14

Bounds for the Ramsey Number of a Disconnected Graph Versus Any Graph

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ABSTRACT

Bounds are determined for the Ramsey number of the union of graphs versus a fixed graph H, based on the Ramsey number of the components versus H. For certain unions of graphs, the exact Ramsey number is determined. From these formulas, some new Ramsey numbers are indicated. In particular, if

$$r(g_i, H) = [\,|V(g_i)| - 1][\chi(H) - 1] + t_1(H) + \beta_i,$$

$$G = \bigcup_{i=1}^{k} g_i,$$

$$p = \max_{1 \le j \le k}\left((j-1)[\chi(H) - 2] + \sum_{i=j}^{k} ik_i\right) + t_1(H) - 1,$$

where k_i is the number of components of order i and $t_1(H)$ is the minimum order of a color class over all critical colorings of the vertices of H, then

$$p \le r(G, H) \le p + \max_i(\beta_i).$$

INTRODUCTION

All graphs in this article are without loops and multiple edges. If G is a disconnected graph let $c(G)$ denote the maximum order of a component of G. A coloring of the vertices of G with exactly $\chi(G)$ colors is called a *critical*

Journal of Graph Theory, Vol. 6 (1982) 413–417
 CCC 0364-9024/82/040413-05$01.50

coloring. In any coloring of a graph, all vertices with the same color form a *color class*. Define $t(G)$ to be the minimum number of vertices in any color class of any critical coloring of G. Finally, the *Ramsey number* $r(G_1, G_2)$ is the least positive integer p such that in any factorization of $K_p = R \oplus B$ [i.e., $V(K_p) = V(R) = V(B)$ and $E(R) \cap E(B) = \emptyset$ and $E(R) \cup E(B) = E(K_p)$], either $G_1 \subseteq R$ or $G_2 \subseteq B$. Ramsey numbers have been studied extensively. Some results of interest include the following.

Theorem A (Burr [3]). If G is a connected graph of order $n \geq t(H)$ then

$$r(G, H) \geq (n - 1)[\chi(H) - 1] + t(H).$$

Theorem B (Chvátal [5]). If T_m is a tree of order m and K_n a complete graph of order n then

$$r(T_m, K_n) = (m - 1)(n - 1) + 1.$$

Theorem C (Stahl [7]). If F is a forest then

$$r(F, K_n) = \max_{1 \leq j \leq c(F)} \left((j - 1)(n - 2) + \sum_{i=j}^{c(F)} ik_i \right),$$

where k_i is the number of components of order i.

Theorem D ([6]). If P_m is the path of order m, $m \geq 4$, and G is a graph of order $n + 2$, $n \geq 3$, with clique number n, then

$$r(P_m, G) = (m - 1)(n - 1) + 1.$$

In this paper we present bounds related to that in Theorem A for $G = \bigcup_{i=1}^{k} G_i$. We use these bounds and others to obtain a generalization of Theorem C, and from this, determine some new Ramsey numbers.

2. UPPER AND LOWER BOUNDS

Let H be a graph, then define $t_i(H)$ to be the minimum, over all critical colorings of the vertices of H, of the order of the ith smallest color class. (Note that the first smallest is the smallest.) Also define $\mathscr{G}_\beta(H) = \{g \mid g$ is a connected graph and $r(g, H) = [|V(g)| - 1][\chi(H) - 1] + t_1(H) + \beta\}$. This set is clearly well defined for all non-negative integers β.

(i) Suppose $g_1, g_2, \ldots, g_k \in \mathscr{G}_\beta(H)$ with smallest graph of order m_0 and largest graph of order m_1. Let $G = \bigcup_{i=1}^{k} g_i$ and let k_i be the number of components of order i. Choose j_0 such that

$$(j_0 - 1)[\chi(H) - 2] + \sum_{i=j_0}^{m_1} ik_i$$

$$= \max_{m_0 \le j \le m_1} \left((j - 1)[\chi(H) - 2] + \sum_{i=j}^{m_1} ik_i \right).$$

We prove the following.

Theorem 1. Suppose conditions (i) hold. If $j_0 \ge t_1(H)$ then

$$r(G, H) \ge (j_0 - 1)[\chi(H) - 2] + \sum_{i=j_0}^{m_1} ik_i + t_1(H) - 1.$$

Proof. Let $p = (j_0 - 1)[\chi(H) - 2] + p_0 + t_1(H) - 2$, where $p_0 = \Sigma_{i=j_0}^{m_1} ik_i$. Consider the factorization of $K_p = R \oplus B$, where $R = K_{p_0-1} \cup [\chi(H) - 2]K_{j_0-1} \cup K_{t_1(H)-1}$. To show that $G \not\subseteq R$ we will concentrate on the subgraph G_{j_0} of G which consists of all components of G which have j_0 or more vertices. Clearly $G_{j_0} \not\subseteq K_{p_0-1}$ since there are not enough vertices. Further, K_{j_0-1} is too small to contain any component of G_{j_0}, and since $j_0 \ge t_1(H)$ it is clear that $K_{t_1(H)-1}$ is also too small to contain any component of G_{j_0}. Thus $G_{j_0} \not\subseteq R$; hence $G \not\subseteq R$. Since B is a complete $\chi(H)$-partite graph with $t_1(B) = t_1(H) - 1$ this implies that $H \not\subseteq B$, and the theorem follows. ■

Theorem 2. Suppose conditions (i) hold. Then

$$r(G, H) \le (j_0 - 1)[\chi(H) - 2] + \sum_{i=j_0}^{m_1} ik_i + t_1(H) + \beta - 1.$$

Proof. It will be convenient for $m_0 \le j \le m_1$ to let G_j denote the subgraph of G consisting of all components of order at least j, so that G_j has order $p_j = \Sigma_{i=j}^{m_1} ik_i$. Let $p = (j_0 - 1)[\chi(H) - 2] + p_{j_0} + t_1(H) + \beta - 1$. Consider an arbitrary factorization of $K_p = R \oplus B$ in which $H \not\subseteq B$. We show that $G \subseteq R$ by descending induction on j.

First assume $G = G_{m_1}$. By an easy induction on k, the number of components of G, we show $G = G_{m_1} \subseteq R$. This is clear for $k = 1$. If $k > 1$ and g is an arbitrary component of G, then the factorization of $K_p = R \oplus B$ induces a factorization on $K_p - V(g)$ with $|V(K_p) - V(g)| = (m_1 - 1) \times [\chi(H) - 2] + m_1(k - 1) + t_1(H) + \beta - 1$. By induction $G - V(g) \subset (K_p - V(g)) \cap R$ since $H \not\subseteq B$. Therefore $G = G_{m_1} \subseteq r$.

To complete the induction on j assume $G_{j+1} \subseteq R$, $m_0 \le j \le m_1$. Clearly $G_j \subseteq R$ when $G_j = G_{j+1}$, so that $G_j - V(G_{j+1})$ consists of k_j components, each of order j. Again the factorization of $K_p = R \oplus B$ induces a factorization on $K_p - V(G_{j+1})$ with

$$|V(K_p) - V(G_{j+1})| = p - \sum_{i=j+1}^{m_1} ik_i \geq (j-1)[\chi(H) - 2] + jk_j$$

$$+ t_1(H) + \beta - 1.$$

As in the argument of the preceding paragraph $G_j - V(G_{j+1}) \subseteq (K_p - V(G_{j+1}) \cap R$. Therefore $G_j \subseteq R$ and the induction is complete. ∎

We now note some useful special cases of Theorems 1 and 2.

Corollary 3. Suppose $g_1, g_2, \ldots, g_k \in \mathscr{G}_\beta(H)$ and $|V(g_i)| = m$ $(i = 1, 2, \ldots, k)$. Let $G = \cup_{i=1}^{k} g_i$. If $m \geq t_1(H)$ then $r(G,H) \geq (m-1)[\chi(H) - 2] + mk + t_1(H) - 1$.

Corollary 4. Suppose $G = \cup_{i=1}^{k} g_i$, where $g_i \in \mathscr{G}_\beta(H)$ and $|V(g_i)| = m$. Then

$$r(G, H) \leq (m-1)[\chi(H) - 2] + mk + t_1(H) + \beta - 1.$$

We now note that Theorems 1 and 2 yield a generalization of Theorem C.

Corollary 5. If $g_1, g_2, \ldots, g_k \in \mathscr{G}_0(H)$ and $G = \cup_{i=1}^{k} g_i$ then

$$r(G, H) = \max_{1 \leq j \leq c(G)} \left((j-1)[\chi(H) - 2] + \sum_{i=j}^{c(G)} ik_i \right) + t_1(H) - 1.$$

We also note that $T_m \in \mathscr{G}_0(K_n)$ from Theorem B, so Theorem C now follows as a corollary to Theorem B and Corollary 5.

3. APPLICATIONS AND CONCLUSION

In [1] it was shown that $C_m \in \mathscr{G}_0(K_n)$ when $m > n^2 - 2$. Since $T_m \in \mathscr{G}_0(K_n)$ as well, we may use Corollary 5 to obtain the Ramsey number for $G = (\cup_{i=1}^{k_1} T_{m_i}) \cup (\cup_{j=1}^{k} C_{m_j})$, where each $m_j > n^2 - 2$, versus K_n. Similarly, $C_m \in \mathscr{G}_0(K_{s(n)})(K_{s(n)}$ denotes the complete n-partite graph $K_{s,s,\ldots,s}$, with n subscripts) when s, n, and m are sufficiently large (see [1]). Thus Corollary 5 may be applied to unions of cycles (sufficiently large) versus $K_{s(n)}$ as well.

In fact, Burr and Erdös [4] have shown that any sufficiently large graph homeomorphic to a connected graph is in $\mathscr{G}_0(K_n)$. They further conjecture that "large" connected graphs with "small" edge density are in $\mathscr{G}_0(K_n)$.

As a final application of Corollary 5 to Theorem D we can produce the Ramsey numbers for stripes (unions of paths) with smallest stripe of order 4 versus any graph G with order $n+2$ having clique number n ($n \geq 3$).

Clearly the list of applications is far from exhausted. We merely mention a few to point out possible applications of Corollary 5.

Finally, we state a theorem bounding the Ramsey number and allowing one to vary the $\mathscr{G}_\beta$ classes.

Theorem 6. If $g_i \in \mathscr{G}_{\beta_i}(H)$ and $G = \cup_{i=1}^{k} g_i$, let

$$p = \max_{1 \leq j \leq c(G)} \left((j-1)[\chi(H) - 2] + \sum_{i=j}^{c(G)} i k_i \right) + t_1(H) - 1;$$

then

$$p \leq r(G, H) \leq p + \max_i (\beta_i).$$

The proof of Theorem 6 would be exactly the same as that for Theorems 1 and 2 with $\max_i (\beta_i)$ substituted for β.

We also feel that investigation of $t_i(H)$ for $i > 1$ may result in new and improved bounds. We direct the reader to [2].

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