NEW ORE-TYPE CONDITIONS FOR H-LINKED GRAPHS

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Abstract. For a fixed (multi)graph H, a graph G is *H-linked* if any injection $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. The notion of an H-linked graph encompasses several familiar graph classes, including k-linked, k-ordered and k-connected graphs. In this paper, we give two sharp Ore-type degree sum conditions that assure a graph G is H-linked for arbitrary H. These results extend and refine several previous results on H-linked, k-linked and k-ordered graphs.

Keywords: H-linked graph, k-linked graph, k-ordered graph

All graphs in this paper are finite. For notation not defined here we refer the reader to [1]. If $X \subseteq V(G)$ is a vertex set, we will often just write X for the induced subgraph $G[X]$ if the context is clear. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively, and let $\sigma_2(G)$ denote the minimum degree sum of nonadjacent vertices in G. Throughout the paper, we will often refer to σ_2 conditions as *Ore-type* conditions in light of Ore's classical theorem on hamiltonian graphs. We will also let $n_i(G)$ be the number of vertices of degree i in G.

Given a vertex v and set $A \subseteq V(G)$, we let $d_A(v)$ denote the number of neighbors of v in the set A. For $A, B \subseteq V(G)$ we let $\delta_B(A)$ and $\Delta_B(A)$ denote the minimum and maximum respectively of $d_B(v)$ taken over all vertices in A. We will let $E_G(A, B)$ denote the number of edges in G with one endpoint in A and the other in B, and let $e_G(A, B) = |E_G(A, B)|$. We will frequently write $E(A, B)$ and $e(A, B)$ when the context is clear.

Given an integer-valued graph parameter p and a graph property $\mathcal{P}$, the *p-threshold for $\mathcal{P}$* is the minimum $k = k(n)$ such that any graph G of order n with $p(G) \geq k$ has property $\mathcal{P}$. We will frequently consider p-thresholds restricted to specific graph classes, such as sufficiently large graphs, or graphs with a prescribed number of edges.

A graph G is *k-linked* if for any ordered subset of $2k$ vertices $S = \{s_1, t_1, \ldots, s_k, t_k\}$ there exist disjoint paths $P_1, \ldots, P_k$ such that for each i, P_i is an $s_i - t_i$ path. We will

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refer to this collection of paths as an *S-linkage* in G. We also say that G is *k-ordered* if for any list of k vertices $v_1, \dots, v_k$ in G, there exists a cycle that visits these vertices in the given order.

A *subdivision* of a (multi)graph H, or an *H-subdivision* is any simple graph obtained by replacing the edges of H with internally disjoint paths. The vertices originally in H are called the *ground vertices* or the *ground set* of the subdivision. For a fixed graph H, a graph G is *H-linked* if any injection $f : V(H) \to V(G)$ can be extended to an H-subdivision in G with ground vertices prescribed by f. We refer to the injection f as an *H-linkage problem* (in G). The notion of an H-linked graph generalizes those of k-linked, k-ordered and k-connected graphs, as G is kK_2-linked if and only if G is k-linked, G is C_k-linked if and only if G is k-ordered and G is k-connected if and only if G is $K_{1,k}$-linked.

1. Degree Conditions for H-Linked Graphs

Kawarabayashi, Kostochka and Yu [8] determined sharp minimum degree and degree sum conditions for a graph G of order at least $2k$ to be k-linked.

Theorem 1. *Let G be a graph on $n \geq 2k$ vertices. If*

$$\delta(G) \geq \begin{cases} \frac{n+2k-3}{2}, & \text{if } n \geq 4k-1 \\ \frac{n+5k-5}{3}, & \text{if } 3k \leq n \leq 4k-2 \\ n-1, & \text{if } 2k \leq n \leq 3k-1 \end{cases}$$

or

$$\sigma_2(G) \geq \begin{cases} n+2k-3, & \text{if } n \geq 4k-1 \\ \frac{2(n+5k)}{3}-3, & \text{if } 3k \leq n \leq 4k-2 \\ 2n-3, & \text{if } 2k \leq n \leq 3k-1 \end{cases}$$

then G is k-linked. These bounds are best possible.

For sufficiently large graphs, the relevant portion of these conditions were obtained independently in [6]. Sharp minimum degree and degree sum conditions for k-ordered graphs were determined in [2] and [9], respectively.

Theorem 2. *Let G be a graph of order n and $k \geq 2$ be an integer. If*

(a) *$n \geq 11k-3$ and $\delta(G) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$, or*

(b) *$n \geq 53k^2$ and $\sigma_2(G) \geq n + \lceil \frac{3k-9}{2} \rceil$,*

then G is k-ordered.

Turning our attention the the broader class of H-linked graphs, minimum degree conditions that assure a graph G is H-linked for arbitrary connected H were first given (independently) in [3] and [10]. These were subsequently strengthened in [5] to include arbitrary multigraphs H, thereby extending Theorem 1. Similar conditions concerned with finding (strong) H-immersions in a graph G appear in [4]. In order to discuss these results, we must first introduce a useful parameter.

For a (multi-)graph H, let

$$b(H) = \max_{\substack{A\cup B\cup C=V(H)\\ e(A,B)\geq 1}} e(A,B) + |C|$$

where we take this maximum over all partitions A, B, C of $V(H)$. As every graph G has a bipartite subgraph with at least half of the edges in G, $b(H) \geq |E(H)|/2$. When H is connected, it is straightforward to see that we may choose C to be empty in any optimal partition, so that $b(H)$ is equal to the maximum number of edges in a bipartite subgraph of H. As was noted in [4] and [5], when H is disconnected, $b(H)$ depends not only on the maximum size of a bipartite subgraph of H, but also on the number of components of H without even cycles.

The following result of Gould, Kostochka and Yu [5] gives the δ-threshold for H-linkedness.

Theorem 3. *Let H be a (multi-)graph with $c(H)$ components that do not contain even cycles and G be a graph of order $n \geq 9.5(|E(H)| + c(H) + 1)$. If*

$$\delta(G) \geq \tfrac{1}{2}(n + b(H) - 2),$$

then G is H-linked. This result is sharp.

Kostochka and Yu [11] gave Ore-type conditions, dependent on k, implying that a graph G is H-linked for *every* graph H with k edges.

Theorem 4. *Let G be a graph of order n and let H be a simple graph with k edges and minimum degree at least two. If*

$$\sigma_2(G) \geq \begin{cases} \left\lceil n + \frac{3k-9}{2} \right\rceil & n > 2.5k - 5.5 \\[2ex] \left\lceil n + \frac{3k-8}{2} \right\rceil & 2k \leq n \leq 2.5k - 5.5 \\[2ex] 2n-3 & k \leq n \leq 2.5k - 1, \end{cases}$$

then G is H-linked.

In light of Theorem 2, one interesting consequence of Theorem 4 is that amongst those graphs H with k edges, C_k has the largest σ_2-threshold for H-linkedness when n is sufficiently large.

The goal of this paper is to refine Theorem 4 by giving sharp Ore-type conditions that assure a graph G is H-linked for an arbitrarily chosen H. We note here that the σ_2-threshold for H-linkedness is not, in general, twice the minimum degree given in Theorem 3, as Theorem 2 demonstrates that this is not the case for $H = C_k$ when n is sufficiently large. Our first result demonstrates that twice the minimum degree in Theorem 3 *does* suffice if we add only a mild minimum degree condition to G.

Theorem 5. *Let H be a multigraph and G be a graph with $|G| \geq 20|E(H)| + n_0(H)$. If*

$$\delta(G) \geq 4|E(H)| + n_0(H), \text{ and}$$
$$\sigma_2(G) \geq |G| + b(H) - 2,$$

then G is H-linked. This result is sharp.

We also utilize Theorem 5 to give a sharp σ_2 bound that, without any additional minimum degree condition, assures a graph G is H-linked for any simple graph H. Let

$$a(H) = \max_{A \cup B = V(H)} \left(e(A, B) + |B| - \Delta_B(A) \right).$$

Theorem 6. *Let H be a simple graph and G be a graph of order $n > 20|E(H)|$. If*

$$\sigma_2(G) \geq n + a(H) - 2,$$

then G is H-linked. This result is sharp.

Observe that for arbitrary H, $a(H) \geq b(H)$. To see this, suppose that $V(H) = A \cup B \cup C$ with $e(A, B) + |C| = b(H)$. Then, if we let $B^* = B \cup C$, it follows that

$$a(H) \geq e(A, B^*) + |B^*| - \Delta_{B^*}(A) \geq e(A, B) + |C| = b(H).$$

There are a number of graphs H, including C_k ($k \geq 5$), for which $a(H) > b(H)$. As such, Theorem 6 demonstrates that there are many choices of H for which the σ_2-threshold for H-linkedness is more than twice the δ-threshold.

2. Preliminary Lemmas

A version of the following Lemma originally appears in [12], pertaining to directed graphs. The proof for undirected graphs is analogous and, hence, omitted.

Lemma 7. *Let G be a graph, $k \geq 1$ and $v \in V(G)$ with $d(v) \geq 2k - 1$. If $G - v$ is k-linked, then G is k-linked.*

Thomas and Wollan [14] used the following to prove that every $10m$-connected graph is m-linked, which represents the current best bound on connectivity sufficient to assure k-linkedness.

Theorem 8. *Let $k \geq 2$ and G be a $2k$-connected graph. If $|E(G)| \geq 5k|G|$, then G is k-linked.*

Corollary 9. *Let $k \geq 2$ and G be a $2k$-connected graph of order n. If $\sigma_2(G) \geq n$ and $n \geq 20k$, then G is k-linked.*

Proof. This follows from the observation that any graph G of order n with $\sigma_2(G) \geq n$ must have at least $\frac{n^2}{4}$ edges. Indeed, if $\delta = \delta(G) \geq \frac{n}{2}$ then there is nothing to prove so assume otherwise and let v be a vertex of G with $d(v) = \delta$. Counting the degrees of the neighbors and non-neighbors of v respectively, we have that

$$|E(G)| \geq \frac{1}{2}\left(\delta(\delta+1) + (n-\delta-1)(n-\delta)\right)$$

which is greater than $\frac{n^2}{4}$ for $\delta < \frac{n}{2}$. □

We close with the following fact and lemma, both of which are straightforward to prove and will be useful as we proceed.

Fact 10. *Let G be a graph and H a (multi-)graph with $|E(H)| = m$ and $n_0(H) = 0$. If G is m-linked, then G is H-linked.*

Lemma 11. *Let H be a multigraph, and let G be an edge maximal non-H-linked graph. Then for every $m \geq |E(H)|$ and $X \subseteq V(G)$ with $|X| \geq 2m$:*

$$G[X] \text{ is } m\text{-linked} \iff G[X] \text{ is complete.}$$

Proof. Assume that X is as given, that there are nonadjacent x and y in X and that f is an H-linkage problem in G. By maximality, $G + xy$ is H-linked, so let F be a solution to the H-linkage problem f in $G+xy$. Let P_i be the path in F corresponding to the edge e_i in H for $1 \leq i \leq |E(G)|$ and when traversing P, let x_i and y_i be the extreme (first and last) vertices in $V(P) \cap X$ (note that x_i and y_i may not be distinct). For each i, delete all vertices of P_i that lie strictly between x_i and y_i to create F', a partial solution to the H-linkage problem f in G. As $G[X]$ is m-linked and $xy \notin F'$, we can extend F' to an H-linkage in G, contradicting the assumption that G is not H-linked. □

3. Proofs of Theorems 5 and 6

We are now ready to prove our main results.

Proof of Theorem 5. Sharpness is established by the following example, which is identical to the sharpness example for Theorem 3. Let $A \cup B \cup C$ be a partition of $V(H)$ such that $e(A, B) + |C| = b(H)$. Create G by first adding $e(A, B) - 1$ vertices to C to obtain C^*, and then adding vertices to A and B to create sets A^* and B^*, each of size $\frac{n-|C^*|}{2}$. The edges of G are all possible edges in $(A^* \cup C^*)$ and $(B^* \cup C^*)$. It is straightforward to see that G is not H-linked, as there is not a sufficient number of vertices in C^* to create paths representing the edges in $E(A, B)$.

Let $n = |G|$ and $m = |E(H)|$. Note that the statement is trivial for $m \leq 1$, so we may also assume that $m \geq 2$. For the sake of contradiction, we assume that there is no H-linkage in G, and furthermore that Theorem 5 holds for every proper subgraph $H' \subsetneq H$. Further, assume that G is edge maximal without an H-linkage.

If $v \in V(H)$ is isolated in H, then solving the H-linkage problem in G is equivalent to solving the $(H-v)$-linkage problem $f\,|_{V(H)-\{v\}}$ in $G-v$. As $G-v$ satisfies all of the conditions in Theorem 5 (note that $b(H-v) = b(H) - 1$), this yields a contradiction, so H does not contain any isolated vertices.

If G is $2m$-connected, we are done by Corollary 9, so we may assume that there is a minimal cut set Z in G with $|Z| \leq 2m-1$. The degree conditions on G imply that $G - Z$ has exactly two components, call them X and Y and we assume without loss of generality that $|X| \leq |Y|$. Let $x \in X$ and $y \in Y$, then

$$n + b(H) - 2 \leq d(x) + d(y) \leq |X| + |Y| + 2|Z| - 2 \leq n + |Z| - 2,$$

so

$$\delta_X(X) + \delta_Y(Y) \geq |X| + |Y| - |Z| + b(H) - 2.$$

Therefore,

$$\delta_X(X) \geq \max\{|X| - |Z| + b(H) - 1, \delta(G) - |Z|\} \geq |X| - \tfrac{3}{2}m.$$

We now wish to show that both X and Y are m-linked. If $|X| \geq 5m$, then $\delta_X(X) \geq \frac{|X|}{2} + m$, so X is m-linked by Theorem 1. Suppose then that $|X| < 5m$, so $2(|X| + |Z|) < |G|$ and X is complete by the degree sum condition. Since $|X| \geq \delta(G) + 1 - |Z| \geq 2m + 2$, the fact that X is complete implies that X is m-linked. Analogously, we also conclude that Y is m-linked.

Let $z \in Z$, and suppose there are vertices $x \in X$ and $y \in Y$ such that $xz, yz \notin E(G)$. Then

$$n + |Z| + 2d(z) \geq d(x) + 2d(z) + d(y) \geq 2n + m - 4,$$

so

$$d(z) \geq \tfrac{1}{2}(n + m - |Z| - 4) \geq \tfrac{1}{2}(n - m - 4) > 6m.$$

Thus, for every $z \in Z$, we have $d_X(z) \geq 2m$ or $d_Y(z) \geq 2m$. Let $V(G) = A \cup B$ be a partition with

$$X \subseteq A \subseteq \{v \in V(G) \ : \ d_X(v) \geq 2m-1\}, \text{ and}$$
$$Y \subseteq B \subseteq \{v \in V(G) \ : \ d_Y(v) \geq 2m-1\}.$$

Then, A and B are m-linked by Lemma 7, and therefore complete by Lemma 11. Let A^H, B^H be the partition of $V(H)$ induced by this partition of $V(G)$.

Choose $ab \in E(H)$, let $H' = H - ab$, and let $F \subseteq G$ be a solution to the H'-linkage problem. As A and B are complete, we may choose F such that every path in F corresponding to an edge in $E(H')$ contains at most two vertices in A and at most two vertices in B. In particular, as $n_0(H) = 0$, this implies that $|F \cap A| \leq 2m$ and $|F \cap B| \leq 2m$, so $A \setminus F \neq \emptyset$ and $B \setminus F \neq \emptyset$. We conclude that $a \in A$ and $b \in B$, and in particular, $E(H) = E_H(A,B)$. We also have that $|E_F(A,B)| = |E_{H'}(A,B)| = |E(H)| - 1$. There are three types of paths in F corresponding to edges $yz \in E_{H'}(A,B)$:

1: $yuvz$ with $u \in A, v \in B$,
2: yuz with $u \in A \cup B$, and
3: yz.

Choose F such that the number of type 1 paths is maximized.

Let $w \in A \setminus F$ and $x \in B \setminus F$. If $wx \in E(G)$, then we can extend F to a solution of the H-linkage problem using the path $awxb$, so we conclude that $wx \notin E(G)$. Similarly, if there exists an $c \in (N(w) \cap N(x)) \setminus F$, we can extend F to a solution of the H-linkage problem using $awcxb$, so $N(x) \cap N(w) \subseteq F$.

If $yuvz$ is a path of type 1 in F, then we claim that

$$N(w) \cap N(x) \cap \{y, u, v, z\} \subsetneq \{u, v\}.$$

Indeed, if $wz \in E(G)$, then we can replace $yuvz$ by ywz and use the path $auvb$ to complete an H-linkage. If $wv, xu \in E(G)$, then we can replace $yuvz$ with $yuxz$ and use the path $awvb$ to complete an H-linkage. The case where $xy \in E(G)$ is handled similarly.

Now if yuz is a path of type 2 in F, then $u \notin N(w) \cap N(x)$, as otherwise we could replace yuz by $yuxz$ or $ywuz$ and increase the number of type 1 paths in F.

Let F_1 be the edges in H' corresponding to type 1 paths in F, and $F_2 := E(H') \setminus F_1$. Furthermore, let $H_1 \subseteq V(H)$ be the vertices in H incident to F_1, and let $H_2 := V(H) \setminus H_1$. Then

$$n - 2 + b(H) \leq d(w) + d(x) \leq n - 2 + |F_1| + |H_2 \cap N(w) \cap N(x)|. \tag{1}$$

If $H_2 = \emptyset$, observe that $|F_1| \leq |E_H(A, B)| - 1 < b(H)$, so (1) gives a contradiction.

Also, if $F_1 = \emptyset$, note that $b(H) \geq |H| - 1 = |H_2| - 1$, and thus by (1), $N(w) \cap N(x)$ may miss at most one vertex in $V(H)$. Therefore, $a \in N(x)$ or $b \in N(w)$. But then, we can complete the H-linkage via axb or awb, a contradiction.

Finally, suppose that $F_1 \neq \emptyset$ and $H_2 \neq \emptyset$, so that $b(H) \geq |F_1| + |H_2|$, with the lower bound realized by a partition of H with all vertices of H_2 in C, and the remaining vertices partitioned according to their membership in A^H and B^H. Therefore by (1), $H_2 \subseteq N(w) \cap N(x)$ for every pair of vertices $w \in A \setminus V(F)$ and $x \in B \setminus V(F)$. If H contains no edge between two vertices in H_2, then $|H_2| \leq |F_2|$, and $|F_1| + |F_2| = |E_H(A, B)| - 1 < b(H)$, so (1) gives a contradiction. Thus, there are two vertices $y \in H_2 \cap A$ and $z \in H_2 \cap B$. As $n \geq 20m$, we may assume by symmetry that $|A| \geq 10m$, and therefore since z is in $N(w) \cap N(x)$ for all $w \in A \setminus V(F)$ and $x \in B \setminus V(F)$, that $d_A(z) \geq 8m$. By Lemma 11, z is connected to all vertices in $V(G)$. But now, there is an $(H - z)$-linkage in $G - z$ by the minimality of H, and this linkage can trivially be expanded to an H-linkage in G. □

Proof of Theorem 6. Sharpness follows from the following example. Starting from a partition $A \cup B$ of $V(H)$ with $(e(A, B) + |B| - \Delta_B(A)) = a(H)$, add a set C of $e(A, B) - 1$ vertices. Blow up B to B^* by adding $n - |A| - |B| - |C|$ vertices to B and then add all edges in $A \cup C$, $B^* \cup C$, and all edges between A and B except for the edges in H. This graph is not H-linked, as there is not a sufficient number of vertices in C to create paths representing the edges in $E(A, B)$, and has $\sigma_2 = n + a(H) - 3$.

As in the proof of Theorem 5, assume that H is a minimal counterexample to the statement, and furthermore that G is edge maximal without creating an H-linkage.

Let $m = |E(H)|$ and $n = |G|$. Again, we have $n_0(H) = 0$ as isolated vertices in H contribute 2 to $|G| + a(H)$ and at most 2 to $\sigma_2(G)$.

If $\delta(G) \geq 4m$, we are done by Theorem 5 (as $b(H) \leq a(H)$), so there is a vertex v with $d(v) < 4m$. Let $Y := V(G) \setminus N[v]$. Then $|Y| > 16m$ and, since for any $y \in Y$ we have that $d(v) + d(y) \geq n + a(H) - 2$, it follows that

$$\delta_Y(Y) > |Y| - 4m > \tfrac{1}{2}|Y| + m.$$

Therefore Y is m-linked by Theorem 1. Let $B \supseteq Y$ be maximal such that B is m-linked, and $A := V(G) \setminus B \subseteq N[v]$. If $A = \emptyset$ we are done, so assume that $A \neq \emptyset$. By Lemma 7 no vertex in A has $2m$ neighbors in B, so since $|A| \leq |N[v]| \leq 4m$, we have that $\Delta_G(A) < 6m$. Therefore A is complete by the degree sum condition. We now continue in a manner similar to the proof of Theorem 5.

Let $A^H \cup B^H$ be the partition of $V(H)$ induced by A and B. Note that B is complete by Lemma 11. If there is an edge $e \in E(H) \cap E(G)$, we can extend any solution of the $(H - e)$-linkage problem trivially to a solution of the H-linkage problem, so we conclude that $E(H) \cap E(G) = \emptyset$, and in particular, $E(H) = E_H(A, B)$.

Let $a \in A^H$ maximize $|E_H(a, B)|$, and choose $ab \in E(H)$. For $H' = H - ab$, let $F \subseteq G$ be a solution of the H'-linkage problem of minimum order, so that in particular $|E_F(A, B)| = |E(H')|$. Further assume that $|F \cap A|$ is minimized.

Next, choose $w \in B \setminus F$. If $aw \in E(G)$, then we can extend F to a solution of the H-linkage problem using the path awb, so we conclude that $aw \notin E(G)$. Similarly, if there exists an $x \in (N(a) \cap N(w)) \setminus F$, we can extend F to a solution of the H-linkage problem using $axwb$, so $N(a) \cap N(w) \subseteq F$.

Now we consider paths $P \subset F$ corresponding to edges in H' with types identical to those described in the proof of Theorem 5. If $P = a'uvb'$ is of type 1, then $a'w, uw \notin E(G)$ by the minimality of $|F \cap A|$. Similarly, if $P = a'ub'$ is of type 2 with $u \in A$, then $a'w, uw \notin E(G)$. If $P = a'vb'$ is of type 1 with $v \in B$, and $a'w, av \in E(G)$, then we can replace av by aw in F and complete the H-linkage via avb.

Therefore, for every path $P \subset F$ corresponding to an edge in H', we have

$$|(V(P) \cap N(a) \cap N(w)) \setminus (B^H \setminus N_H(a))| \leq 1.$$

But this yields a contradiction, as then

$$\begin{aligned} a(H) \leq |N(a) \cap N(w)| &\leq |E_F(A, B)| + |B^H \setminus N_H(a)| \\ &= |E(H)| - 1 + |B^H| - \Delta_{B^H}(A^H) \leq a(H) - 1. \end{aligned}$$

□

We note here that Theorem 6 does not extend to arbitrary multigraphs H. To see this, let $k \geq 6$, $r = 2(k-1)$, and let H be the disjoint union of a star having center c and leaves $\ell_1, \ldots, \ell_r$ with an edge uv of multiplicity k. As defined above, $a(H) = 3k - 1$ (let B consist of u and all of the ℓ_i). However, consider the following example. Let $A = \{c, u, v\}$ be a triangle and X be a clique of order $n-3$ containing disjoint subsets L, X_u and X_v of X with $|X_v| = r, |X_u| = r-1$ and $L = \{\ell_1, \ldots, \ell_r\}$.

Construct G from A and X by adding all edges from u to $X_u \cup L$, v to $X_v \cup L$ and c to $X_u \cup X_v$ and note that $\sigma_2(G) = n + (4k-4) - 2 > n + a(H) - 2$. If we let the vertex labels in G define an H-linkage problem ρ, then we require at least one vertex from $X_u \cup X_v$ to construct the r desired paths from c to L and at least two vertices from $X_u \cup X_v$ to construct each of the remaining $k-1$ paths from u to v. This is a total of at least $2k-4$ additional vertices, which exceeds the $2k-5$ vertices in $X_u \cup X_v$. Hence G is not H-linked.

Theorems 5 and 6 also allow us to obtain a number of interesting results on k-linked and k-ordered graphs as corollaries. In particular, we obtain the degree conditions for sufficiently large k-linked, k-ordered and H-linked graphs found in Theorems 2, 3 and 4, respectively. In most cases, our bounds on $|G|$ are reasonable, but slightly larger than those in the original theorems due to the more general nature of our results.

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