Distributing Vertices on Hamiltonian Cycles

Ralph J. Faudree,[1] Ronald J. Gould,[2] Michael S. Jacobson,[3] and Colton Magnant[4]

[1] *DEPARTMENT OF MATHEMATICAL SCIENCES UNIVERSITY OF MEMPHIS, MEMPHIS TENNESSEE 38152*

[2] *DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE EMORY UNIVERSITY ATLANTA, GEORGIA 30322*

[3] *DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO DENVER DENVER, COLORADO 80217*

[4] *DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY BETHLEHEM, PENNSYLVANIA 18015*

E-mail: rg@mathcs.emory.edu; dr.colton.magnant@gmail.com

Received July 15, 2008; Revised September 25, 2009

Published online 9 February 2011 in Wiley Online Library (wileyonlinelibrary.com). DOI 10.1002/jgt.20564

Abstract: Let G be a graph of order n and $3 \leq t \leq n/4$ be an integer. Recently, Kaneko and Yoshimoto [J Combin Theory Ser B 81(1) (2001), 100–109] provided a sharp $\delta(G)$ condition such that for any set X of t vertices, G contains a hamiltonian cycle H so that the distance along H between any two vertices of X is at least $n/2t$. In this article, minimum degree and connectivity conditions are determined such that for any graph G of sufficiently large order n and for any set of t vertices $X \subseteq V(G)$, there is a hamiltonian cycle H so that the distance along H between any two consecutive vertices of X is approximately n/t. Furthermore, the minimum degree threshold is determined for the existence of a hamiltonian cycle H such that the vertices of X appear in a prescribed order

at approximately predetermined distances along H.
J Graph Theory 69: 28–45, 2012

Keywords: *hamiltonian cycles; distances*

1. INTRODUCTION

In this paper, we use the following notation. For a graph G, let $\delta(G)$ be the minimum degree, $\kappa(G)$ be the connectivity of G, $N(v)$ be the set of neighbors of a vertex $v \in V(G)$, $d(v) = |N(v)|$ and $d_A(v)$ be $|N(v) \cap A|$ for any set $A \subseteq V(G)$. Also let $G[A]$ denote the induced subgraph of G on A. For two subsets of vertices $A, B \subseteq G$, let $e(A,B)$ be the number of edges between A and B.

We denote a path from u to v by $P[u,v]$ or $(u,\ldots,v)$, while a path from u to v, along a path or cycle A, is denoted by $(u,v)_A$. When the context is clear, we may simply use the notation P. Within a path $(v_1.v_2,\ldots)$, the vertex v_{i-1} is called the predecessor (likewise v_{i+1} is called the successor) of the vertex v_i.

The distance between u and v is denoted by $dist(u,v)$, while the distance, along a path or cycle A, is denoted by $dist_A(u,v)$. For any subgraph $H \subseteq G$, we define the order of H as the number of vertices in H, that is, $|H|$. All other notation may be found in [2].

In 2001, Kaneko and Yoshimoto [6] proved the following result.

Theorem 1. *Let G be a graph of order n, $d \leq n/4$ a positive integer and A a set of at most $n/(2d)$ vertices. If $\delta(G) \geq n/2$, then there exists a hamiltonian cycle in G with the distance, along the cycle, between any pair of vertices of A at least d.*

The key restriction here is that $\delta(G) \geq n/2$ only guarantees that G is 2-connected. Consider the graph $G_0 = (A \cup B) + \{u,v\}$ where $A = B = K_{(n-2)/2}$ and suppose we select $n/(2d)$ vertices in A (see Fig. 1). This graph has minimum degree $n/2$, but the chosen vertices cannot be spread evenly around a hamiltonian cycle.

In 1984, El-Zahar [5] conjectured the following.

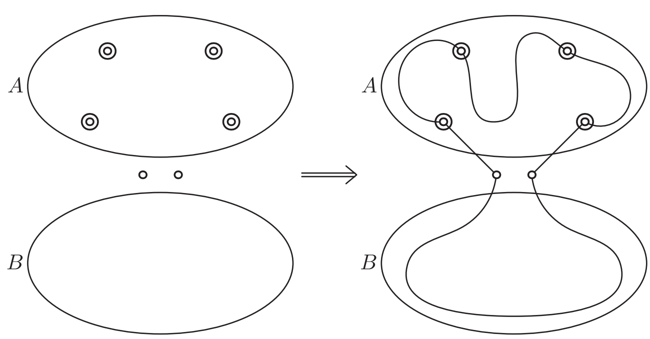

FIGURE 1. Sharpness of Theorem 1.

Conjecture 1. *Any graph G on n vertices and minimum degree at least* $\lceil \frac{1}{2}n_1 \rceil + \cdots + \lceil \frac{1}{2}n_k \rceil$, *where* $n_1 + \cdots + n_k = n$, *has a* 2*-factor with cycle lengths* $n_1, \ldots, n_k$.

This conjecture is known to be true for large values of n (see S. Abbasi's dissertation [1]). Our results combine the ideas of Conjecture 1 and Theorem 1.

A graph is said to be *k-linked* if for every choice of $2k$ vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, there exists a collection of vertex disjoint paths $P_i = (x_i, y_i)$ for all i. We use, in our proofs, the following result of Thomas and Wollan [7].

Theorem 2. *If a graph G is* $10k$*-connected, then G is k-linked.*

A graph G is said to be *panconnected* if for each pair of vertices $u, v \in V(G)$, there exists a path of length l in G for each l satisfying $dist_G(u,v) \leq l \leq n-1$. Finally, we also make use of the following result of Williamson [8].

Theorem 3. *If* $\delta(G) \geq (n+2)/2$, *then the graph G is panconnected.*

2. MAIN RESULTS

For a given integer t, for ease of notation, we consider all indices modulo t. Using the above results, we prove the following theorems.

Theorem 4. *Let* $t \geq 3$ *be an integer and let* $0 < \varepsilon < 1/(2t)$. *For* $n \geq 7t^6 \times 10^{10}/\varepsilon^6$, *let G be a graph of order n having* $\delta(G) \geq n/2$ *and* $\kappa(G) \geq 2\lceil t/2 \rceil$. *For every* $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, *there exists a hamiltonian cycle H such that* $dist_H(x_i, x_j) \geq (1/t - \varepsilon)n$ *for all* $1 \leq i < j \leq t$. *Furthermore, the minimum degree and connectivity conditions are sharp.*

The lower bound on n in this and all of our results comes from Lemma 2. One may easily verify that this particular bound dominates all bounds from other inequalities. By choosing $\varepsilon' = \varepsilon/t$, the following corollary to Theorem 4 becomes obvious.

Corollary 5. *Let* $t \geq 3$ *be an integer and let* $0 < \varepsilon < 1/(2t^2)$. *For* $n \geq 7t^{12} \times 10^{10}/\varepsilon^6$, *let G be a graph of order n having* $\delta(G) \geq n/2$ *and* $\kappa(G) \geq 2\lceil t/2 \rceil$. *For every* $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, *there exists a hamiltonian cycle H and an ordering of the elements of X such that* $(1/t - \varepsilon)n \leq dist_H(x_i, x_j) \leq (1/t + \varepsilon)n$ *for all* $1 \leq i \leq t$. *Furthermore, the minimum degree and connectivity conditions are sharp.*

We also consider the case in which we would like the chosen vertices $\{x_1, \ldots, x_t\}$ to appear in order along the hamiltonian cycle and at approximately predetermined distances.

Theorem 6. *Let* $t \geq 3$ *be an integer and* $\varepsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ *positive real numbers having* $\sum_{i=1}^{t} \gamma_i = 1$ *and* $0 < \varepsilon < \min\{\gamma_i^2/2\}$. *For* $n \geq 7t^{12} \times 10^{10}/\varepsilon^6$, *let G be a graph of order n having* $\delta(G) \geq (n+t-1)/2$ *or* $\delta(G) \geq n/2$ *and* $\kappa(G) \geq 3t/2$. *For every* $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, *there exists a hamiltonian cycle H containing the vertices of X in order such that* $(\gamma_i - \varepsilon)n \leq dist_H(x_i, x_{i+1}) \leq (\gamma_i + \varepsilon)n$ *for all* $1 \leq i \leq t$. *Furthermore, the minimum degree and connectivity conditions are sharp.*

The sharpness of Theorems 4 and 6 are established, along with the proofs, in Section 4. Aside from placing vertices on one long cycle, we also place chosen vertices on different cycles of a 2-factor with prescribed cycle lengths. The following theorem finds such a 2-factor.

Theorem 7. *Let $t \geq 3$ be an integer and $\varepsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \varepsilon < \min\{\gamma_i/2\}$. For $n \geq 7t^6 \times 10^{10}/\varepsilon^6$, let G be a graph of order n having $\delta(G) \geq (n+t-1)/2$. For every set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a spanning collection $\mathscr{C}$ of vertex disjoint cycles C_i with $x_i \in C_i$ such that $(\gamma_i - \varepsilon)n \leq |C_i| \leq (\gamma_i + \varepsilon)n$ for all $1 \leq i \leq t$. Furthermore, the minimum degree condition is sharp for many choices of $\gamma_1, \ldots, \gamma_t$ and ε.*

The degree condition in this theorem is sharp because of the following example. Let $G_1 = K_t + (K_{(n-t)/2} \cup K_{(n-t)/2})$ when $n-t$ divisible by 2. Clearly $\delta(G_1) = (n+t)/2 - 1$. If we choose S to be the vertices of the K_t and we choose $\gamma_1, \gamma_2, \ldots, \gamma_t$ so there is no subset $I_0 \subset [t]$ of the index set such that $\frac{1}{2} - \varepsilon t \leq \sum_{i \in I_0} \gamma_i \leq \frac{1}{2} + \varepsilon t$, then this graph cannot contain the desired collection of cycles.

Also similar to the above results, the following theorem finds a spanning linkage with approximately prescribed lengths on the paths of the linkage.

Theorem 8. *Let $t \geq 3$ be an integer and $\varepsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \varepsilon < \min\{\gamma_i/2\}$. For $n \geq 7t^6 \times 10^{10}/\varepsilon^6$, let G be a graph of order n having $\delta(G) \geq (n+2t-1)/2$. For every set $X = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\} \subseteq V(G)$ of $2t$ vertices, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = (x_i, \ldots, y_i)$ such that $(\gamma_i - \varepsilon)n \leq |P_i| \leq (\gamma_i + \varepsilon)n$ for all $1 \leq i \leq t$. Furthermore, the minimum degree condition is sharp for many choices of $\gamma_1, \ldots, \gamma_t$ and ε.*

The sharpness of Theorem 8 is given by the following construction. Let $G_2 = K_{2t} + (K_{(n-2t)/2} \cup K_{(n-2t)/2})$ when $n-2t$ divisible by 2. Clearly $\delta(G_1) = (n+2t)/2 - 1$. If we choose S to be the vertices of the K_{2t} and we choose $\gamma_1, \gamma_2, \ldots, \gamma_t$ so there is no subset $I_0 \subset [t]$ of the index set such that $\frac{1}{2} - \varepsilon tn \leq \sum_{i \in I_0} \gamma_i \leq \frac{1}{2} + \varepsilon tn$, then this graph cannot contain the desired linkage.

The following is an easy corollary to Theorem 8. The sharpness is also given by the same example as above.

Corollary 9. *Let $t \geq 3$ be an integer and $\varepsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \varepsilon < \min\{\gamma_i/2\}$. For $n \geq 7t^6 \times 10^{10}/\varepsilon^6$, let G be a graph of order n having $\delta(G) \geq (n+2t-3)/2$. For every set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$ and $Y \subseteq V(G)$ with $t \leq |Y| \leq n/8$, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = (x_i, \ldots, y_i)$ where $y_i \in Y$ such that $(\gamma_i - \varepsilon)n \leq |P_i| \leq (\gamma_i + \varepsilon)n$ for all $1 \leq i \leq t$. Furthermore, the minimum degree condition is sharp for many choices of $\gamma_1, \ldots, \gamma_t$ and ε.*

Given a subgraph $H \subset G$ with $2t$ chosen vertices $X = x_1, x_2, \ldots, x_t$, $y_1, y_2, \ldots, y_t \subseteq H$, the following corollary to Theorem 8 constructs a spanning collection of vertex disjoint paths from x_i to y_i in $G \backslash H$ of lengths within the prescribed range.

Corollary 10. *Let $t \geq 3$ be an integer and $\varepsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \varepsilon < \min\{\gamma_i/2\}$. For $n \geq 7t^6 \times 10^{10}/\varepsilon^6$, let G be a graph of order n and let $H \subset G$ with $|H| = r$. Suppose $\delta(G) \geq (n+r-3)/2$. For every set*

$X=\{x_1,x_2,\ldots,x_t,y_1,y_2,\ldots,y_t\}\subseteq V(H)$ of $2t$ vertices, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i=(x_i,\ldots,y_i)\subset(G\setminus H)$ such that $(\gamma_i-\varepsilon)(n-r)\le|P_i|\le(\gamma_i+\varepsilon)(n-r)$ for all $1\le i\le t$. Furthermore, the minimum degree condition is sharp for many choices of $\gamma_1,\ldots,\gamma_t$ and ε.

In particular, this corollary implies that one may place a linear forest on a hamiltonian cycle in a prescribed order with a given orientation on each path and with approximately given distances between the paths of the linear forest. Similar work may be found in [3].

The main work of the proofs is contained in the four lemmas presented in the next section. In Section 4, we bring together the lemmas to prove Theorems 4 and 6. The proofs of Theorems 7 and 8 are omitted since they are almost identical to the proof of Theorem 6.

Each proof begins with the application of a Setup Lemma (see Lemma 4) which provides a hamiltonian cycle similar to the desired cycle but with only rough bounds on the distances between vertices. With the structure provided by the Setup Lemma, if certain conditions are satisfied, we then apply a Swapping Lemma (see Lemma 2) to adapt the structure to make it closer to the desired cycle. If the above conditions are not satisfied, we ignore the structure previously built and apply a Rebuilding Lemma (see Lemma 3) to build the desired cycle directly. We also make use of a technical Absorbing Lemma (see Lemma 1), within the proofs of the other lemmas, to clean up any vertices that are “misbehaving”.

3. LEMMAS

We now provide lemmas which are necessary for the proofs of Theorems 4 and 6. The first lemma tells how to absorb vertices into a long cycle. By “absorbing” we mean making a cycle larger by including more vertices. Since adding vertices to a cycle sometimes involves removing other vertices, we must be careful in the absorbing process.

Lemma 1 (Absorbing). *Let $t\ge 1$, $n\ge 5t$ be integers, and let G be a graph of order n having $\delta(G)\ge n/2$ and let $X=\{x_1,\ldots,x_t\}$ be an ordered set of t vertices in G. If there exists a cycle C of order at least $3n/4+t$ containing the vertices of X in the given order, then there exists a hamiltonian cycle H containing the vertices of X in the given order such that $dist_H(x_i,x_{i+1})\ge dist_C(x_i,x_{i+1})$ for all $1\le i\le t$.*

Proof. Proceed by contradiction. Let J be a smallest collection of vertices that cannot be absorbed into C while maintaining $dist_C(x_i,x_{i+1})$ for all i. Let J' be a component of smallest order in J. If J' is the single vertex v, then since $\delta(G)\ge n/2$ it follows that $d_C(v)\ge n/2$. Since $|C|\le n-1$, it follows that v is adjacent to two consecutive vertices u and u^+ of C; hence v can be absorbed into C, a contradiction. Thus, $|J'|\ge 2$.

If $|J'|=2$ with $J'=\{u,v\}$, then since we have assumed J' is connected, $G[J']$ contains the edge uv. If one of u or v is adjacent to consecutive vertices along the cycle, then we can make the same insertion as above. Also if u and v are adjacent to vertices u' and v', respectively, with $u'v'\in E(C)$, then we may replace

$u'v'$ with $u'uvv'$ to absorb u and v into C. Both cases lead to a contradiction as before.

Since $d_C(u), d_C(v) \geq n/2-1$ and $|C| \leq n-2 = 2(n/2-1)$, and $N(u)$ and $N(v)$ satisfy neither of the above cases, we know u and v must both be adjacent to every other vertex along C and $N_C(u) = N_C(v)$. Therefore, since $n \geq 5t$, there must exist some vertex $w \in C-(N(u) \cup N(v))$ with $w \notin X$. Let w^- and w^+ be the vertices adjacent to w along C and, without loss of generality, select the edges uw^- and vw^+. Then $C' = (\ldots, w^-, u, v, w^+, \ldots)$ contradicts the maximality of C.

Finally, suppose $|J'| \geq 3$. Then there exists a path (v_1, v_2, v_3) on three vertices in J'. Clearly $d_C(v_i) > n/2 - |J'|$ for all i. Also note that $3n/4+t \leq |C| \leq n-|J'|$. Therefore, since $|J'| \leq |J| \leq n/4-t$, we get:

$$d_C(v_1)+d_C(v_2)+d_C(v_3) > \frac{3n}{2} - 3|J'| > n-|J'|+t \geq |C|+t.$$

From the previous cases, we may assume v_i, and similarly v_i and v_j together are not adjacent to consecutive vertices for any $1 \leq i < j \leq 3$. Therefore, there must exist at least $t+1$ vertices, w_k such that $v_i w_k^+, v_j w_k^- \in E(G)$ for some $i \neq j$ and for $1 \leq k \leq t+1$. Hence, there exists at least one vertex $w \in G \backslash X$ with $v_i w^+, v_j w^- \in E(G)$. We may now replace the path (w_1, w, w_2) with the path (w_1, P, w_2) (where P is any subpath of (v_1, v_2, v_3) of order at least 2) to again contradict the choice of J' and finish the proof of Lemma 1. ■

The next two lemmas are the main components of our proofs. This first lemma explains how to move path segments from one subpath of H to another. We will use A, B, A^* and B^* to denote both paths and the vertex set of the corresponding path. The context will make the usage clear to the reader.

The function $f(c,c')$ is given by

$$f(c,c') = \max\left\{\frac{64}{c'c^2}, \frac{16}{c^3}\right\}.$$

Lemma 2 (Swapping). *Given constants $c, c' > 0$, let G be a graph of order $n > f(c,c')$. If $A[a,a']$ and $B[b,b']$ are disjoint paths with $e(A,B) \geq cn^2$, then there exist two other disjoint paths $A^*[a,a']$, $B^*[b,b'] \subseteq G[A \cup B]$ such that $|B| < |B^*| < |B| + c'n$ and $|A^*| + |B^*| \geq |A| + |B| - 64/c^2$.*

The lower bound on n comes from inequalities (3) and (4), respectively.

Proof. Let $A' \subseteq A$ denote the set of vertices $v \in A$ with $d_B(v) \geq cn^2/(2|A|)$. Since $e(A,B) \geq cn^2$, we find

$$|A'| \geq \frac{cn^2 - (|A|-|A'|)\left(\dfrac{cn^2}{2|A|}\right)}{|B|} \geq \frac{cn^2}{2|B|}.$$

Assign a labeling $l(v)$ to the vertices of A (and B) given by their distance from a (or b), respectively, along A (or B). Define a *crossing pair* to be a pair of edges uy

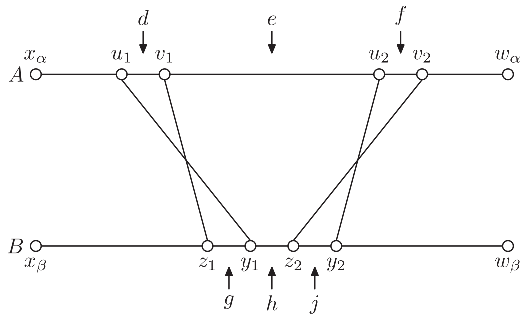

FIGURE 2. Swapping.

and vz with $u,v\in A$ and $y,z\in B$ such that $l(u)<l(v)$ and $l(z)<l(y)$. We call the vertices $u,v\in A$ of a crossing pair the *base* of the pair and $y,z\in B$ the *terminal vertices* of the pair. Define the *gap* of a crossing pair to be $l(y)-l(z)>0$. We will concern ourselves only with crossing pairs with gap length at most $4/c$.

Consider Figure 2 consisting of two crossing pairs u_1y_1, v_1z_1 and u_2y_2, v_2z_2 where $d=l(v_1)-l(u_1)>0$, $e=l(u_2)-l(v_1)>0$, $f=l(v_2)-l(u_2)>0$, $g=l(y_1)-l(z_1)>0$, $h=l(z_2)-l(y_1)\geq 0$, and $j=l(y_2)-l(z_2)>0$. The goal of this lemma is to find two such crossing pairs with

$$g+h+j\leq e\leq c'n-(g+h+j) \tag{1}$$

and

$$d+f+g+j\leq \frac{64}{c^2}. \tag{2}$$

Within this structure, the new paths $A^*=a,\ldots,u_1,y_1,\ldots,z_2,v_2,\ldots,a'$ and $B^*=b,\ldots,z_1,v_1,\ldots,u_2,y_2,\ldots,b'$ yield the desired pair of paths.

Partition the vertices of A' into collections of $\lceil 4/c\rceil$ consecutive (within A' but not necessarily consecutive within A) vertices. Notice there are at least $c^2n/8$ such collections. Call each such collection a *chunk*.

Claim 1. *Given a chunk C, there are at least $c|B|/2$ crossing pairs based in C, whose corresponding pairs of terminal vertices are pairwise disjoint in B, with gap length at most $4/c$.*

Proof of Claim 1. Given a crossing pair, removing all edges incident to the terminal vertices of this pair decreases $\delta_B(C)$ by at most 2. Using the fact that $\delta_B(C)\geq cn^2/(2|A|)\geq 2c|B|$, it suffices to prove that as long as $\delta_B(C)\geq c|B|$, there exists a crossing pair with gap length at most $4/c$. This would imply that there exist at least $c|B|/2$ crossing pairs with gap length at most $4/c$. Since all edges from the terminal pairs are removed as the pairs are being chosen, the pairs are vertex disjoint within B.

Suppose Claim 1 is not true. By the above arguments, we may assume $\delta_B(C)\geq c|B|$ and there exists no crossing pair with gap length at most $4/c$. Index the vertices of C as $v_1,v_2,\ldots$ such that $l(v_i)<l(v_j)$ for all $i<j$. We know $d_B(v_1)\geq c|B|$ but no vertex of $C\setminus v_1$ may be adjacent to any vertex that is the immediate predecessor of an adjacency

of v_1. Therefore, there are at least $c|B|-1$ vertices of B to which none of the vertices of $C \setminus v_1$ may be adjacent.

For any i and j with $i<j$, $d_B(v_i) \geq c|B|$, but we claim that v_i can share at most $|B|/(4/c) = c|B|/4$ neighbors with v_j. If v_i and v_j share $c|B|/4+1$ neighbors in B, then two such neighbors u_1 and u_2 must have $dist_B(u_1,u_2) \leq 4/c$. Hence, $\{v_i,u_1,v_j,u_2\}$ form a crossing pair with gap length at most $4/c$ which is a contradiction.

Therefore, v_i has at least $c|B|-c|B|/4 = 3c|B|/4$ unshared adjacencies. This implies that each vertex v_i of C (aside from v_1) forces at least $3c|B|/4-1$ vertices of B to have no adjacencies in $C \setminus \{v_1,v_2,\ldots,v_i\}$.

Recall that $n \geq f(c,c') \geq 16/c^3$ and $|B| \geq cn \geq 16/c^2$. After considering $\lceil 4/(3c) \rceil$ vertices of C, there remain at most

$$|B|-(c|B|-1)-\left(\frac{4}{3c}-1\right)\left(\frac{3c|B|}{4}-1\right) \leq 0 \tag{3}$$

vertices of B remaining to which a vertex of C may be adjacent. Because $|C| = 4/c > 4/(3c)$, this is a contradiction and completes the proof of Claim 1. ■

Given two crossing pairs u_1y_1, v_1z_1 and u_2y_2, v_2z_2, we say these pairs form a *swapping structure* if $l(u_i) > l(v_j)$ and $l(z_i) > l(y_j)$ for some choice of $i,j \in \{1,2\}$. For this choice of i and j, define the *gap* of the swapping structure to be $l(z_i)-l(y_j)$ (i.e. the distance in B between the vertices of the crossing pairs).

Claim 2. *Any collection of* $\lceil 8/c \rceil$ *chunks contains a swapping structure with gap* h *for some* $0 \leq h \leq 16/c^2$.

Proof of Claim 2. This claim employs a proof almost identical to that of Claim 1. A chunk C is said to be to the *right* of another chunk C' if the maximum index of a vertex in C' is smaller than the minimum index of a vertex in C.

First consider the left-most chunk R^*. The chunk R^* has at least $c|B|/2$ crossing pairs and consequently at least $c|B|/2$ "y"-vertices of these crossing pairs. It follows that at least $c|B|/2-1$ vertices of B cannot be "z"-vertices of any crossing pairs from other chunks, since this would create a swapping structure.

Suppose some number of chunks have been considered and, as before, consider the left-most chunk R of the remaining collection. For every segment of length $16/c^2$ in B, we claim that R may share at most $4/c$ "y"-vertices with chunks to its right. Otherwise, there exists a desired swapping structure within such a segment.

Therefore, this chunk may share a total of at most $(4/c)(|B|/(16/c^2)) = c|B|/4$ "y"-vertices with chunks to its right. This implies that at least $c|B|/4$ "y"-vertices are unshared.

Recall that $n \geq f(c,c') \geq 64/(c'c^2)$, so $|B| \geq cn \geq 64/(c'c)$. Hence, after considering $4/c$ chunks, there are

$$|B|-\left(\frac{c|B|}{2}-1\right)-\left(\frac{4}{c}-1\right)\left(\frac{c|B|}{4}\right) \leq 0 \tag{4}$$

vertices available in B for "z"-vertices of crossing pairs, which is again a contradiction completing the proof of Claim 2. ■

Given a chunk C, define the *span* of C to be the number of vertices $v \in A$ with $l(v_1) \leq l(v) \leq l(v_2)$ for some $v_1, v_2 \in C$. Since the chunks are subsets of A' of order $4/c$

and $|A'| \geq (c/2)|A|$, the average span of the chunks is at most $8/c^2$. Recall that there are at least $(c^2/2)|A|$ chunks. From this, we see that the number of chunks of span at most $16/c^2$ is at least $(c^2/16)|A|$. We call a chunk *good* if its span is at most $16/c^2$. Since there are many good chunks, we will only consider good chunks for the remainder of this proof.

Our goal is to mark good chunks that are a particular distance apart within A. Start at the beginning of A (in terms of the original labeling) and mark the first good chunk. Since $(g+h+j) \leq [2(4/c)+16/c^2]$ (recall our goal is inequality (1)), we skip the next $32(c+1)/c^2 > [2(4/c)+16/c^2]$ vertices of A. This bounds the distance between marked chunks. We then mark the next (complete) good chunk and repeat this process until we have crossed the entire length of A.

Since at most $8/c+9$ chunks may intersect each skipped segment, there are at least $c^2|A|/(16((8/c)+9))$ marked chunks. Consider any segment of $c'n/2$ consecutive vertices of A. The average number of marked chunks that we see in such a segment is $(c^2c'/(32((8/c)+9)))n$; so if n is sufficiently large, there must exist a segment containing at least $8/c$ marked chunks.

By Claim 2, there exists a swapping structure within these marked chunks. This is the desired swapping structure (see inequality (1)) since, using the notation from Figure 2, we have shown that $d,f,g,j \leq 4/c$ and $h \leq 16/c^2$ in Claims 1 and 2. Our bounds on e give us

$$\begin{aligned} e &\geq \left\lceil \frac{32(c+1)}{c^2} \right\rceil \\ &\geq (g+h+j) \end{aligned}$$

and

$$\begin{aligned} e &\leq \frac{c'n}{2} \\ &\leq c'n - \left(\frac{16}{c^2}\right) - 2\left(\frac{4}{c}\right) \\ &\leq c'n - (g+h+j) \end{aligned}$$

for n sufficiently large. Also we have shown that $d+f+g+j \leq 64/c^2$ which proves inequality (2). This completes the proof of Lemma 2. ■

If there exists a partition of G into two sets with very few edges from one set to the other, the following lemma constructs the desired hamiltonian cycle directly.

Lemma 3 (Rebuilding). *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \varepsilon < \min\{\gamma_i/2\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq (n+t-1)/2$ or $\delta(G) \geq n/2$ and $\kappa(G) \geq 3t/2$. If there exists a partition of $V(G)$ into sets A and B with $|A|, |B| \geq \varepsilon n$ and $e(A,B) < \varepsilon^2 n^2/1600$, then for every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in the given order such that $(\gamma_i - \varepsilon)n \leq dist_H(x_i, x_{i+1}) \leq (\gamma_i + \varepsilon)n$ for all $1 \leq i \leq t$.*

Proof. Let D_A (or D_B) be the set of vertices in A (respectively B) with each vertex of D_A (or D_B) having more than $(\varepsilon/40)n$ edges into B (respectively A) and let $D = D_A \cup D_B$. From the hypotheses of the lemma, $|D_A|, |D_B| < (\varepsilon/40)n$.

Claim 1. *For any set $A' \subseteq (A \setminus D_A)$ with $|A'| \geq (\varepsilon/5)/n + 2$, $G[A']$ is panconnected. For any set $B' \subseteq (B \setminus B_D)$ with $|B'| \geq (\varepsilon/5)n + 2$, $G[B']$ is panconnected.*

Proof of Claim 1. Since $\delta(G[A \setminus D_A]) \geq n/2 - (\varepsilon/40)n - (\varepsilon/40)n = n/2 - (\varepsilon/20)n$ and, by symmetry, $\delta(G[B \setminus D_B]) \geq n/2 - (\varepsilon/20)n$, we see that $|A|, |B| \leq n/2 + (\varepsilon/20)n$. Hence:

$$\begin{aligned}
\delta(G[A \setminus D_A]) &\geq \frac{n}{2} - \frac{\varepsilon}{20}n \\
&= \frac{|A| + |B|}{2} - \frac{\varepsilon}{20}n \\
&\geq \frac{|A| + |A| - \frac{\varepsilon}{10}n}{2} - \frac{\varepsilon}{20}n \\
&= |A| - \frac{\varepsilon}{10}n \\
&\geq |A \setminus D_A| - \frac{\varepsilon}{10}n.
\end{aligned}$$

Thus given $A' \subseteq A \setminus D_A$ with $|A'| \geq (\varepsilon/5)n + 2$,

$$\begin{aligned}
\delta(A') &\geq |A'| - \frac{\varepsilon}{10}n \\
&\geq \left(\frac{|A| + 2}{2} + \frac{\varepsilon}{10}n\right) - \frac{\varepsilon}{10}n \\
&\geq \frac{|A| + 2}{2}.
\end{aligned}$$

Hence, by Theorem 3, we know $G[A']$ is panconnected. By symmetry, $G[B'] \subseteq (B \setminus D_B)$ is also panconnected. This completes the proof of Claim 1. ■

The proof of this lemma is divided into cases based on the connectivity.

Case 1. *Suppose $\kappa(G) \geq 5t$.*

Choose a system $X' = \{u_1, v_1, u_2, v_2, \ldots, u_t, v_t\}$ of two distinct representatives for each of the vertices of X with $x_i u_i, x_i v_i \in E(G)$ for all i such that $X' \subseteq G \setminus (X \cup D)$. By our degree conditions, there exists such a set X'. Since G is $5t$-connected, we know there exists a set of $2t$ vertex disjoint paths from $A \setminus D$ to $B \setminus D$ in $G \setminus (X \cup X')$. Let M be the collection of shortest such paths (see Fig. 3).

Suppose we have constructed paths $P_1, \ldots, P_{i-1}$ for some $1 \leq i < t$ where $P_j = P[v_j, \ldots, u_{j+1}]$ for $j < i$. Further suppose $|P_j| = \lfloor \gamma_j n \rfloor$. Let $v_i, u_{i+1} \in X'$ and, without loss of generality, suppose $v_i \in A$. Let $Q_i = V(P_1) \cup \cdots \cup V(P_{i-1})$ and let $A' = [A \setminus (D \cup X \cup X' \cup M \cup Q_i)] \cup \{v_i, u\}$ for some $u \in A \cap M \setminus (D \cup Q_i)$. If $\gamma_i n \leq |A'| - ((\varepsilon/3)n + 3t + 2) - (2t - i + 1)$,

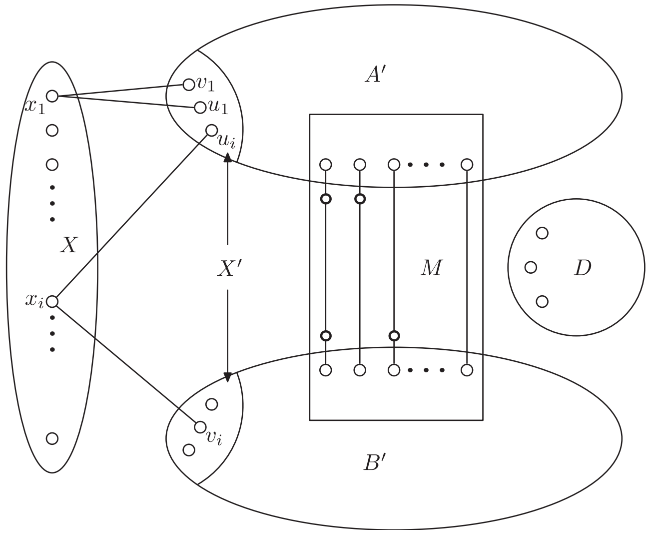

FIGURE 3. Graph G.

we use the fact that $G[A']$ is panconnected to construct a path P_i' of order $\gamma_i n - \lambda_i$ in A' from v_i to u where $\lambda_i = dist(u, u_{i+1})$.

By Claim 1, since $|A' \setminus P_i'| \geq (\varepsilon/3)n+2 > (\varepsilon/5)n+2$, it follows that $G[A' \setminus P_i']$ is panconnected. Construct the path P_i by using P_i' and the path of length λ_i from u to u_{i+1}. Note that this uses at most 2 vertices from A' or B'; so $|A' \setminus P_i| \geq (\varepsilon/3)n$ which means the resulting graph is still panconnected by Claim 1.

If $\gamma_i n > |A'| - ((\varepsilon/3)n + 3t + 2) - (2t - i + 1)$, we again use the fact that $G[A']$ is panconnected to create a path of length 2 from v_i to u. Let v be the vertex of $M \cap B$ such that $u, \ldots, v$ is a path of M. First suppose $\gamma_i n \leq |B'| - ((\varepsilon/3)n + 3t + 2) - (t - i)$, where $B' = [B \setminus (D \cup X \cup X' \cup M \cup Q_i)] \cup v$. We take the path of length 2 from v_i to u followed by the path from u to B through M, and finally, using the panconnectivity of $G[B']$, complete our path with a path of the desired length within B'.

Again breaking this into cases as above, based on whether u_{i+1} is in A or B, construct P_i. This process may involve crossing from A to B and back (using M) at most twice per path; so since M originally contains $2t$ paths, the construction process will never run out of paths.

If $\gamma_i n > |B'| - ((\varepsilon/3)n + 3t + 2) - (t - i)$, we mark v_i as reserved and construct the associated path later. This reservation of vertices happens at most twice. Note that if we had reserved at least three vertices, then since $\varepsilon < \min\{\gamma_i n/2\}$, we could have constructed one of the reserved paths as before.

Suppose, without loss of generality, that v_t is the single remaining vertex in X' (whether it was reserved or not) and $v_t \in A$ and let u be a remaining vertex of $A \cap M \setminus (D \cup Q_t)$. If $u_1 \in B$, then use the panconnectivity of $G[A']$ to connect v_t to u using all of A', take the path in M from u to B and use the panconnectivity of B' to cover B'. This creates a path of order l_t for $\gamma_t n \geq l_t > \gamma_t n - |D| - |M| > (\gamma_t - (\varepsilon/2))n$ as long as n is sufficiently large. If $u_1 \in A$, we take a path $v_t, \ldots, u$ of length 2, cover all of B' on a path between two vertices of M, come back to A and cover A' to again construct the desired path (see Fig. 4).

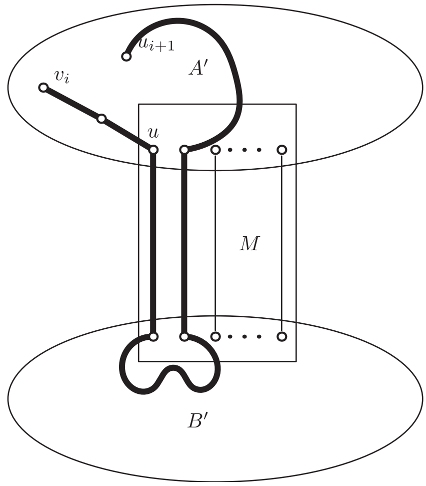

FIGURE 4. Path construction.

Finally, suppose v_{t-1} and v_t are the two reserved vertices of X'. One may show that $|A'|, |B'|, \lceil\gamma_{t-1}n\rceil$, and $\lceil\gamma_t n\rceil$ are all within $(\varepsilon/2)n$ of each other. As above, we create short paths from v_{t-1} and u_t to vertices $v'_{t-1}, u'_t \in A$ and from v_t and u_1 to vertices $v'_t, u'_1 \in B$. We now use the panconnectivity of $G[A']$ and $G[B']$ to construct a path $P'_{t-1} = v'_{t-1}, \ldots, u'_t$ of length $|A'|$ and a path $P'_t = v'_t, \ldots, u'_1$ of length $|B'|$. We then let $P_i = v_i, \ldots, v'_i, \ldots, u'_{i+1}, \ldots, u_{i+1}$ for $i = t-1, t$. Because $|A'|, |B'|, \lceil\gamma_{t-1}n\rceil$, and $\lceil\gamma_t n\rceil$ are all within $(\varepsilon/2)n$ of each other, one may easily check that these paths are of length l_i with:

$$\left(\gamma_i - \frac{\varepsilon}{3}\right)n + 3t + 2 - |D_A| < l_i < \left(\gamma_i + \frac{\varepsilon}{3}\right)n + 3t + 2,$$

so we get:

$$\left(\gamma_i - \frac{\varepsilon}{2}\right)n < l_i < \left(\gamma_i + \frac{\varepsilon}{2}\right)n.$$

Notice, in this process, we can miss at most $|D_A| + |D_B| + (\varepsilon/5)n + 2 + 6t < (\varepsilon/2)n$ vertices for n sufficiently large. Applying Lemma 1, the desired hamiltonian cycle results.

Case 2. *Suppose* $(3t/2) \leq \kappa(G) < 5t$.

Let K be a minimum cutset of G with $3t/2 \leq |K| < 5t$. Since $\delta(G) \geq n/2$, there cannot be more than two components of $G \backslash K$. Call these components A and B.

We call a vertex $v \in K$ *blocked to* A (or B) if for every edge e from v into A (respectively, B), $e = vx_i$ for some $x_i \in X$. For each vertex $v \in K \backslash X$ which is blocked to A, we choose a distinct vertex $x_i \in N(v) \cap X \cap A$. Call this the *blocking vertex*. We call the vertices of $K \cap X$ with only one edge to either $A \backslash X$ (or $B \backslash X$) *half-blocked* to A (or B).

For $v \in K \backslash X$ which is blocked by a vertex $x_i \in A \cap X$, remove all edges to $A \cap X \backslash x_i$ and move v to B and move x_i to K. By the choice of these removed edges, the connectivity will not be affected. We have now eliminated all the blocked vertices of $K \backslash X$ and possibly created more half-blocked vertices.

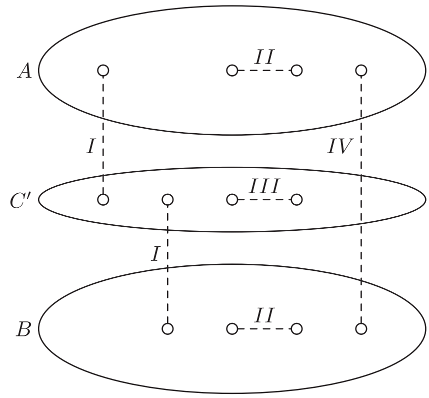

FIGURE 5. Types of paths.

We next remove all edges between vertices of X to create a new graph G'. Note that these edges are useless in the construction of the desired cycle. Let K' be a minimum cutset in G' containing the maximum number of vertices of X and observe that we have the following facts about G':

- There are no blocked vertices in K'.
- $\kappa(G') \geq \kappa(G) - (t/2) \geq t$.
- No half-blocked vertices could also have been blocked or blocking.

For the sake of notation, we distinguish four different types of paths that we would like to construct. A path P_i from x_i to x_{i+1} is of Type I if $x_i \in K'$ and $x_{i+1} \notin K'$ or $x_i \notin K'$ and $x_{i+1} \in K'$. A path P_i is of Type II if $x_i, x_{i+1} \in A$ or both are in B. A path P_i is of Type III if $x_i, x_{i+1} \in K'$. Finally, a path P_i is of Type IV if $x_i \in A$ and $x_{i+1} \in B$ or $x_i \in B$ and $x_{i+1} \in A$. See Figure 5.

Since $\delta(G) \geq n/2$ and $|K'| < 5t$, we know $n/2 - 5t \leq |A|, |B| \leq n/2 + 5t$ and, by Claim 1, $G[A]$ and $G[B]$ are panconnected. Using the same argument as in the previous case, as long as there are enough paths from A to B, we may construct all paths as desired. If $\kappa(G') > t$, the reader may verify that there are enough paths from A to B to complete the above argument. If $\kappa(G') = t$, we know every vertex of X was either blocked or blocking. This implies that all the paths are of Types *II* or *IV*. By tedious case analysis, the paths may be constructed as above to get the desired hamiltonian cycle.

Case 3. *Suppose* $\kappa(G) < 3t/2$.

Let $k = k_a + k_b$ where k_a is the number of blockings or half-blockings into A and likewise k_b for B. From the previous case, we know that if $\kappa(G) \geq t + 1 + k$ then we may construct the paths to get the desired hamiltonian cycle. Consider a vertex $v \in A$ and a vertex $w \in B$ which are not involved in any half-blocking. The vertex v is adjacent to at most $\kappa(G) - k_a$ vertices of K and w is adjacent to at most $\kappa(G) - k_b$ vertices of K. Therefore, $|A| \geq d(v) + 1 - (\kappa(G) - k_a) \geq (n + t + 1)/2 - \kappa(G) + k_a$ and similarly $|B| \geq (n + t + 1)/2 - \kappa(G) + k_b$. Hence, $n = |A| + |B| + \kappa(G) \geq n + t + 1 - \kappa(G) + k_a + k_b$ or $\kappa(G) \geq t + k_a + k_b + 1$ and we have our result.

This completes the proof of Lemma 3. ■

Our final lemma provides some structure similar to that in Theorem 1 but with the chosen vertices in a given order on the hamiltonian cycle.

Lemma 4 (Setup). *Let $t \geq 3$ be an integer and for sufficiently large n, let G be a graph of order n having $\delta(G) \geq (n+t-1)/2$ or $\delta(G) \geq n/2$ and $\kappa(G) \geq 3t/2$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in order such that $dist_H(x_i, x_{i+1}) \geq (1/(6400t^3(1-1/(2t))))n$ for all $1 \leq i \leq t$.*

Proof. Let n be sufficiently large and G be as stated. Let $\{x_1, \ldots, x_t\} \subseteq V(G)$ and let $\varepsilon = 1/(2t)$. If there exists a partition of $V(G)$ into two sets A and B, having $|A|, |B| \geq \varepsilon n$ such that $e(A,B) < (\varepsilon^2/1600)n^2$, then we may apply Lemma 3 to get the desired hamiltonian cycle. Subsequently, we need to only show how to proceed if such a partition does not exist.

Claim 1. *Suppose we are given a graph G of sufficiently large order n with $\delta(G) \geq n/2$ and a real number $\varepsilon > 0$. If, for every partition of $V(G)$ into two sets A and B with $|A|, |B| \geq \varepsilon n$, we have $e(A,B) \geq (\varepsilon^2/1600)n^2$, then $\kappa(G) \geq (\varepsilon^2/1600(1-\varepsilon))n$.*

Proof of Claim 1. We proceed by contradiction. Let K be a cutset of order less than $(\varepsilon^2/(1600(1-\varepsilon))n$. Since $\delta(G) \geq n/2$, we know there are only two components (call them A and B) of $G \setminus K$ and since $|K| < \varepsilon^2/(1600(1-\varepsilon))n$ and $\delta(G) \geq n/2$, we know $|A|, |B| > \varepsilon n$. Let $A' = A \cup K$. By assumption, $e(A', B) \geq (\varepsilon^2/1600)n^2$ and all these edges must be incident to vertices in K. Therefore, there exists a vertex $v \in K$ such that,

$$d_B(v) \geq \frac{\frac{\varepsilon^2 n^2}{1600}}{\frac{\varepsilon^2 n}{1600(1-\varepsilon)}} = (1-\varepsilon)n.$$

However, since $|A| > \varepsilon n$, this is a contradiction, completing the proof of Claim 1. ■

Since $\delta(G) \geq n/2$, we may choose a system X' of two distinct representatives from the neighborhood of each vertex of X. Also since $\delta(G) \geq n/2$, we wish to create a collection $\mathscr{P}$ of $2t$ vertex disjoint paths in $G \setminus X$ two of which start at each vertex of X. Further, we wish each path to have length $\lfloor \varepsilon^2 n/(1600(1-\varepsilon)2t) \rfloor - 6$ which ensures that $\kappa(G \setminus V(P)) \geq 10t$. Note, these paths are easily constructed using a greedy approach within the neighborhood of the end vertex of the path under construction.

Let $P = \cup V(P_i)$ for $P_i \in \mathscr{P}$. By Theorem 2, we know $G \setminus (P \cup X)$ is t-linked. This implies that we may link, using only vertices of $G \setminus (P \cup X)$, the ends of the paths of P to create a cycle of length at least $(\varepsilon^2/(1600(1-\varepsilon)))n - 11t$ containing the vertices of X in the given order.

Choose a longest cycle H having $dist_H(x_i, x_j) \geq \varepsilon^2 n/(1600(1-\varepsilon)t) - 11$ for $i \neq j \leq t$. We may assume $|H| < 3n/4 + t$; otherwise applying Lemma 1, the desired hamiltonian cycle results.

First suppose $|H| \leq (n+t-1)/2$. This implies $d_{G \setminus H}(v) \geq 1$ for all $v \in H$. Also any vertex of $G \setminus H$ may not be adjacent to consecutive vertices of H, so $\delta(G[G \setminus H]) \geq (n+t-1-|H|)/2 > |G \setminus H|/2$. By Dirac's Theorem [4], this implies $G[G \setminus H]$ is hamiltonian connected. At this point, we simply choose two consecutive vertices $v, v^+ \in V(H)$ and neighbors of these vertices $u, u' \in G \setminus H$. Now create H' from H by removing the edge vv^+ and inserting the path $v,u,\ldots,u',v^+$ using all of $G\backslash H$. Notice $|H'|>|H|$, which contradicts the choice of H.

Now suppose $(n+t-1)/2<|H|<3n/4+t$. If $J=G\backslash H$, then $n/4-t<|J|<(n-t+1)/2$. By assumption, $e(H,J)\geq\varepsilon^2n^2/1600$; hence, it follows that there exists a path $P_i=(x_i,x_{i+1})_H$ such that $e(P_i,J)\geq\varepsilon^2n^2/(1600t)$. Consequently, there are at least $\varepsilon^2n^2/(1600t|J|)-1\geq\varepsilon^2n/(800t)-1$ vertices $v\in P_i$ with $d_J(v)\geq 2$. Since $|P_i|<3n/4$, the average distance between vertices v such that $d_J(v)\geq 2$ is at most:

$$\frac{|P_i|}{\frac{\varepsilon^2 n}{800t}-1}<\frac{300t}{\varepsilon^2}<\frac{n}{4}-t<|J|,$$

if n is sufficiently large. Therefore, there exist two vertices $u,v\in P_i$ with $dist_{P_i}(u,v)<|J|$ such that $d_J(u),d_J(v)\geq 2$.

Recall that no vertex of J may be adjacent to consecutive vertices of H; so $\delta(J)\geq(n+t-1)/2-|H|/2\geq(|J|+2)/2$; so, by Theorem 3, J is panconnected. Let $u'\in J\cap N(u)$ and let $v'\in J\cap N(v)\backslash\{u'\}$. There exists a hamiltonian path P of J from u' to v'. We now replace $P_i=x_i,\ldots,u,\ldots,v,\ldots,x_{i+1}$ with the path $P_i'=x_i,\ldots,u,u',P,v',v,\ldots,x_{i+1}$. Because $|J|>dist_H(u,v)$, it follows that $|P_i'|>|P_i|$ contradicting the choice of H and completing the proof of Lemma 4. ■

4. PROOFS OF THE MAIN RESULTS

Theorem 4. *Let $t\geq 3$ be an integer and let $0<\varepsilon<1/(2t)$. For $n\geq 7t^6\times 10^{10}/\varepsilon^6$, let G be a graph of order n having $\delta(G)\geq n/2$ and $\kappa(G)\geq 2\lceil t/2\rceil$. For every $X=\{x_1,x_2,\ldots,x_t\}\subseteq V(G)$, there exists a hamiltonian cycle H such that $dist_H(x_i,x_j)\geq(1/t-\varepsilon)n$ for all $1\leq i<j\leq t$.*

Proof. By Theorem 1, we know there exists a hamiltonian cycle H in G such that, for a given set of t vertices $X=\{x_1,\ldots,x_t\}$, $dist_H(x_i,x_j)\geq n/(2t)$ for all $i\neq j$. Let $\mathscr{H}$ be the set of hamiltonian cycles which satisfy Theorem 1.

For each H in $\mathscr{H}$, let $\mathscr{P}$ be the set of path segments of H between the vertices of X including both end vertices. Order the paths $P_i\in\mathscr{P}$ from shortest to longest and let $k=\min\{i:|P_i|\leq|P_{i+1}|-(\varepsilon/t^2)n\}$. If no such k exists, one may easily verify that H is the desired hamiltonian cycle. Let I be the set of indices of paths of order less than $n/t-\varepsilon n$ and define $\mu(H)=\sum_{i\in I}t^{n/t-\varepsilon n-|P_i|}$.

Let H be the graph in $\mathscr{H}$ with $\mu(H)$ minimum. If $I=\emptyset$, this cycle H is the desired hamiltonian cycle; so suppose not. Partition the path segments into $\mathscr{B}=\bigcup_{i=1}^k P_i$ (where k is defined above) and $\mathscr{A}=\mathscr{P}\backslash\mathscr{B}$. If the number of edges between $\mathscr{A}$ and $\mathscr{B}$ is at least $(\varepsilon^2/1600)n^2$, there must exist a pair of paths $A\in\mathscr{A}$ and $B\in\mathscr{B}$ with $e(A,B)\geq(\varepsilon^2/1600t^2)n^2$. First applying Lemma 2 with $\gamma_i=1/t$ for all i, $c=\varepsilon^2/(1600t^2)$ and $c'=\varepsilon n/(2t^2)$, and then Lemma 1 to reabsorb any lost vertices yields a hamiltonian cycle $H'\in\mathscr{H}$ with $\mu(H')<\mu(H)$, which is a contradiction. Note that the lower bound on the value of n comes from the application of Lemma 2.

Suppose the number of edges between $\mathscr{A}$ and $\mathscr{B}$ is less than $(\varepsilon^2/1600)n^2$. Let K be a minimum cutset of G. If $|K|\geq 3t/2$, then we may apply Lemma 3 to complete the

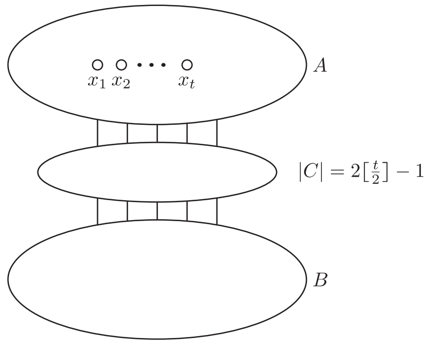

FIGURE 6. The graph G_1.

proof; so assume $|K|<3t/2$. By the minimum degree condition, there can only be two components A and B of $G\setminus K$; furthermore, $|K|\geq 2\lceil t/2\rceil$. For every vertex $x_i\notin K$, make a short path (by the degree condition, this path has length at most 2) to a vertex v_i of K and contract the path to a new vertex $x_i\in K$. Notice we have only removed at most $2t$ vertices and since $|K|<3t/2$, A and B are very dense; so we have not decreased the connectivity of G below $2\lceil t/2\rceil$. If t is even, we may connect x_1 to x_2 through A, x_2 to x_3 through B, and so on to get a hamiltonian cycle with all vertices essentially equally spaced.

If t is odd, there exists at least one vertex $v\in K\setminus X$ and we may again connect all but one of the paths as above. There will be one path $P_t=(x_t,x_1)$ remaining that cannot fit into only one of A or B. For this path, we must use the vertex v to cross between A and B to complete the desired hamiltonian cycle.

To see the sharpness of the minimum degree condition, consider the graph G_0 consisting of two cliques of order $(n+1)/2$ sharing a common vertex. The graph G has $\delta(G_0)\geq n/2-1$ and G_0 is not hamiltonian.

To see the sharpness of the connectivity condition, consider the graph G_1 in Figure 6. This graph consists of two sets $A=B=K_{(n-t)/2}$ and an A,B separating set C with $|C|=2\lceil t/2\rceil-1$ (for n of the correct parity). Notice $|C|$ is always odd and $|C|\leq t$. If all of the vertices X are in A (or B), we would need at least $\lceil t/2\rceil$ of our path segments to cross into B and hence use at least $2\lceil t/2\rceil$ vertices of $C\setminus X$ but $|C|=2\lceil t/2\rceil-1$; so it is impossible to construct the desired hamiltonian cycle.

This completes the proof of Theorem 4. ∎

Theorem 6. *Let $t\geq 3$ be an integer and $\varepsilon,\gamma_1,\gamma_2,\ldots,\gamma_t$ positive real numbers having $\sum_{i=1}^{t}\gamma_i=1$ and $0<\varepsilon<\min\{\gamma_i^2/2\}$. For $n\geq 7t^{12}\times 10^{10}/\varepsilon^6$, let G be a graph of order n having $\delta(G)\geq(n+t-1)/2$ or $\delta(G)\geq n/2$ and $\kappa(G)\geq 3t/2$. For every $X=\{x_1,x_2,\ldots,x_t\}\subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in order such that $(\gamma_i-\varepsilon)n\leq dist_H(x_i,x_{i+1})\leq(\gamma_i+\varepsilon)n$ for all $1\leq i\leq t$.*

Proof. By Lemma 4, there exists a hamiltonian cycle H with the vertices of X in the given order and $dist_H(x_i,x_{i+1})\geq\varepsilon n$ for all $x_i\in X$. Applying Lemmas 2 and 3 as in the previous proof, the desired result follows. Note that the lower bound on the value of n comes from the application of Lemma 2.

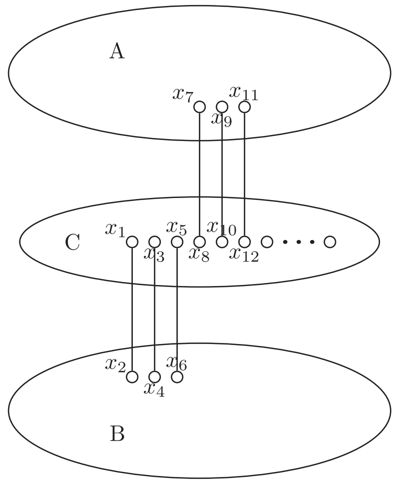

FIGURE 7. The graph G_2.

Again the sharpness of the minimum degree bound follows by considering the graph G_0 used in Theorem 4. To see the sharpness of the connectivity bound, consider the graph $G_2 = A \cup B \cup C$ where $A = B = K_{(n-3t/2+1)/2}$ and $C = K_{3t/2} - 1$ (see Fig. 7).

Choose a set S_A of $t/4$ vertices in A and a set S_B of $t/4$ vertices in B, as well as two disjoint sets of $t/4$ vertices S_{C_1} and S_{C_2} in C. Now join each vertex of S_A to each vertex of S_{C_2} and each vertex of S_B to each vertex of S_{C_1}. The set S_{C_1} will contain $\{x_1, x_3, \ldots, x_{t/2-1}\}$, S_B will contain $\{x_2, x_4, \ldots, x_{t/2}\}$, S_{C_2} will contain $\{x_{t/2+2}, x_{t/2+4}, \ldots, x_t\}$, and S_A will contain $\{x_{t/2+1}, x_{t/2+3}, \ldots, x_{t-1}\}$ (as shown in Fig. 7).

For each vertex $v \in C$ which does not already have any edges to A (likewise B), all edges are added from v to A (respectively B). Notice, each path we construct must use a vertex of $C \setminus X$; hence, we need $|C| \geq 3t/2$.

This completes the proof of Theorem 6. ■

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their helpful comments and corrections.

REFERENCES

[1] S. Abbasi, Spanning subgraphs of dense graphs and a combinatorial problem on strings, Doctoral dissertation, Rutgers University, New Brunswick, 1998.

[2] G. Chartrand and L. Lesniak, Graphs & Digraphs, 4th edn, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[3] G. Chen, R. J. Faudree, R. J. Gould, M. S. Jacobson, L. Lesniak, and F. Pfender, Linear forests and ordered cycles, Discuss Math Graph Theory 24(3) (2004), 359–372.

[4] G. A. Dirac, Some theorems on abstract graphs, Proc London Math Soc (3) 2 (1952), 69–81.
[5] M. H. El-Zahar, On circuits in graphs, Discrete Math 50(2–3) (1984), 227–230.
[6] A. Kaneko and K. Yoshimoto, On a Hamiltonian cycle in which specified vertices are uniformly distributed, J Combin Theory Ser B 81(1) (2001), 100–109.
[7] R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, Eur J Combin 26(3–4) (2005), 309–324.
[8] J. E. Williamson, Panconnected graphs. II, Period Math Hungar 8(2) (1977), 105–116.