Combinatorics, Probability and Computing (2012) 21, 129–139.
doi:10.1017/S0963548311000708

Minimum Degree and Disjoint Cycles in Claw-Free Graphs

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Received 11 February 2011; revised 4 November 2011; first published online 2 February 2012

This paper is dedicated to the memory of our friend and colleague, Richard Schelp

A graph is claw-free if it does not contain an induced subgraph isomorphic to $K_{1,3}$. Cycles in claw-free graphs have been well studied. In this paper we extend results on disjoint cycles in claw-free graphs satisfying certain minimum degree conditions. In particular, we prove that if G is claw-free of sufficiently large order $n = 3k$ with $\delta(G) \geqslant n/2$, then G contains k disjoint triangles.

1. Introduction

In this paper we consider only graphs without loops or multiple edges. We let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The *order* of G, usually denoted by n, is $|V(G)|$ and the *size* of G is $|E(G)|$. For any vertex v in G, let $N(v)$ denote the set of vertices adjacent to v and $N[v] = N(v) \cup v$. The *degree* of a vertex v is $|N(v)|$, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a vertex in G, respectively. If $U \subset V(G)$, we will use $G[U]$ to denote the subgraph of G induced by the vertices in U, and let $E(U_1, U_2)$ denote the set of edges with one end in U_1 and one end in U_2.

Let G and H be graphs. We say that G is *H-free* if H is not an induced subgraph of G. In this paper, we are interested in determining the number of disjoint cycles possible in a claw-free graph which satisfies certain minimum degree conditions.

Disjoint cycles in claw-free graphs have been studied in a variety of papers. For example, H. Wang [5] showed the following.

Theorem 1.1. *For any integer $k \geqslant 2$, if G is a claw-free graph of order $n \geqslant 6(k-1)$ with $\delta(G) \geqslant 3$, then G contains at least k disjoint triangles or belongs to a special family that has only $k-1$ disjoint triangles.*

Chen, Markus and Schelp studied independent cycles based on edge density [2].

Theorem 1.2. *Let $k \geqslant 1$ and G be a $K_{1,r}$-free graph of order n and size q.*
(1) *If $r = 3$ and $q \geqslant n + \frac{1}{2}(3k-1)(3k-4) + 1$, then G contains k disjoint cycles.*
(2) *If $r \geqslant 4$ and $q \geqslant n + 16rk^2$, then G contains k disjoint cycles.*

The range of values for the number of cycles in a 2-factor of a 2-connected claw-free graph was studied in [1].

Theorem 1.3. *If G is a 2-connected claw-free graph with $\delta(G) \geqslant \frac{n-2}{3}$, then G contains a 2-factor with exactly k cycles for $1 \leqslant k \leqslant \frac{n-24}{3}$. Furthermore, this result is sharp in the sense that if we lower $\delta(G)$ we cannot obtain the full range of values for k.*

We will also need the following results.

Theorem 1.4 (Corradi and Hajnal [3]). *If G is a graph of order $n = 3k$ with $\delta(G) \geqslant 2n/3$, then G contains k disjoint triangles.*

Theorem 1.5 (Li, Rousseau and Zang [4]). *For k fixed and $n \to \infty$, the Ramsey number satisfies*

$$r(K_k, K_n) \leqslant (1 + o(1)) \frac{n^{k-1}}{(\log n)^{k-2}}.$$

2. Key examples

We now examine several key examples that will show the sharpness of our later results.

Example 1. Suppose we are interested in covering our claw-free graph with disjoint triangles; hence n must be divisible by 3. We will show that the minimum degree is $n/2$, where $n \geqslant 9$ is required. First note that at least nine vertices are required. Consider the wheel on six vertices; that is, a C_5 with another vertex adjacent to each vertex of the cycle. This graph has order six, is claw-free, and has minimum degree 3, but clearly does not contain two vertex-disjoint triangles.

Next, to see that minimum degree $n/2$ is required, consider the graph G_1 formed by taking a copy of $H_1 = K_{3t+2}$ and a copy of $H_2 = K_{3t+1}$ and adding edges that match each vertex of H_2 with a distinct vertex of H_1. It is clear that any attempt to cover the vertex set of G_1 with disjoint triangles requires either two vertices of H_1 to be in a triangle with one vertex of H_2 or two triangles with single vertices in H_1 and two vertices in H_2. Neither case is possible as no two vertices of H_i have a common neighbour in H_j, for $i \neq j$. Finally, note that $\delta(G_1) = (n-1)/2$; hence, $n/2$ is required. This is the sharpness example for Theorem 4.1.

Example 2. Next we ask the question of what minimum degree is required for a claw-free graph G to have a 2-factor with exactly two cycles? The graph G_2 of Figure 1 has order n and $\delta(G_2) = \frac{n-1}{3}$, but clearly cannot be covered by two cycles. Thus $\delta(G) \geqslant n/3$ is required. This

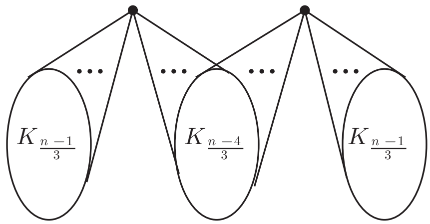

Figure 1. Claw-free graph G_2, with no 2-factor consisting of two cycles.

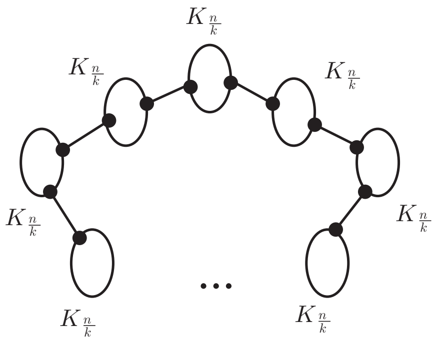

Figure 2. The graph G_3 with only kt cycles possible in a 2-factor.

is a sharpness example for Theorem 3.5 at the low end of the range for k cycles, namely two cycles.

Example 3. Now we ask the question of how many cycles are possible in a 2-factor of a claw-free graph with minimum degree at least n/k for some $k \geqslant 2$? The graph G_3 of Figure 2 consists of k copies of the graph $K_{n/k}$ with an edge between two copies forming them into a ring. This graph has minimum degree $n/k - 1$. If $n/k = 3t + 2$, then $n = 3kt + 2k$ and the graph could possibly contain as many as $kt + \lfloor \frac{2k}{3} \rfloor$ cycles in a 2-factor. But clearly only kt disjoint cycles are possible here. This example applies to Theorem 3.5 and shows the upper limit on the number of disjoint cycles, namely $n/3 - 2$, when $k = 3$.

Example 4. For $n - 1$ divisible by k, consider the graph G_4 formed from copies of $H_1 = K_{\frac{n-1}{k}}$ and copies of $H_2 = K_{\frac{n-1}{k}-1}$ as follows. Take a new vertex and join it to all the vertices of a copy of H_1 and to all vertices in a copy of H_2. Now take another new vertex and join it to our last copy of H_2 and to a new copy of H_2. Repeat this process until $k - 2$ copies of H_2 have been included. We complete the construction by taking another new vertex and join it to all vertices of the last copy of H_2 and to all the vertices of a new copy of H_1, obtaining (see Figure 3) a graph composed of a 'path' of k complete subgraphs with $k - 1$ connecting vertices. The graph G_4 has order n and $\delta(G_4) = \frac{n-1}{k}$, and clearly has no 2-factor composed of $k - 1$ cycles. This is the sharpness example for Theorem 3.4.

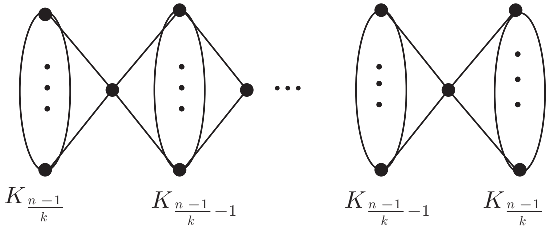

Figure 3. G_4 composed of k blocks with no 2-factor with $k-1$ cycles.

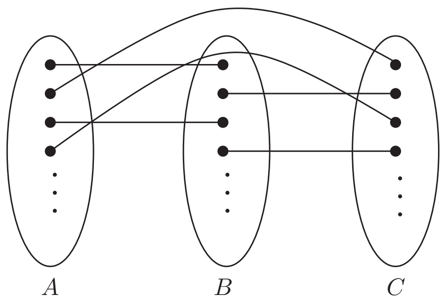

Figure 4. G_5 with $\delta = n/3$ and $n/3 - 2$ triangles.

Example 5. Consider the graph G_5 formed by taking three copies of the graph $K_{n/3}$, say A, B, C. Suppose that $V(A) = \{a_1, a_2, \ldots, a_{n/3}\}$, $V(B) = \{b_1, b_2, \ldots, b_{n/3}\}$, and $V(C) = \{c_1, c_2, \ldots, c_{n/3}\}$. Now place a matching between the odd labelled vertices of A and B, that is, match $a_{2i-1}b_{2i-1}$. Next we match the even labelled vertices of A with the odd labelled vertices of C, namely, match $a_{2i}c_{2i-1}$. Finally, match the even labelled vertices of B to the even labelled vertices of C (see Figure 5). The graph G_5 has minimum degree $n/3$. If $n = 9k + 6$, then each of A, B and C contains $3k + 2$ vertices. Further, it is clear that the only triangles in this graph lie entirely within one of A, B or C. Thus, there are exactly k disjoint triangles in any of A, B or C and hence $3k$ disjoint triangles in any largest collection in G_5. Further, $3k = n/3 - 2$. This is the sharpness example for Corollary 3.2 and for the upper limit on the range of disjoint cycles in Theorem 3.5.

3. Disjoint cycles

We begin with a look at disjoint triangles.

Theorem 3.1. *If G is a claw-free graph of order n and minimum degree δ, then G contains at least $F(n) = (\frac{\delta-2}{\delta+1})\frac{n}{3}$ disjoint triangles.*

Proof. Select a disjoint cycle system T composed of the maximum number, say t, of triangles. Let $H = G - T$ be the subgraph of G that remains after removing T. The subgraph H must have $\Delta(H) \leqslant 2$, and hence H is the disjoint union of paths and/or induced cycles of length at least four. Let $E = E(H, T)$ and note that $|E| \geqslant (\delta - 2)(n - 3t)$.

Claim. Each vertex in T has at most three adjacencies in H.

Assume this is not true and there is a $v \in T$ such that $\deg_H(v) \geqslant 4$. Suppose $v \in T$ is on a triangle with v_1 and $v_2 \in T$ and v has four adjacencies to H, say x_1, x_2, x_3, x_4. Then, among $x_1, x_2, x_3, x_4, v_1, v_2$ there must be a triangle (as $r(K_3, K_3) = 6$). Then there are three remaining vertices, all adjacent to v, and a claw would exist unless at least two of these remaining vertices are adjacent. But then we have another triangle formed, contradicting our choice of T, and completing the proof of the claim.

From this we see that $|E| \leqslant 9t$, and so

$$(n-3t)(\delta-2) \leqslant |E| \leqslant 9t.$$

Thus,

$$(\delta-2)n - 3t\delta + 6t \leqslant 9t,$$

and hence

$$(\delta-2)n - 3t(\delta+1) \leqslant 0,$$

so that

$$t \geqslant \left(\frac{\delta-2}{\delta+1}\right)\frac{n}{3}. \qquad \square$$

Note that when $\delta \geqslant n/3$, it follows from Theorem 3.1 that $t > n/3 - 3$ for all positive integers $n \geqslant 3$. Consequently, we state the following corollary, which is sharp in view of Example 5.

Corollary 3.2. *If G is a claw-free graph of order n with minimum degree $\delta(G) \geqslant n/3$, then G contains at least $n/3 - 2$ disjoint triangles and this result is best possible.*

We note, however, that for $\delta < n/3$, Theorem 3.1 fails to provide a sharp result. This can be observed for $\delta = 3$, $n = 6(k-1)$; Theorem 1.1 implies at least $k-1$ disjoint triangles, while Theorem 3.1 only ensures $(k-1)/2$ disjoint triangles.

Lemma 3.3. *If $k \geqslant 2$ and G is a claw-free graph with $\delta(G) \geqslant n/k$, then the independence number $\alpha(G) \leqslant 2k-1$.*

Proof. Choose an independent set S with $\alpha = \alpha(G)$. Let $H = G - S$ be the remaining subgraph of order $n - \alpha$. Any vertex of H has degree at most two into S as G is claw-free. Further, each vertex of S has all its neighbours in H. If $E = E(S, H)$, then

$$\alpha\left(\frac{n}{k}\right) \leqslant |E| \leqslant 2(n-\alpha),$$

so that

$$\alpha \leqslant \frac{2kn}{n+2k} = 2k\left(\frac{n}{n+2k}\right) < 2k,$$

and hence

$$\alpha(G) \leqslant 2k-1. \qquad \square$$

Theorem 3.4. *Let $k \geqslant 2$ be a positive integer. If G is a claw-free graph of order*

$$n \geqslant 10k^4$$

with $\delta(G) \geqslant n/k$, then G contains a 2-factor with $k-1$ components. Further, this value of $\delta(G)$ is best possible.

Proof. Select an independent set, say F, of $k-1$ cycles, $C_1, C_2, \ldots, C_{k-1}$, where $|\cup_{i=1}^{k-1} V(C_i)|$ is as large as possible. We know such a set exists from Theorem 1.1. Let $H = G - \cup_{i=1}^{k-1} V(C_i)$.

Note that to any one cycle C_i, a vertex $x \in V(H)$ has at most $2k-2$ adjacencies or else there would exist an independent set (predecessors of adjacencies along with x) of order at least $2k$, a contradiction to Lemma 3.3. Thus, $\delta(H) \geqslant n/k - 2(k-1)^2$.

But the bound on $\delta(H)$ implies that H contains a cycle of length at least $\delta(H)$. Thus, as F is as large as possible, each cycle C_i, $(1 \leqslant i \leqslant k-1)$, contains at least $\delta(H)$ vertices.

This implies that for each i,

$$n/k - 2(k-1)^2 \leqslant |C_i| \leqslant n - (k-1)(n/k - 2(k-1)^2) = n/k + 2(k-1)^3.$$

Also, $|H| \leqslant n/k + 2(k-1)^3$. If $\delta(H) \geqslant \frac{n/k+2(k-1)^3+1}{2}$, then H is Hamiltonian-connected. This is true since $2(n/k - 2(k-1)^2) \geqslant n/k + 2(k-1)^3 + 1$, or equivalently $n \geqslant 2k(k-1)^3 + 4k(k-1)^2 + k$.

Claim 1. No cycle in F has two independent edges to H.

Suppose this were not the case: say C_b has edges w_ih_i and w_jh_j with $w_i, w_j \in V(C_b)$ and $h_i, h_j \in V(H)$. Without loss of generality suppose that $w_i, w_{i+1}, \ldots, w_j$ contains more than half the vertices of C_b. Then, consider the cycle

$$C^* : w_i, w_{i+1}, \ldots, w_j, h_j, P, h_i, w_i,$$

where P is a Hamiltonian path connecting h_i and h_j in H. The claim is that $|C^*| > |C_b|$. If this is not true, then $\frac{|C_b|+1}{2} + |H| \leqslant |C_b|$, which implies $2|H| + 1 \leqslant |C_b|$. However, this implies $2(n/k - 2(k-1)^2) \leqslant n/k + 2(k-1)^3$, or equivalently, $n \leqslant 2k(k-1)^3 + 4k(k-1)^2$, a contradiction.

Claim 2. No two cycles of F have three independent edges between them.

Suppose instead that C_a and C_b had three independent edges between them. Without loss of generality, say that a_1b_1, a_2b_2 and a_3b_3 are these edges with $a_i \in C_a$ and $b_i \in C_b$, $i = 1, 2, 3$. Also, without loss of generality, suppose that the segment (a_1, a_2) contains at most $|C_a|/3$ vertices and (b_1, b_2) contains less than $|C_b/2|$ vertices. Then, a new cycle

$$C_a' : a_2, a_2^+, \ldots, a_1, b_1, b_1^-, \ldots, b_2, a_2$$

replaces C_a and H replaces C_b to form a new system.

If $(2/3)|C_a| + (1/2)|C_b| + |H| > |C_a| + |C_b|$, then the new system would have more vertices than F. Assume not, and so $|H| \leqslant \frac{|C_a|}{3} + \frac{|C_b|}{2}$. This implies $n/k - 2(k-1)^2 \leqslant \frac{5}{6}(n/6 + 2(k-1)^3)$, since $|C_a|, |C_b| \leqslant n/k + 2(k-1)^3$. Hence, $n \leqslant 12k(k-1)^2 + 10k(k-1)^3$, a contradiction.

By Claim 1 we see that some cycles may have a vertex of large degree to H, but then no other vertices of that cycle have any adjacencies in H.

Claim 3. Every vertex of H has at least two edges to F.

First suppose that $\deg_F(x) = 0$ for some $x \in V(H)$. Then, as $\deg(x) \geqslant n/k$, we see that $|H| \geqslant n/k + 1$. But since every cycle in F is at least as large as H, this implies that $n \geqslant k(n/k+1) = n + k$, a contradiction.

Next suppose that some vertex has only one edge to F. If this were the case then $|H| \geqslant n/k - 1 + 1 = n/k$, and hence $|H| = n/k = |C_i|$ for $i = 1, 2, \ldots, k-1$. Also, no vertex of a cycle C_i can have as many as $2k$ adjacencies into C_j for $j \neq i$. Indeed, as $\alpha(G) \leqslant 2k-1$ and if successors of neighbours of a vertex $x \in C_i$ on C_j together with x do not form an independent set, then the cycle C_j can be extended to $n/k + 1$ vertices. At most $4k$ vertices of C_j can have edges from each C_i or from H, and so many vertices of C_j are left with no adjacencies outside C_j, contradicting the minimum degree.

Claim 4. $|H| \geqslant \frac{n}{k} - k + 2$.

It follows from Claim 1 that at most one vertex of H has more than one adjacency in C_i. Hence, at most $k-1$ vertices of H have degree greater than $k-1$ to F. But this implies that there are vertices in H adjacent to at most $k-1$ vertices of F. Let v be such a vertex. Then $\deg_H(v) \geqslant n/k - (k-1)$. But this implies that $|H| \geqslant n/k - (k-1) + 1 = n/k - k + 2$.

To complete the proof of the theorem, as noted above, for a given cycle C_i in F there are at most two independent edges between C_i and C_j for $j \neq i$, and at most one independent edge from C_i to H. Thus, there are at most $2k-3$ such independent edges. This implies there are at least $|C_i| - 2k - 3$ vertices of C_i, which we denote by C_i^*, that have at most $2k-3$ adjacencies outside C_i. Thus, the graph induced by C_i^* is a dense graph for $1 \leqslant i \leqslant k-1$. This implies that between any pair of vertices of C_i there is a path of length at least $n/k - 2k + 3$. From Claim 4 we see that

$$\frac{n}{k} - k + 2 \leqslant |C_i| \leqslant \frac{n}{k} + (k-1)(k-2).$$

For convenience let $H = C_0$. Consider a maximal set of independent edges M between the C_i for $0 \leqslant i \leqslant k-1$, and assume that $C_0, C_1, \ldots, C_s$ are cycles that contain at least one vertex of these independent edges. Form a new graph in which vertices $v_0, v_1, \ldots, v_s$ correspond to the cycles $C_0, C_1, \ldots, C_s$ and $v_i v_j$ is an edge if M contains an edge from C_i to C_j. Among these $s+1$ cycles there are at least $s+1$ independent edges, hence the graph contains a cycle.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_r}, v_{i_1}$ be the vertices of this cycle. Then, starting in C_{i_1} we may traverse all but possibly $2k$ vertices before we cross to C_{i_2}. In C_{i_2} we traverse all but at most $2k$ vertices before we cross to C_{i_3}, where we only traverse at most $2k$ vertices before we cross to C_{i_4}. Continuing in this manner we return to C_{i_1}, completing a cycle in G. Now on the subgraphs of the cycles C_{i_j}, for $j \geqslant 3$, we form new cycles using all the remaining vertices in the dense subgraph C_i^*. Thus, at most $2k$ vertices have been lost from any of the original cycles.

We now form F' to include all these new cycles, as well as H if it is not part of these cycles, and all the unchanged cycles from F. This is a system of $k-1$ cycles that includes all but $6(k-1)k$ vertices of G, contradicting our choice of F and completing the proof. □

The value of δ is best possible, as seen from Example 4.

Theorem 3.5. *If G is a claw-free graph of sufficiently large order n with $\delta(G) \geqslant n/3$, then G contains a 2-factor with k disjoint cycles, for $2 \leqslant k \leqslant \lfloor n/3 - 2 \rfloor$.*

Proof. When $k = 2$, the result holds by Theorem 3.4. Suppose we select a disjoint cycle system $F : C_1, C_2, \ldots, C_t$ for each $t \geqslant 3$ in the range. We know such a system exists by Corollary 3.2. Also suppose that F is chosen to contain the maximum number of vertices. Let $H = G - F$.

Note that if $\deg_H(v) > n/(t+1)$ for all $v \in V(H)$, then H contains a cycle of length greater than $n/(t+1)$, and hence each cycle in F has length greater than $n/(t+1)$, or else we could find a system larger than F. But then, $|V(G)| = n > (t+1)(n/(t+1)) = n$, a contradiction. Thus, there exists a vertex $x \in V(H)$ such that $\deg_F(x) \geqslant n/3 - n/(t+1)$. Also, by Lemma 3.3 we know that $\alpha(G) \leqslant 5$.

Now consider a vertex $x \in V(H)$ such that $\deg_F(x) \geqslant cn$, for some constant c. We note that x has at most one adjacency to any triangle in F and x has at most $\alpha(G) - 1$ adjacencies to any longer cycle, or else we could extend the cycle to include x and contradict our choice of F. Thus, either x is adjacent to a large number of triangles in F or x is adjacent to a large number of longer cycles.

Case 1. Suppose x is adjacent to at least nine longer cycles.

Choose a collection of nine disjoint longer cycles in F such that x has at least one adjacency to each of the cycles. Note that as G is claw-free, the predecessor and successor of neighbours of x on the long cycles must themselves be adjacent, or again we could extend F. Now by Ramsey theory, if we consider a set of at least six cycles where x has an adjacency, then there will be a triangle formed on three of these neighbours (or else a claw would exist). Also, considering the set of longer cycles not involved in the triangle with adjacencies from x, we will find an edge between two successors of neighbours of x, or else we would have an independent set of size larger than $\alpha(G)$.

We now form a new disjoint cycle system, F', by combining x and the two long cycles where there is an edge between successors of neighbours to form one cycle, using the triangle formed on neighbours of x to form a second cycle, keeping the rest of each of the three cycles where a vertex was deleted to form the triangle, and all the remaining cycles. This system clearly has more vertices than F (by exactly one), again a contradiction to our choice of system. This contradiction completes this case.

Consider the Ramsey numbers $r_1 = r(K_3, K_6)$, $r_2 = r(K_6, K_{r_1})$ and $r = r(K_3, K_{r_2})$.

Case 2. Suppose x is adjacent to r triangles.

Let $\{a_i, b_i, c_i\}$, for $1 \leqslant i \leqslant r$, be a set of r triangles such that $xc_i \in E(G)$. Since G is claw-free, there is no independent set of three vertices in $\{c_i : 1 \leqslant i \leqslant r\}$, and so there is a clique, say $\{c_1, c_2, \ldots, c_{r_2}\}$ of r_2 vertices. As $\alpha(G) \leqslant 5$, there is no independent set of six vertices in $\{b_1, b_2, \ldots, b_{r_2}\}$, and so there is a clique with r_1 vertices, say $\{b_1, b_2, \ldots, b_{r_1}\}$. Similarly, $\alpha(G) \leqslant 5$ implies there is a triangle, say $\{a_1, a_2, a_3\}$ in $\{a_1, a_2, \ldots, a_{r_1}\}$.

Then $G[a_1, a_2, a_3]$, $G[b_1, b_2, b_3]$ and $G[x, c_1, c_2, c_3]$, along with the remaining cycles of F, form a system of t cycles larger than F, a contradiction.

Thus, in either case we reach a contradiction, and hence F must cover $V(G)$, completing the proof. □

4. More on disjoint triangles

Our goal in this section is to prove the following result, which is best possible by Example 1.

Theorem 4.1. *If G is a claw-free graph of sufficiently large order $n = 3k$ with $\delta(G) \geqslant n/2$, then G contains k disjoint triangles.*

Proof of Theorem 4.1. By Lemma 3.3, $\alpha(G) \leqslant 3$. Theorem 1.5 implies that

$$r(K_t, K_3) \leqslant c\frac{t^2}{(\log\ t)}.$$

Since G does not contain a claw, and $\delta(G) \geqslant n/2$, the application of Theorem 1.5 implies for sufficiently large n that G contains a large clique: in fact, G contains $K_{n^{\frac{1}{2}-\epsilon}}$, for any $\epsilon > 0$. Select such a clique and call it A. Now let $B \subseteq G - A$ be those vertices of $G - A$ whose degree to A is at most 32. Also let $C = G - (A \cup B)$.

Note that

$$|E(A, C)| \geqslant |A|(n/2 - |A|) - 32|B|.$$

Thus,

$$|C| \geqslant \frac{|A|(n/2 - |A|) - 32|B|}{|A|},$$

as this measures the average degree into C for a vertex in A.

But then, as $|A| \geqslant n^{1/2-\epsilon}$,

$$|C| \geqslant n/2 - |A| - \frac{32|B|}{|A|} \geqslant n/2 - o(n).$$

Let

$$B_2 = \{b \in B \mid \deg_C(b) \geqslant 33\}$$

and let $B_1 = B - B_2$. Note that each vertex in B_1 has at most 64 adjacencies in $A \cup C$ (32 and 32).

Now we consider the partition $V(G) = B_1 \cup R$, where $R = V(G) - B_1$.

Claim. B_1 can be covered with triangles using at most 16 vertices not in B_1.

Note that if $|B_1| \geqslant n/2 - c\sqrt{n}$, c a constant, then $\delta(G[B_1]) \geqslant n/2 - O(n^{1/2+\epsilon})$.

Case 1. Suppose $|B_1| > n/2 + 1$.

This implies that $|R| < n/2 - 1$, and thus every vertex of R is adjacent to at least three vertices of B_1. If $|B_1| \equiv 0 \bmod 3$, Theorem 1.4 implies a disjoint triangle cover. If $|B_1| \equiv 1 \bmod 3$, then any vertex of B_1 with three or more adjacencies in R can be covered by a triangle (using two of these R neighbours) and again Theorem 1.4 allows us to cover the rest of B_1 with disjoint triangles. Similarly, if $|B_1| \equiv 2 \bmod 3$, then two vertices of B_1 can be covered by a triangle with one vertex each from R and Theorem 1.4 allows us to complete the disjoint triangle cover of the rest of B_1.

Case 2. Suppose n is even and $|B_1| = n/2 + 1$, or n is odd and $|B_1| = (n+1)/2$ or $|B_1| = (n-1)/2$.

If n is even and $|B_1| = n/2 + 1$, then $|B_1| \equiv 1 \bmod 3$, as n is divisible by 3. This implies that $|R| = n/2 - 1$. Hence, every vertex of R is adjacent to at least two vertices of B_1. Consequently, either a triangle can be formed covering one vertex of B_1 and two vertices of R, or it must be the case that each vertex of B_1 has at most two adjacencies in R and none of these adjacencies form a triangle with the vertex of B_1. But then we can easily find two disjoint triangles covering two vertices of B_1 and one vertex of R. A similar argument applies when n is odd and $|B_1| = (n-1)/2$.

If n is odd and $|B_1| = (n+1)/2$, then $|B_1| \equiv 2 \bmod 3$. This implies that $|R| = (n-1)/2$. Hence, every vertex of R is adjacent to at least two vertices of B_1. Consequently, there are two vertices of B_1 adjacent to two disjoint 2-element subsets of vertices of R. Then two disjoint triangles can be formed, each covering one vertex of B_1 and two vertices of R.

In any case, the number of remaining vertices of B_1 is congruent to zero mod 3. Then Theorem 1.4 allows us to cover these remaining vertices with disjoint triangles.

Case 3. Suppose $|B_1| = n/2$.

It follows that $n/2 \equiv 0 \bmod 3$, and hence B_1 is dense and can be covered by disjoint triangles using Theorem 1.4.

Case 4. Suppose $n/2 - c\sqrt{n} \leqslant |B_1| \leqslant n/2 - 1$, c a constant.

Then vertices of B_1 have at least two adjacencies in R, and one or two disjoint triangles can be found to cover vertices of B_1, leaving the number of remaining vertices in B_1 congruent to zero mod 3. Then Theorem 1.4 allows us to complete the disjoint covering of B_1.

Case 5. Suppose $0 \leqslant |B_1| \leqslant n/2 - c\sqrt{n}$, c a constant.

Cover B_1 with as many disjoint triangles as possible. There remains at most $8 = r(K_3, K_4) - 1$ vertices uncovered. But each of these vertices has at least $c\sqrt{n}$ adjacencies to R and so they can be covered with at most eight disjoint triangles. This completes the proof of the claim.

To complete the proof of the theorem we show that the vertices of R that have not been covered by disjoint triangles can be covered.

Since we have shown that there exists a system of disjoint triangles that covers B_1 and uses at most 16 vertices of R, we extend this collection by adding disjoint triangles from B_2. Since $\alpha(G) \leqslant 3$, this leaves at most 8 vertices of B_2, say B^*, uncovered, as $r(K_3, K_4) = 9$. Since every vertex of B_2 has at least 33 adjacencies in C, and considering the neighbours of vertices of B^* in C, the vertices of B^* can be covered by disjoint triangles.

Similarly, the remaining vertices of C can be covered by disjoint triangles, with the exception of at most eight such vertices. But each of these vertices has at least 16 neighbours in what remains of A, and so each of these vertices can be placed on disjoint triangles with two vertices of A.

The remaining vertices all belong to A and can clearly be covered by disjoint triangles, completing the proof of the theorem. □

Acknowledgement

The authors would like to thank the referee for careful reading and good suggestions that improved the manuscript.

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