_{d,m}$.

The wide diameter of a graph is a frequently studied network parameter. In many recent papers involving the wide diameter, the focus has been on finding or bounding the wide diameter of particular classes of graphs. See [13, 21, 22] for examples. Several other measures of recent interest such as the Rabin number, the fault tolerant diameter, and the edge wide diameter also share a strong relation to the concept of property $P_{d,m}$ as they bring together the ideas of diameter and containers between vertices in an attempt to measure the reliability of networks. See [11, 12, 22] for examples of studies

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of these measures. The recent book by Xu [20] provides a detailed look at each of these measures including links to containers and property $P_{d,m}$.

With these connections in mind, we present several results regarding property $P_{d,m}$ in this article. We prove some basic relationships between $P_{d,m}$ and properties $D_{d,m}$ and $D'_{d,m}$ (defined below) which were introduced after property $P_{d,m}$. The majority of the article focuses on extremal questions about graphs with property $P_{d,m}$. Instead of starting with a particular class of graphs and then determining the appropriate length d so that the graph satisfies $P_{d,m}$ (which is essentially determining the wide diameter), we start with the assumption that a graph has property $P_{d,m}$ and try to determine the minimum number of edges in the graph (or a bound on this number) and the structure of the graph with the minimum number of edges.

2. NOTATION AND DEFINITIONS

Unless stated otherwise, all graphs in this article are simple graphs (i.e., no multiedges and no loops). For any other notation not defined here, see [5]. Define $N(A, B)$ to be set of neighbors of A inside the set B. Let $N[A, B] = A \cup N(A, B)$ be the closed neighborhood of A in B. When $A = \{x\}$, then the neighborhood of x in G will typically be denoted $N(x, G)$. Let $d_G(A, B) = |N(A, B)|$ be the degree of a set A in the set B. In the case when $A = \{x\}$, we use $d_G(x, B)$. If the context is clear, typically we write $d(A, B)$ for $d_G(A, B)$. For distinct vertices $u, v \in V(G)$, let $\text{dist}_G(u, v)$ denote the distance between u and v in G. The join $G_1 + G_2$ of disjoint graphs G_1 and G_2 is obtained from $G_1 \cup G_2$ by adding an edge from every vertex in G_1 to every vertex in G_2. For any positive integer m, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. For any integers a, b with $a \le b$, let $[a, b]$ denote the set $\{a, a+1, a+2, \ldots, b\}$.

The origins of property $P_{d,m}$ go back to several different efforts to quantify the reliability of a network. One of the first efforts was by Murty and Vijayan [17] who introduced the sets $G_V(n, k, \lambda, s)$ and $G_E(n, k, \lambda, s)$. For positive integers n, k, λ, and s such that $n > \lambda \ge k$ and $n > s$, denoted by $G_V(n, k, \lambda, s)$, the set of graphs with n vertices and diameter at most k with the property that the subgraph obtained by deleting any s or fewer vertices has diameter at most λ. Let $G_E(n, k, \lambda, s)$ be the corresponding set when s or fewer edges are deleted.

The sets $G_V(n, k, \lambda, s)$ and $G_E(n, k, \lambda, s)$ contain graphs (or networks) on n vertices which can be considered reliable. That is, even if s vertices or s communication links fail in the network, the remaining vertices are all within distance λ. Limiting the postfailure diameter in the network can ensure that data can be transmitted between any two vertices quickly even after several failures and ensure that the probability of error introduction is low. Thus, a short distance λ might act as a way to keep the data error-free even after s failures.

In 1966, Murty posed the following question:

Problem 2.1 ([15]). *Given n centers of communication $x_1, x_2, \ldots, x_n$ and an $n \times n$ matrix $C = \{c_{ij}\}$, which describes the cost of connecting center x_i with center x_j in the network, what is the optimum set of links which minimizes the total establishment cost when it is required that the resulting network be either a member of $G_V(n, k, \lambda, s)$ or $G_E(n, k, \lambda, s)$?*

Because of the very difficult nature of Problem 2.1, nearly all the early research focused on the case when $c_{ij} = c$ for all $i \neq j$ and for some constant c. In this case, Problem 2.1 is equivalent to finding a graph in $G_V(n, k, \lambda, s)$ or $G_E(n, k, \lambda, s)$ which has the fewest number of edges. Such graphs are called extremal graphs. However, despite this simplification of the problem, extremal graphs could be found only in limited cases such as (i) $k = 2$, $\lambda = 2$ for large n, (ii) $k = 2$, $\lambda > 2$, and (iii) $k \ge 3$ and $s = 1$. For an extensive overview of these results and their extremal graphs, see [3] and Chapter 4 of [2].

In 1986, Ordman introduced the aforementioned property $P_{d,m}$ in [18]. We formally state the definition:

Definition 2.2. *For $d \ge 2$ and $m \ge 2$, a graph G has property $P_{d,m}$ if there exist at least m vertex-disjoint paths of length at most d between any pair of vertices.*

This property provides another way to quantify reliability. As before, a network which satisfies property $P_{d,m}$ has the property that if $m - 1$ vertices or $m - 1$ links fail, then the remaining vertices are less than distance d apart. The benefits of this were mentioned in the discussion of $G_V(n, k, \lambda, s)$ and $G_E(n, k, \lambda, s)$. However, a network which satisfies $P_{d,m}$ has an additional property. In such a network, if there are no failures, then data can be sent quickly and error free in parallel using the guaranteed m short paths. Although it is not obvious that this characteristic distinguishes graphs which satisfy property $P_{d,m}$ from the graphs in $G_V(n, d, d, m - 1)$ or $G_E(n, d, d, m - 1)$, we will prove that the set of graphs which satisfy $P_{d,m}$ is not, in general, the same set as $G_V(n, d, d, m-1)$ or $G_E(n, d, d, m-1)$. However, in 1990, an equivalent notion to being in one of these latter sets was introduced in [4]:

Definition 2.3. *Let $d \ge 1$ and $m \ge 0$ be integers, G be a graph of order n, and for each $m' \le m$, let $V_{m'} = \{v_i | 1 \le i \le m'\}$ and $E_{m'} = \{e_i | 1 \le i \le m'\}$ be arbitrary collections of m' vertices and edges of G, respectively. We say G satisfies* property $D_{d,m}$ *if for each $m' \le m$ and for each choice of $V_{m'}$, the graph $G - V_{m'}$ has a path of length at most d between each pair of vertices. We say G satisfies* property $D'_{d,m}$ *if for each $m' \le m$ and each choice of $E_{m'}$, the graph $G - E_{m'}$ has a path of length at most d between each pair of vertices.*

Note that G has property $D_{d,m-1}$ if and only if $G \in G_V(n, d, d, m - 1)$, and G has property $D'_{d,m-1}$ if and only if $G \in G_E(n, d, d, m - 1)$. As the results of interest to us regarding $G_V(n, k, \lambda, s)$ and $G_E(n, k, \lambda, s)$ all have $k = \lambda = d$ and $s = m - 1$, we will use the simpler notation of $D_{d,m-1}$ and $D'_{d,m-1}$, respectively.

We now turn our attention to property $P_{d,m}$. In Refs. [7,8], various combinations of connectivity, minimum degree, degree sum, and neighborhood conditions which imply that a graph satisfies $P_{d,m}$ were studied. We will primarily be interested in the minimum size of a graph on n vertices which satisfies $P_{d,m}$. To present these results, some additional notation will be helpful. For a graph property P, let $\text{ext}(n; P)$ be the minimum number of edges in a graph G on n vertices which satisfies property P. Note that if G satisfies $P_{d,m}$, then G must be m-connected and so every vertex must have at least m edges incident to it. Thus, we get the following easy lower bound:

Observation 2.4. *For $n \geq m + 1$,*

$$\text{ext}(n; P_{d,m}) \geq \left\lceil \frac{nm}{2} \right\rceil.$$

This simple observation will prove useful particularly as we focus on finding extremal graphs.

3. PROPERTY COMPARISONS

In this section, we prove several results regarding the relationships between the properties $P_{d,m}$, $D_{d,m-1}$, and $D'_{d,m-1}$. We begin with a definition.

Definition 3.1. *For two graph properties, Property A and Property B, we say Property A implies Property B, written $A \implies B$, if every graph with Property A also has Property B.*

Theorem 3.2. $P_{d,m} \implies D'_{d,m-1}$ *and* $P_{d,m} \implies D_{d,m-1}$.

Proof. If G satisfies property $P_{d,m}$, then there exist m vertex-disjoint paths of length d or less between each pair of vertices. The removal of either $m-1$ vertices or $m-1$ edges could destroy at most $m-1$ of the m short paths between any pair of vertices. Thus, after the removal of $m-1$ vertices or edges, the resulting graph has the property that every vertex is within distance d. Thus, G satisfies properties $D_{d,m-1}$ and $D'_{d,m-1}$. ∎

Thus, the set of graphs which satisfies $P_{d,m}$ is contained in the set of graphs which satisfy $D_{d,m-1}$ and the set of graphs which satisfy $D'_{d,m-1}$. When $d = 2$, we can say more with regard to $D'_{d,m-1}$.

Theorem 3.3. $P_{2,m} \iff D'_{2,m-1}$.

Proof. Theorem 3.2 gives the forward implication. For the reverse implication, suppose G is a graph which does not satisfy $P_{2,m}$ but does satisfy $D'_{2,m-1}$. Then for some $x, y \in V(G)$, there exist m edge-disjoint paths of length two or less between them but there do not exist m vertex-disjoint paths of length two or less between them. However, this implies that G must be a multigraph, which is a contradiction. ∎

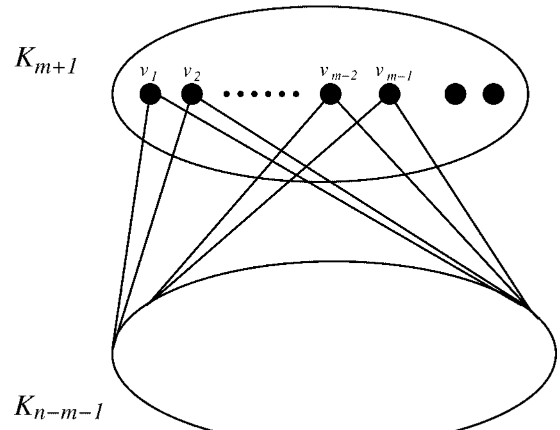

FIG. 1. Construction for Theorems 3.4 and 3.5.

The next theorem shows that this equivalence does not hold for $d \geq 3$.

Theorem 3.4. *For $d \geq 3$, $D'_{d,m-1} \not\Longrightarrow P_{d,m}$.*

Proof. Assume $n \geq m + 3$, and let $d \geq 3$. Let $G_1 = K_{m+1}$ and $G_2 = K_{n-m-1}$. Let T be a set of $m-1$ of the vertices of G_1. Now, let G be the graph consisting of G_1 and G_2 where every vertex in G_2 is adjacent to every vertex of T (see Fig. 1). Now, G satisfies $D'_{d,m-1}$. However, as G is clearly $(m-1)$-connected, there do not exist m vertex-disjoint paths between x and y for every $x \in V(G_2)$ and $y \in V(G_1) - T$. Thus, G does not satisfy $P_{d,m}$. ∎

Not only are $D'_{d,m-1}$ and $P_{d,m}$ not equivalent for $d \geq 3$ but also $D'_{d,m-1}$ and $D_{d,m-1}$ are not equivalent for $d \geq 3$.

Theorem 3.5. *For $d \geq 3$, $D'_{d,m-1} \not\Longrightarrow D_{d,m-1}$.*

Proof. Let $d \geq 3$ and $n \geq m + 3$. The graph in Figure 1 satisfies $D'_{d,m-1}$. However, since G is not m-connected, it does not satisfy $D_{d,m-1}$. ∎

Other relationships involving $D_{d,m-1}$ are more difficult to determine, partially because very little is known about graphs which satisfy $D_{d,m-1}$ for $d \geq 3$. However, a relationship can be established when $d = 2$. First, however, we state the following result by Murty:

Theorem 3.6 ([17]). *If $n \geq 2m$, then $\text{ext}(n; D_{2,m-1}) = m(n-m)$ and the unique extremal graph is $K_{m,n-m}$ (Fig. 2).*

Due to Theorem 3.6, we get the following two results.

Theorem 3.7. $D_{2,m-1} \not\Longrightarrow D'_{2,m-1}$.

Proof. Let $n \geq 2m$, and consider $G = K_{m,n-m}$ (see Fig. 2). By Theorem 3.6, G satisfies $D_{2,m-1}$. Assume the vertices of the $\overline{K}_m$ are $v_1, v_2, \ldots, v_m$, and the vertices of the $\overline{K}_{n-m}$ are $x_1, x_2, \ldots, x_{n-m}$. Note that if the $m-1$

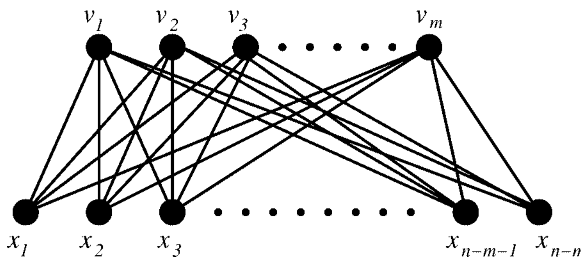

FIG. 2. $K_{m,n-m}$ for Theorems 3.6, 3.7, 3.8, and 6.1.

edges $x_1v_1, x_1v_2, \ldots, x_1v_{m-1}$ are removed, then x_1 and v_1 are distance three apart. Thus, G does not satisfy $D'_{2,m-1}$. ∎

Theorem 3.8. $D_{2,m-1} \not\Longrightarrow P_{2,m}$.

Proof. Consider the graph from Theorem 3.7 (Fig. 2). By Theorem 3.6, G satisfies $D_{2,m-1}$. However, it is straightforward to see that G does not satisfy $P_{2,m}$. ∎

Note that Theorem 3.8 can also be proven by assuming the result is not true and applying Theorem 3.2 to reach a contradiction to Theorem 3.7.

4. NEW CONSTRUCTION FOR $d = 2$

For n sufficiently large relative to m, the value of $ext(n; P_{2,m})$ is known. In [16], Murty proved the following:

Theorem 4.1 ([16]). *For any $m \geq 2$, $ext(n; P_{2,m}) = m(n - m) + \binom{m}{2}$ provided that $n \geq \frac{m(3+\sqrt{5})}{2}$. Furthermore, the graph $K_m + \overline{K}_{n-m}$ is the unique extremal graph.*

Murty's original theorem involves $G_E(n, 2, 2, m-1)$. Note that we have used the equivalence in Theorem 3.3 to update the terminology of Theorem 4.1. The unique extremal graph is illustrated in Figure 3.

In [1], Bollobás and Erdős investigated whether or not the bound on n in Theorem 4.1 could be lowered. They were able to show that the graph in Figure 3 is essentially never the extremal graph when n is smaller than the bound in Theorem 4.1. In addition, they were able to find the order of magnitude of $\text{ext}(n; P_{2,m})$ for smaller n:

Theorem 4.2 ([1]). *Let c be in the range $1 \leq c \leq \frac{(3+\sqrt{5})}{2}$ and let $n = \lfloor cm \rfloor$. Then*

$$ext(n; P_{2,m}) = \frac{1}{2}c^{\frac{3}{2}}m^2 + o(m^2).$$

In this section, we present a new construction and determine $\text{ext}(n; P_{2,m})$ for many cases when $n = m + 2k + 1$. To describe the construction, we need the following lemma. The proof can be found in many graph theory texts (such as Ch. 2 of [6]):

Lemma 4.3. *Suppose $n \geq 2k$. Then, there exists a k-regular graph on n vertices if and only if nk is even.*

Using Lemma 4.3, we now present a new construction and in the process, determine an upper bound for $ext(n; P_{2,m})$.

Theorem 4.4. *Let $n = m + 2k + 1$ where nk is even. Then,*

$$ext(n; P_{2,m}) \leq \frac{(m + 2k + 1)(m + k)}{2}.$$

Proof. Let B be a k-regular graph of order n. We know that such a graph exists by Lemma 4.3. Consider the graph $D = K_{m+2k+1} - E(B)$. Let x and y be two vertices of D. In K_{m+2k+1}, these vertices are adjacent and have $m + 2k - 1$ common neighbors. As each vertex loses exactly k edges on the removal of $E(B)$, then in D either (i) $xy \notin E(D)$ and x and y have at least m common neighbors or (ii) $xy \in E(D)$ and x and y have at least $m - 1$ common neighbors. So, D satisfies $P_{2,m}$ and $|E(D)| = \frac{(m+2k+1)(m+k)}{2}$. ∎

With the upper bound determined, we now proceed to show that the same number in Theorem 4.4 is also a lower bound.

Theorem 4.5. *Let $n = m + 2k + 1$ where nk is even. Then for $m > 2k^2 + 2k + 3$,*

$$ext(n; P_{2,m}) \geq \frac{(m + 2k + 1)(m + k)}{2}.$$

Proof. Let G be a graph on $n = m + 2k + 1$ vertices which satisfies $P_{2,m}$ and has $ext(n; P_{2,m})$ edges where $m > 2k^2 + 2k + 3$. If $k = 0$, then G has $m + 1$ vertices and must be the complete graph to satisfy $P_{2,m}$. In this case, the lower bound is proven. Assume now that $k > 0$ and $\delta(G) < m + k$. Recall that $\delta(G) \geq m$.

Let $H = \overline{G}$. Thus, $\Delta(H) = k + t$ for some t where $1 \leq t \leq k$. Let n_i be the number of vertices of H of degree i, and let u, v be two vertices of H. Note that if $d(v, H) = k + t$ and $d(u, H) = (k - t) + r$ for some r, then u is adjacent to at least $r - 1$ vertices of $N(v, H)$. To see this, suppose the opposite is true. That is, assume there exist u and v where u is adjacent to less than $r - 1$ vertices of $N(v, H)$. Thus, $|N(v, H) \cap N(u, H)| = d < r - 1$. Then, there are two cases.

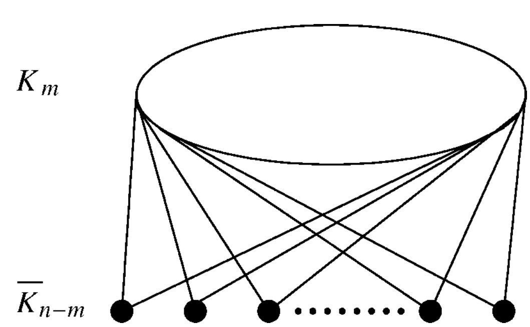

FIG. 3. Extremal graph $K_m + \overline{K}_{n-m}$ for Theorem 4.1.

If $uv \notin E(H)$, then the number of common neighbors of u and v in G is

$$\begin{aligned}|V(H) - (N[v,H] \cup N[u,H])| &= m + 2k + 1 - (k + t + 1 + k - t + r + 1 - d)\\ &= m - 1 - (r - d)\\ &< m - 2.\end{aligned}$$

So, in this case, u and v have at most $m - 3$ common neighbors in G and, since $uv \in E(G)$, there can exist at most $m - 2$ paths of length two or less between these two vertices in G. However, this contradicts our assumptions on G.

If $uv \in E(H)$, then the number of common neighbors of u and v in G is

$$\begin{aligned}|V(H) - (N[v,H] \cup N[u,H])| &= m + 2k + 1 - (k + t + k - t + r - d)\\ &= m + 1 - (r - d)\\ &< m.\end{aligned}$$

So, in this case, u and v have at most $m - 1$ common neighbors in G and, as $uv \notin E(G)$, there can exist at most $m - 1$ paths of length two or less between these two vertices in G. Once again, this contradicts our assumptions on G.

Therefore, as these are the only two cases, u must be adjacent to at least $r - 1$ vertices in $N(v, H)$.

With this in mind, we will now consider the number of edges incident to $N(v, H)$ where $d(v, H) = \Delta(H) = k + t$ for some $1 \le t \le k$. Assume first that $t \ge 2$. We know from the work above that each vertex of degree $(k - t) + r$ has at least $r - 1$ edges into $N(v, H)$ where, as $\Delta(H) = k + t$, we have $0 \le r \le 2t$. So the number of edges from vertices of degree $(k - t) + r$ into $N(v, H)$ is at least

$$\sum_{r=0}^{2t} (r - 1)n_{k-t+r} = \sum_{i=k-t}^{k+t} (i - k + t - 1)n_i.$$

However, as $\Delta(H) = k + t$, the number of edges incident to the vertices in $N(v, H)$ is less than or equal to $(k + t)^2$. Thus, we have

$$\sum_{i=k-t}^{k+t} (i - k + t - 1)n_i \le (k + t)^2.$$

Rearranging the terms, we get the following useful expression:

$$\sum_{i=k-t}^{k+t} in_i \le (k + t)^2 + (k - t + 1) \sum_{i=k-t}^{k+t} n_i. \tag{4.1}$$

By Theorem 4.4, $|E(G)| \le \frac{(m+2k+1)(m+k)}{2}$ and so $|E(H)| \ge \frac{k(m+2k+1)}{2}$. Therefore,

$$\sum_{i=0}^{k+t} in_i \ge (m + 2k + 1)k. \tag{4.2}$$

Now by breaking the sum of the in_i's in (4.2) at $i = k - t - 1$ and applying (4.1), we get

$$\sum_{i=0}^{k-t-1} in_i + (k + t)^2 + (k - t + 1) \sum_{i=k-t}^{k+t} n_i \ge (m + 2k + 1)k.$$

In the first sum, the largest i is $i = k - t - 1$. So,

$$(k - t - 1) \sum_{i=0}^{k-t-1} n_i + (k + t)^2 + (k - t + 1) \sum_{i=k-t}^{k+t} n_i \ge (m + 2k + 1)k. \tag{4.3}$$

Recall that $n = m + 2k + 1$. Let x be the number of vertices which have degree greater than $k - t - 1$ in H. Note that $x \le n$. Then, inequality (4.3) becomes

$$(k - t - 1)(n - x) + (k + t)^2 + (k - t + 1)x \ge nk.$$

Hence,

$$n(k - t - 1) + (k + t)^2 + 2x \ge nk,$$

or

$$(k + t)^2 + 2x \ge nk - n(k - t - 1).$$

Thus, as $n \ge x$,

$$(k + t)^2 + 2n \ge n(t + 1),$$

and so

$$(k + t)^2 \ge n(t - 1).$$

As $t \ge 2$, we get

$$n \le \frac{(k + t)^2}{t - 1}.$$

If we let $f(t) = \frac{1}{t-1}(k + t)^2$, then

$$f'(t) = \frac{(t + k)(t - k - 2)}{(t - 1)^2}.$$

So, $f(t)$ is a decreasing function of t over the interval $[2, k]$ and, consequently, is maximized when $t = 2$. Therefore, $n \le (k + 2)^2$. Then,

$$\begin{aligned}n \le (k + 2)^2 &\Longrightarrow m + 2k + 1 \le k^2 + 4k + 4\\ &\Longrightarrow m \le k^2 + 2k + 3.\end{aligned}$$

However, this is a contradiction to our assumption that $m > 2k^2 + 2k + 3$.

Therefore, it must be the case that $t = 1$. Assume that $v \in V(G)$ with $d(v, H) = \Delta(H) = k + t = k + 1$. Note that as $|E(H)| \geq \frac{k(m+2k+1)}{2}$ and $\Delta(H) = k+1$, there can be at most $\frac{n}{2}$ vertices of degree $k-1$ or less in H.

Now, if T is the set of all vertices of degree k or $k+1$, then every non-neighbor of v in H in the set T has at least one neighbor in $N(v,H)$. To see this, assume that $uv \notin E(H)$ and $u \in T$ with $N(v,H) \cap N(u,H) = \emptyset$. Let J be the set of vertices of $V(H) - (N[v,H] \cup N[u,H])$ and let $j = |J|$. Then,

$$\begin{aligned} n &= j + d(v,H) + d(u,H) + 2 \\ &= j + k + 3 + d(u,H) \end{aligned}$$

Now, since $n = m + 2k + 1$, we have

$$m + 2k + 1 = j + k + 3 + d(u,H)$$

which implies that

$$j = m + k - 2 - d(u,H).$$

As $d(u,H)$ is either k or $k+1$, then j is either $m-2$ or $m-3$, respectively. However, since J is essentially the set of common neighbors of u and v in G, this means that u and v have at most $m-2$ common neighbors in G. Even though $uv \in E(G)$, we do not have m paths of length two or less in G between these vertices and this contradicts our assumptions on G. Therefore, every non-neighbor of v in H in the set T has at least one neighbor in $N(v,H)$.

We now wish to consider the edges in H which have exactly one endpoint in $N(v,H)$. As $d(v,H) = k+1$ and $\Delta(H) = k+1$, the number of edges with one endpoint in $N(v,H)$ is at most $(k+1)^2$. Now, there are at least $(n-1) - (k+1) - \frac{n}{2}$ vertices in $V(H) - \{v\}$ which are non-neighbors of v and have degree k or greater. Each of these vertices has at least one edge into $N(v,H)$. So, along with the $k+1$ edges from v into $N(v,H)$, the number of edges with one endpoint in $N(v,H)$ is at least

$$(k+1) + \left[(n-1) - (k+1) - \frac{n}{2}\right].$$

Thus,

$$(k+1) + \left[(n-1) - (k+1) - \frac{n}{2}\right] \leq (k+1)^2$$

which implies that $n \leq 2k^2 + 4k + 4$. As $n = m + 2k + 1$, we have $m \leq 2k^2 + 2k + 3$, which contradicts our assumption on m.

As these are the only possibilities for t, we have $\delta(G) \geq m + k$ and $|E(G)| \geq \frac{1}{2}(m+2k+1)(m+k)$. ∎

Theorems 4.4 and 4.5 give us the following:

Corollary 4.6. *Suppose that $n = m + 2k + 1$ where nk is even. Then, for $m > 2k^2 + 2k + 3$,*

$$ext(n; P_{2,m}) = \frac{(m+2k+1)(m+k)}{2}.$$

5. SOME BEST-POSSIBLE CONSTRUCTIONS WHEN $d \geq 3$

In this section, we present several best-possible constructions for $d \geq 3$, where best-possible means that the trivial lower bound in Observation 2.4 is achieved. We first focus on $n \leq (d-1)m$. To describe the constructions, we need another definition.

Definition 5.1. *For $k \geq 1$, the kth power G^k of a connected graph G is the graph with $V(G^k) = V(G)$ and with $uv \in E(G^k)$ if and only if $1 \leq \mathrm{dist}_G(u,v) \leq k$.*

The next two theorems use graph powers for their constructions which depend on the parity of m. We present first the result for even m.

Theorem 5.2. *If $d \geq 3$, m is an even integer with $m \geq 2$, and $m + 1 \leq n \leq \frac{dm}{2} + 1$, then*

$$ext(n; P_{d,m}) = \frac{nm}{2}.$$

Proof. As we have the lower bound in Observation 2.4, we prove the upper bound. Consider $G = (C_n)^{\frac{m}{2}}$. Note that $|E(G)| = \frac{nm}{2}$. We now wish to show that G satisfies $P_{d,m}$. Let $x, y \in V(G) = \{x_1, x_2, \ldots, x_n\}$, where the indices correspond to their clockwise cyclic order. Assume without loss of generality that $x = x_1$ and $y = x_j$ for some $j \leq \lceil \frac{n+1}{2} \rceil$.

Let T be the set of neighbors of x_j. We now define a special type of path. We call a path from x_1 to x_j a largest jump possible (LJP-T) path from x_1 to x_j through x_k if it is of the form $x_1 x_k x_{k+\frac{m}{2}} \ldots x_t x_j$ or $x_1 x_k x_{k-\frac{m}{2}} \ldots x_t x_j$ where each step after the first step is either to that vertex's farthest neighbor along the cycle or, in the case of x_t, that vertex's farthest neighbor in T which is not already used by a previously constructed path. We will call an LJP-T path where the indices of the path vertices increase a forward LJP-T path. An LJP-T path where the indices decrease will be called a backward LJP-T path.

As we are taking the largest step possible from each vertex, we can ensure that these LJP-T paths are vertex-disjoint. That is, if x_1 is adjacent to x_2, x_3, x_4, then the vertices distance two away from x_1 on each forward LJP-T path from x_1 to x_j through x_2, x_3, and x_4 are distinct. For each step until x_j is reached, the vertices on each of these different paths will be unique.

Note that in the paths below, consecutive vertices may be the same. That is, the path $x_1x_2x_3$ may have $x_2 = x_3$ but in this event, the actual path is x_1x_3.

Suppose first that $2 \leq j \leq \frac{m}{2} + 1$. In particular, suppose $j = \frac{m}{2} + 1 - i$ for some i. Then, we can find m paths of length d or less by taking:

- the $\frac{m}{2}$ paths of the form $x_1 x_k x_j$ for $k \in [2, \frac{m}{2} + 1]$,
- the i paths of the form $x_1 x_k x_j$ for $k \in [n - i + 1, n]$, and
- the $\frac{m}{2} - i$ backward LJP-T paths from x_1 to x_j through x_k for $k \in [n - \frac{m}{2} + 1, n - i]$.

Note that since LJP-T paths are used in this system, the paths should be constructed in the order listed above.

Suppose now that $j > \frac{m}{2} + 1$; then, we find m paths of length d or less by taking:

- the $\frac{m}{2}$ forward LJP-T paths from x_1 to x_j through x_k for $k \in [2, \frac{m}{2} + 1]$, and
- the $\frac{m}{2}$ backward LJP-T paths from x_1 to x_j through x_k for $k \in [n - \frac{m}{2} + 1, n]$.

For any of the LJP-T paths $x_1 x_k \ldots x_j$, we have

$$|k - j| \le n - \left(\frac{m}{2} + 1\right) \le \frac{dm}{2} + 1 - \left(\frac{m}{2} + 1\right) = \frac{m}{2}(d - 1).$$

So, none of the LJP-T paths have length greater than d. Therefore, the paths above show that G satisfies $P_{d,m}$. ∎

Note that the graph constructed in Theorem 5.2 is a type of circulant graph.

When m is odd and n is even, we use a modification of the construction in Theorem 5.2. This modification allows us to increase the upper bound on n.

Theorem 5.3. *If $d \ge 3$, m is an odd integer with $m \ge 3$, and n is even integer with $2m \le n \le (d-1)(m-1)+2$, then*

$$ext(n; P_{d,m}) = \frac{nm}{2}.$$

Proof. Consider the graph $(C_n)^{\frac{m-1}{2}}$. Label the vertices $x_1, x_2, \ldots, x_n$. Now add the edge $x_i x_{i+\frac{n}{2}}$ for $1 \le i \le \frac{n}{2}$ and call this new graph H. An example of this construction is given in Figure 4. Note that $|E(H)| = \frac{nm}{2}$. We now wish to show that H satisfies $P_{d,m}$. Let x, y be two vertices of H. Assume without loss of generality that $x = x_1$ and $y = x_j$ for some $j \le \frac{n}{2} + 1$. Let $T = N(x_j, H)$ and $T' = N(x_{j+\frac{n}{2}}, H)$.

Suppose that $2 \le j < \frac{m+1}{2}$. In particular, suppose $j = \frac{m-1}{2} + 1 - i$ for some i. Then, we find m paths of length d or less by taking:

- the $\frac{m-1}{2}$ paths of the form $x_1 x_k x_j$ for $k \in [2, \frac{m+1}{2}]$,
- the i paths of the form $x_1 x_k x_j$ for $k \in [n - i + 1, n]$,
- the $\frac{m-1}{2} - i$ paths of the form $x_1 x_k P$ where P is a backward LJP-T path from x_k to x_j through $x_{k-\frac{n}{2}}$ for $k \in [n - (\frac{m-1}{2} - 1), n - i]$,
- the one path of the form $x_1 P x_j$ where P is forward LJP-T' path from x_1 to $x_{j+\frac{n}{2}}$ through $x_{1+\frac{n}{2}}$.

Note that since LJP-T paths are used in this system, the paths should be constructed in the order listed above. This applies also to the cases outlined below that use LJP-T paths.

Suppose now that $j \ge \frac{m+1}{2}$ but $j \le \lfloor \frac{n}{4} \rfloor$. Then, we find m paths of length d or less by taking:

- the $\frac{m-1}{2}$ forward LJP-T paths from x_1 to x_j through x_k for $k \in [2, \frac{m+1}{2}]$,
- the $\frac{m-1}{2}$ paths of the form $x_1 x_k P$ where P is a backward LJP-T path from x_k to x_j through $x_{k-\frac{n}{2}}$ for $k \in [n - (\frac{m-3}{2}), n]$, and

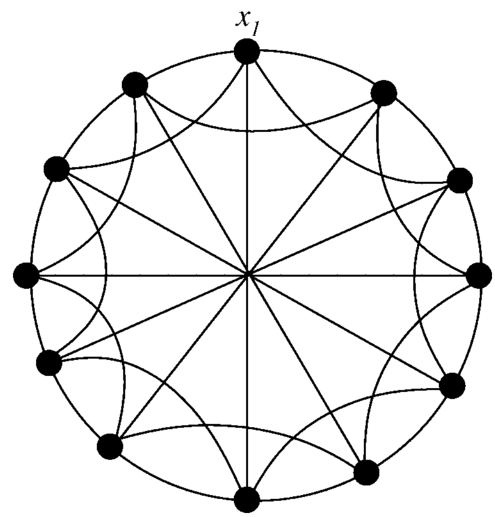

FIG. 4. Construction from Theorem 5.3 with $m = 5$, $n = 12$.

- the one path of the form $x_1 P x_j$ where P is a forward LJP-T' path from x_1 to $x_{j+\frac{n}{2}}$ through $x_{1+\frac{n}{2}}$.

Suppose now that $j \ge \frac{m+1}{2}$, $j > \lfloor \frac{n}{4} \rfloor$, and $x_{j+\frac{n}{2}} \in N(x_1, H)$. In particular, suppose that $x_{j+\frac{n}{2}}$ is x_{n-i} for some $i \in [0, \frac{m-3}{2}]$. Then, we can find m paths of length d or less by taking:

- the $\frac{m-1}{2}$ paths of the form $x_1 x_k x_{k-\frac{n}{2}} x_j$ for $k \in [n - (\frac{m-3}{2}), n]$,
- the $\frac{m-1}{2} - i$ paths of the form $x_1 x_k x_{k+\frac{n}{2}} x_j$ for $k \in [1, \frac{m-1}{2} - i]$, and
- the $i + 1$ forward LJP-T paths from x_1 to x_j through x_k for $k \in [\frac{m+1}{2} - i, \frac{m+1}{2}]$.

Last, suppose that $j \ge \frac{m+1}{2}$, $j > \lfloor \frac{n}{4} \rfloor$, and $x_{j+\frac{n}{2}} \notin N(x_1, H)$. Then, we can find m paths of length d or less by taking:

- the $\frac{m-1}{2}$ paths of the form $x_1 x_k P x_j$ where P is a backward LJP-T path from x_k to x_j through $x_{k-\frac{n}{2}}$ for $k \in [n - (\frac{m-3}{2}), n]$,
- the $\frac{m-1}{2}$ forward LJP-T paths from x_1 to x_j through x_k for $k \in [2, \frac{m+1}{2}]$, and
- the one path of the form $x_1 P x_j$ where P is a forward LJP-T path from $x_{1+\frac{n}{2}}$ to x_j which contains $x_{j+\frac{n}{2}}$.

This takes care of all possibilities.

Note first that the lower bound on n ensures that no chord of the form $x_i x_{i+\frac{n}{2}}$ for $1 \le i \le \frac{n}{2}$ is adjacent to more than one vertex of $N(x_1, H)$. This allows for the construction of the path systems above. We also need to make sure that all these paths have length d or less. The farthest two vertices that must be joined by a path occur in the case when $j = \frac{m-1}{2} + 1 - i$ for some i, and the path is $x_1 x_{n-i} P$ where P is a backward LJP-T path from x_{n-i} to x_j through $x_{n-i-\frac{n}{2}}$. Note that there can be at most $d - 2$ steps of length $\frac{m-1}{2}$ from $x_{n-i-\frac{n}{2}}$ to x_j.

If we subtract the indices, we get the distance along the cycle we must travel. Thus, we need

$$\left(n-i-\frac{n}{2}\right)-\left(\frac{m+1}{2}-i\right)\leq\frac{m-1}{2}(d-2).$$

However, this holds as long as $n \leq (m-1)(d-1)+2$. Thus, all the paths constructed above have length d or less and, consequently, H satisfies $P_{d,m}$. ∎

When $n = m + x = rx$ for some r, x, we can also find a best-possible construction when $d \geq 3$.

Theorem 5.4. *Suppose $n = m + x = rx$ for some integers x, r, and d with $d \geq 3$. Then*

$$ext(n; P_{d,m}) = \frac{nm}{2}.$$

Proof. Consider K_n. Remove edges of r vertex-disjoint copies of K_x from the K_n. The resulting graph is m-regular, and it is straightforward to check that the graph satisfies $P_{3,m}$, and consequently $P_{d,m}$, for any $d \geq 3$. ∎

When $n = 2m$ and $d = 3$, Theorem 5.4 tells us that $K_{m,m}$ is an extremal graph. Note that the graph $K_2 \times K_m$ (Fig. 5), where "×" denotes the Cartesian product of the graphs, is also m-regular and satisfies $P_{3,m}$. However, $K_2 \times K_m$ is not isomorphic to $K_{m,m}$. Thus, in these cases, the extremal graph may not be unique.

We now turn our attention to the case $d \geq 4$. In this final best-possible construction, we assume m is even and $n = tm$ where $t \in [3, d-1]$. Consider the graph $G = C_t \times K_m$. We will refer to the copies of K_m as levels. Assume the levels are labeled from 1 to t and that the vertices of the kth level are labeled $\{x_{k,1}, x_{k,2}, \ldots, x_{k,m}\}$. We will be removing a matching from each level but the type of matching will depend on the location of the level on the cycle. For Level k, define the following types of matchings:

- An M_1 matching consists of the edges

$$x_{k,1}x_{k,2}; x_{k,3}x_{k,4}; \ldots; x_{k,m-1}x_{k,m}.$$

- An M_2 matching consists of the edges

$$x_{k,1}x_{k,m}; x_{k,2}x_{k,3}; \ldots; x_{k,m-2}x_{k,m-1}.$$

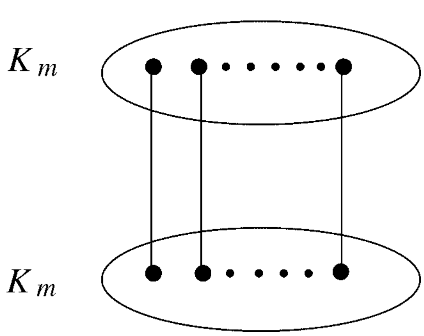

FIG. 5. $K_2 \times K_m$ from discussion following Theorem 5.4.

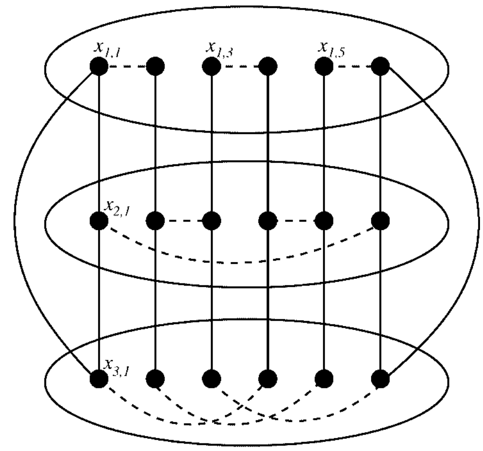

FIG. 6. Construction from Theorem 5.5 with $m = 6$, $n = 18$.

- An M_3 matching consists of the edges

$$x_{k,1}x_{k,m'+1}; x_{k,2}x_{k,m'+2}; \ldots; x_{k,m'}x_{k,m}$$

where $m' = \frac{m}{2}$.

Now, we remove from G a matching from each level. From Level 1, remove an M_1 matching and from Level 2, remove an M_2 matching. Continue alternating in this manner until you reach Level t. If t is even, remove an M_2 matching from Level t. Otherwise, remove an M_3 matching.

Note that the removal of these matchings ensures that no consecutive copies of K_m on the cycle are missing the same edge. Let M be the set of all edges removed. Consider $G - M$. An example when $m = 6$ and $n = 18$ is given in Figure 6, and it shows a different type of matching removed from each level.

Overall, for this construction, note that $G - M$ is m-regular, and so $|E(G - M)| = \frac{nm}{2}$. More importantly, $G - M$ satisfies $P_{d,m}$ as illustrated in the following theorem.

Theorem 5.5. *If $d \geq 4$, m is even with $m \geq 4$, and $n = tm$ for some $t \in [3, d-1]$, then*

$$ext(n; P_{d,m}) = \frac{nm}{2}.$$

Proof. We wish to show that the graph $G - M$ constructed above satisfies $P_{d,m}$. Choose two vertices $x, y \in V(G - M)$ where x is in Level r and y is in Level s. Relabel the graph so that $x = x_{r,1}$ and so that its non-neighbor in the level is labeled $x_{r,2}$. That is, relabel so that an M_1 matching has been removed from Level r. Assume further that the labeling of the graph is such that individual copies of the cycle go through vertices with the same last index number (e.g., $x_{1,1}x_{2,1}x_{3,1}\ldots x_{t,1}$ go through the same copy of C_t). Assume that $y = x_{s,a}$ and its non-neighbor in its level is $x_{s,b}$.

We will call a path a standard path if it is of the form $x_{r,1}x_{r,u}x_{r+1,u}\ldots x_{s,u}x_{s,a}$ or $x_{r,1}x_{r,u}x_{r-1,u}\ldots x_{s,u}x_{s,a}$ for some u. The path systems we find in $G - M$ will consist of standard paths for some set of indices u along with several nonstandard paths. Two standard paths are illustrated in Figure 7.

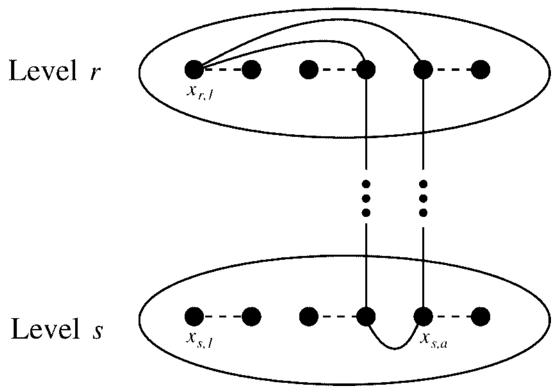

FIG. 7. Standard path for $u = 4, 5$.

Assume that for the first index of each vertex, the addition is performed modulo t. Also assume that the standard paths go through the graph in the direction that minimizes the distance between the target and destination levels. As before, in the paths below, consecutive vertices may be the same. That is, the path $xx_1x_2x_3$ may have $x_1 = x_2$, but in this event, the actual path is xx_1x_3.

Suppose first that $r = s$. Then,

- **For** $a = 2$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with the paths $x_{r,1}x_{r+1,1}x_{r+1,2}x_{s,a}$ and $x_{r,1}x_{r-1,1}x_{r-1,2}x_{s,a}$ to complete the system.
- **For** $a \neq 2$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, b\}$ along with $x_{r,1}x_{r+i,1} \dots x_{r+j,1}x_{r+j,a}x_{r+j-i,a} \dots x_{s,a}$ where j is the least number in absolute value so that $x_{r+j,1}x_{r+j,a} \in E(G)$ and $i \in \{-1, 1\}$ has the same sign as j. Assume without loss of generality that j is negative. Then, take $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r,2}x_{s,a}$ and $x_{r,1}x_{r,b}x_{r+1,b}x_{r+1,a}x_{s,a}$ to complete the system.

Suppose now that $r \neq s$ but that Level s had an M_1 matching removed from it. Then,

- **For** $a = 1$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with $x_{r,1}x_{r+1,1} \dots x_{s,a}$ and $x_{r,1}x_{r-1,1} \dots x_{s,a}$.
- **For** $a = 2$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with $x_{r,1}x_{r+1,1}x_{r+1,2} \dots x_{s-1,2}x_{s,a}$ and $x_{r,1}x_{r-1,1} \dots x_{s+1,1}s_{s+1,2}x_{s,a}$.
- **For** $a \neq 1, 2$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, b\}$ along with $x_{r,1}x_{r,b}x_{r+1,b} \dots x_{s-1,b}x_{s-1,a}x_{s,a}$; $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r+2,2} \dots x_{s,2}x_{s,a}$; and $x_{r,1}x_{r-1,1} \dots x_{s+1,1}x_{s,1}x_{s,a}$.

Suppose now that $r \neq s$ but Level s had an M_2 matching removed from it. Assume that $|\{r, r+1, \dots, s\}| \geq |\{s, s+1, \dots, r\}|$. Then,

- **For** $a = 1$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, m\}$ along with $x_{r,1}x_{r+1,1}x_{r+1,2} \dots x_{s,2}x_{s,a}$; $x_{r,1}x_{r,m}x_{r+1,m}x_{r+2,m} \dots x_{s-1,m}x_{s-1,1}x_{s,a}$; and $x_{r,1}x_{r-1,1} \dots x_{s,a}$.
- **For** $a = 2$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, 3\}$ along with $x_{r,1}x_{r,3}x_{r,2}x_{r-1,2} \dots x_{s,a}$; $x_{r,1}x_{r-1,1} \dots x_{s,1}x_{s,a}$; and $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r+2,2} \dots x_{s,a}$.
- **For** $a = 3$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r+2,2} \dots x_{s-1,2}x_{s-1,a}x_{s,a}$; and $x_{r,1}x_{r-1,1} \dots x_{s,1}x_{s,a}$.
- **For** $a = m$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with $x_{r,1}x_{r-1,1} \dots x_{s+1,1}x_{s,1}x_{s,2}x_{s,a}$; and $x_{r,1}x_{r+1,1} \dots x_{s-1,1}x_{s-1,m}x_{s,a}$.
- **For** $a \notin \{1, 2, 3, m\}$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, b\}$ along with $x_{r,1}x_{r+1,1}x_{r+1,2} \dots x_{s,2}x_{s,a}$; $x_{r,1}x_{r,b}x_{r+1,b} \dots x_{s-1,b}x_{s-1,a}x_{s,a}$; and $x_{r,1}x_{r-1,1} \dots x_{s,1}x_{s,a}$.

Suppose now that $r \neq s$ but Level s had an M_3 matching removed from it. Again, assume that $|\{r, r+1, \dots, s\}| \geq |\{s, s+1, \dots, r\}|$. Then,

- **For** $a = 1$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, b\}$ along with $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r+2,2} \dots x_{s,2}x_{s,a}$; $x_{r,1}x_{r,b}x_{r+1,b} \dots x_{s-1,b}x_{s-1,1}x_{s,a}$; and $x_{r,1}x_{r-1,1} \dots x_{s+1,1}x_{s,a}$.
- **For** $a = 2$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, b\}$ along with $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r+2,2} \dots x_{s-1,2}x_{s,a}$; $x_{r,1}x_{r-1,1} \dots x_{s+1,1}x_{s,1}x_{s,a}$; and $x_{r,1}x_{r,b}x_{r,2}x_{r-1,2} \dots x_{s+1,2}x_{s,a}$.
- **For** $a = m' + 1$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with $x_{r,1}x_{r-1,1} \dots x_{s,1}x_{s,2}x_{s,a}$ and $x_{r,1}x_{r+1,1} \dots x_{s-1,1}x_{s-1,a}x_{s,a}$.
- **For** $a = m' + 2$: Take the $m-2$ standard paths for $u \in [m] - \{1, 2\}$ along with $x_{r,1}x_{r-1,1} \dots x_{s+1,1}x_{s,1}x_{s,a}$ and $x_{r,1}x_{r+1,1} \dots x_{s-1,1}x_{s-1,a}x_{s,a}$.
- **For** $a \notin \{1, 2, m'+1, m'+2\}$: Take the $m-3$ standard paths for $u \in [m] - \{1, 2, b\}$ along with $x_{r,1}x_{r-1,1} \dots x_{s,1}x_{s,a}$; $x_{r,1}x_{r+1,1}x_{r+1,2}x_{r+2,2} \dots x_{s,2}x_{s,a}$; and $x_{r,1}x_{r,b}x_{r+1,b} \dots x_{s-1,b}x_{s-1,a}x_{s,a}$.

Note that in each of the paths, at most two intralevel edges were used. So, each path has length d or less. Therefore, $G - M$ satisfies $P_{d,m}$. ∎

6. NEW BOUNDS FOR $d \geq 3$

In this final section, we present several new upper and lower bounds for $\text{ext}(n; P_{d,m})$ when $d \geq 3$. First, consider the case $d = 3$. Unlike $d = 2$, there is no theorem similar to Theorem 4.1 for $\text{ext}(n; P_{3,m})$ where n is large. However, we can use an earlier construction to get an easy upper bound for $\text{ext}(n; P_{3,m})$.

Theorem 6.1. *For $n \geq 2m$,*

$$ext(n; P_{3,m}) \leq m(n - m).$$

Proof. The graph $K_{m,n-m}$ (see Fig. 2) satisfies $P_{3,m}$ and has $m(n-m)$ edges. ∎

We now turn our attention to new bounds for $\text{ext}(n; P_{d,m})$ when n is large and $d \geq 3$. Note that if a graph is to satisfy $P_{d,m}$ and be C-regular for some constant C, then there is an upper bound on the number of vertices that graph can contain. The following result gives an upper bound for the number of vertices a graph which satisfies $P_{d,m}$ can contain based on the minimum and maximum degree of G.

Theorem 6.2. *Suppose G is a graph on n vertices with maximum degree $\Delta = \Delta(G)$ and minimum degree $\delta = \delta(G)$ where $\Delta \geq \delta \geq m > 2$ for some m. If G satisfies $P_{d,m}$, then*

$$n \leq 1 + \frac{\delta(\Delta-1)^{d-1}(m+\Delta-2) - m\delta}{m(\Delta-2)}.$$

Proof. Suppose G is a graph on n vertices which satisfies $P_{d,m}$ and has minimum degree $\delta \geq m$. Let x be a vertex of G of degree δ. Consider the subtree rooted at the vertex x where the vertices in the ith level are distance i from x in G. Then, level i can contain at most $\delta(\Delta-1)^{i-1}$ vertices for $1 \leq i \leq d-1$.

We can get a better bound on the number of vertices in the dth level of the tree. There are at most $\delta(\Delta-1)^{d-2}$ vertices in the $(d-1)$st level, and there are at most $\Delta - 1$ edges from each of those vertices to the dth level. Now, each vertex in level d must have at least m of these edges. Thus, the number of vertices in the dth level is bounded from above by

$$(\Delta-1)\cdot\delta\cdot(\Delta-1)^{d-2}\cdot\left(\frac{1}{m}\right) = \frac{\delta}{m}(\Delta-1)^{d-1}.$$

Consequently,

$$\begin{aligned}
n &\leq 1 + \delta + \delta(\Delta-1) + \cdots + \delta(\Delta-1)^{d-2} \\
&\quad + \frac{\delta}{m}(\Delta-1)^{d-1} \\
&= 1 + \delta[1 + (\Delta-1) + \cdots + (\Delta-1)^{d-2}] \\
&\quad + \frac{\delta}{m}(\Delta-1)^{d-1} \\
&= 1 + \delta\left[\frac{(\Delta-1)^{d-1}-1}{(\Delta-1)-1}\right] + \frac{\delta}{m}(\Delta-1)^{d-1} \\
&= 1 + \frac{\delta(\Delta-1)^{d-1}(m+\Delta-2) - m\delta}{m(\Delta-2)}.
\end{aligned}$$

■

In the case when a graph is m-regular, we get the following corollary:

Corollary 6.3. *Suppose $m > 2$. If G is an m-regular graph on n vertices which satisfies $P_{d,m}$, then*

$$n \leq \frac{2[(m-1)^d - 1]}{(m-2)}.$$

Thus, the bound in Observation 2.4 cannot be sharp for large n. We now wish to determine a lower bound for large n which takes into account this fact. The following theorem gives a lower bound for the number of additional edges we need above the bound in Observation 2.4 for $\text{ext}(n; P_{d,m})$ when n is very large.

Theorem 6.4. *Suppose that G has n vertices and satisfies $P_{d,m}$ where $d \geq 3$ and $m \geq 3$. Then,*

$$|E(G)| \geq \frac{nm}{2} + \frac{1}{2}\left[\frac{n(m-2) + 2 - m(m-1)^d}{(m-1)[(m-1)^{d-2}-1] + (m-2)}\right].$$

Proof. Select a vertex of G of degree exactly m. If no such vertex exists, then as $\delta(G)$ must be greater than m, we have $|E(G)| \geq \frac{n(m+1)}{2} = \frac{mn}{2} + \frac{n}{2}$. This is greater than the bound in the theorem statement when

$$\frac{n}{2} \geq \frac{1}{2}\left[\frac{n(m-2) + 2 - m(m-1)^d}{(m-1)[(m-1)^{d-2}-1] + (m-2)}\right]$$

which is true when

$$n(m-1)[(m-1)^{d-2} - 1] \geq 2 - m(m-1)^d.$$

As $m \geq 3$ and $d \geq 3$, this inequality is always satisfied, so the result follows in this case.

Thus, assume that there does exist a vertex $v \in V(G)$ which has degree exactly m. Consider the subtree T rooted at the vertex v where the vertices in the ith level of the tree are distance i from v in G. Note that since G satisfies $P_{d,m}$ and hence must be connected, T is a spanning subgraph which contains at most d levels beyond the root v. Assume that T has k levels where $k \leq d$. Let V_i be the vertices of the ith level of the tree, so in particular $V_0 = \{v\}$. We will delete edges from T using two processes.

Process α: Examine the vertices of V_{k-1}. If any vertex has degree strictly greater than m in T, then delete edges from that vertex to the kth level until the vertex has degree exactly m. Then, examine the vertices in V_{k-2}. If any vertex in that set has degree greater than m, then delete edges from the vertex to the $(k-1)$st level until the degree is exactly m. Do this for each V_i for $1 \leq i \leq k-1$.

Let X_j be the set of edges from level j to level $j+1$ deleted in Process α for $0 \leq j \leq k-1$. Assume that $|X_j| = x_j$. Note that $x_0 = 0$. After Process α, a forest remains. Let T_v be the component of this forest which contains the vertex v.

Process β: Delete all edges from all components except T_v.

Each edge $e \in X_j$ for $1 \leq j \leq k-2$ deleted in Process α leaves a new component which does not contain v and may contain as many as

$$(m-1) + (m-1)^2 + \cdots + (m-1)^{k-j-1} = \frac{(m-1)((m-1)^{k-j-1}-1)}{m-2}$$

edges. Consequently, for each edge in X_j with $1 \leq j \leq k-2$, we delete as many as

$$\frac{(m-1)((m-1)^{k-j-1}-1)}{m-2}$$

edges in Process β. So, an upper bound for the total number of edges deleted in both processes is

$$x_{k-1} + \sum_{j=1}^{k-2}\left(x_j\frac{(m-1)[(m-1)^{k-j-1}-1]}{m-2} + x_j\right) \quad (6.1)$$

We now wish to get a lower bound on the total number of edges deleted. After the two processes, $|E(T_v)|$ is at most

$$m + m(m-1) + m(m-1)^2 + \cdots + m(m-1)^{k-1} = \frac{m(m-1)^k - m}{(m-1)-1}.$$

Thus, at least

$$n - 1 - \left(\frac{m(m-1)^k - m}{m-2}\right) \tag{6.2}$$

edges were deleted from our original tree T.

Let

$$\rho = \sum_{j=1}^{k-1} x_j.$$

So, ρ is the number of edges deleted in process α. Consequently, $|E(G)| \geq \frac{mn}{2} + \frac{\rho}{2}$. We wish to get a lower bound on ρ.

Comparing (6.1) and (6.2), we get

$$\begin{aligned} n - 1 - \left(\frac{m(m-1)^k - m}{m-2}\right) &\leq x_{k-1} + \sum_{j=1}^{k-2}\left(x_j \frac{(m-1)[(m-1)^{k-j-1} - 1]}{m-2} + x_j\right) \\ &\leq (\rho - x_{k-1})\left[\frac{(m-1)[(m-1)^{k-2} - 1]}{m-2}\right] + \rho. \end{aligned}$$

As $k \leq d$, we have

$$\begin{aligned} n - 1 - \left(\frac{m(m-1)^d - m}{m-2}\right) &\leq n - 1 - \left(\frac{m(m-1)^k - m}{m-2}\right) \\ &\leq (\rho - x_{k-1})\left[\frac{(m-1)[(m-1)^{k-2} - 1]}{m-2}\right] + \rho \\ &\leq \rho\left[\frac{(m-1)[(m-1)^{k-2} - 1]}{m-2}\right] + \rho \\ &\leq \rho\left[\frac{(m-1)[(m-1)^{d-2} - 1]}{m-2}\right] + \rho \\ &= \rho\left[\frac{(m-1)[(m-1)^{d-2} - 1] + (m-2)}{m-2}\right]. \end{aligned}$$

Consequently,

$$\rho \geq \frac{n(m-2) + 2 - m(m-1)^d}{(m-1)[(m-1)^{d-2} - 1] + (m-2)}.$$

As $|E(G)| \geq \frac{nm}{2} + \frac{\rho}{2}$, the result follows. ∎

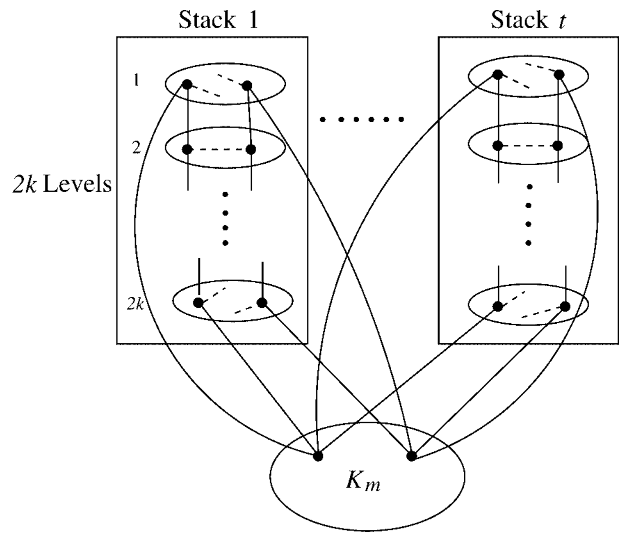

FIG. 8. H' from Theorem 6.5.

In order for Theorem 6.4 to provide any more information than Observation 2.4, n must be quite large. In fact, we need

$$n > \frac{m(m-1)^d - 2}{m-2}.$$

Although Theorem 6.4 provides an improvement in the lower bound when n is large, it does not give any indication of the structure of an extremal graph for large n. In our final result, we present a construction which works for certain values of d and large n and which provides a new upper bound on $ext(n; P_{d,m})$ in these cases.

Theorem 6.5. *Assume that $d = 4k+2$ for $k \geq 1$, m is even with $m \geq 4$, and $n = m(2kt+1)$ for some t. Then*

$$ext(n; P_{d,m}) \leq \frac{nm + m(2t-1)}{2} = \frac{nm}{2} + tm - \frac{m}{2}.$$

Proof. Consider t copies of C_{2k+1}. Now, designate one vertex on each cycle, and identify these designated vertices so that we have the cycles all sharing exactly one vertex v. Call this graph H. Now consider the Cartesian product $H \times K_m$. Let K_m^v be the copy of K_m which is in the position of vertex v. Now, for each copy of K_m, except for K_m^v, remove either an M_1 or an M_2 matching (as described in Theorem 5.5) so that no two adjacent copies have the same matching removed. Call the resulting graph H' (see Fig. 8).

Thus, H' consists of t stacks of $2k$ levels of copies of $K_m - M_i$ ($i = 1$ or 2) each of which route through K_m^v (which we will call the hub). Note that each copy of $K_m - M_i$ ($i =$ 1 or 2) can be referenced using a stack and level number. Also,

$$|E(H')| = \frac{nm + m(2t-1)}{2}.$$

Assume that the vertices in each copy of $K_m - M_i$ ($i =$ 1 or 2) and K_m^v are labeled $x_1, x_2, \ldots, x_m$ so that each copy of

C_{2k+1} goes through each of the vertices labeled with the same index number in every level. Let x be any vertex in Stack i, Level j. Assume also without loss of generality that $x = x_1$ and that Stack i, Level j had an M_1 matching removed. Note that there is a path of length $2k + 1$ or less from x_1 in Stack i, Level j to vertex x_p in K_m^v for $p \in [m]$. For $p \neq 2$, the path is $x_1 x_p P$ where P is the direct path to x_p in K_m^v along one copy of the cycle which goes through Level $j - 1$ or directly to K_m^v if $j \in \{1, 2k\}$. For $p = 2$, the path is $x_1 x_1' x_2' P$ where x_1', x_2' are in Stack i, Level $j + 1$ and P is the direct path to x_2 in K_m^v along one copy of the cycle which does not pass through Level j again. Note that these direct-to-hub paths give m vertex-disjoint paths of length at most $2k + 1$ from any vertex in Stack i, Level j to the hub (where the endpoints in the hub are distinct).

With this in mind, we show that H' satisfies $P_{d,m}$. Let x, y be two vertices of H'. Assume without loss of generality that $x = x_1$.

- Suppose $x, y \in K_m^v$. Then they are adjacent, and they clearly have $m-2$ common neighbors in K_m^v. So, we can find $m-1$ paths of length 2 or less between x and y. The direct-to-hub paths to x and y from any vertex in any one of the stacks and levels supply another path of length d or less between x and y that is disjoint from the other $m - 1$ paths. Thus, we can find m paths of length d or less between x and y.
- Suppose x, y are not in the hub but are both in Stack i. Note that the graph in Theorem 5.5, where $n = (2k + 1)m$, is a subgraph of the graph induced in H' by Stack i and the hub. Thus, by Theorem 5.5, we can find m paths of length $2k + 2$ or less in this subgraph of H'.
- Suppose x is Stack i but y is in the hub. Then, the direct-to-hub paths from x along with the edges from y to every other vertex in K_m^v give us m paths of length $2k + 2$ or less.
- Suppose x is in Stack i but y is in Stack j where $j \neq i$. Then, the direct-to-hub paths from x and the direct-to-hub paths from y give us m vertex-disjoint paths of length $4k + 2$ or less between x and y.

Thus, H' has property $P_{d,m}$, and the result is proven. ∎

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