



## An iterative approach to graph irregularity strength

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### ABSTRACT

An assignment of positive integer weights to the edges of a simple graph  $G$  is called irregular if the weighted degrees of the vertices are all different. The irregularity strength,  $s(G)$ , is the maximal edge weight, minimized over all irregular assignments, and is set to infinity if no such assignment is possible. In this paper, we take an iterative approach to calculating the irregularity strength of a graph. In particular, we develop a new algorithm that determines the exact value  $s(T)$  for trees  $T$  in which every two vertices of degree not equal to two are at distance at least eight.

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### 1. Introduction and notation

Let  $w : E(G) \rightarrow \mathbb{N}$  be an assignment of positive integer weights to the edges of a simple graph  $G$ . This assignment yields a weighted degree  $w(v) := \sum_{v \in e} w(e)$  for all vertices  $v \in V(G)$ , and is called irregular if the weighted degrees of the vertices are all different. Let  $I(G)$  denote the set of irregular labelings of  $G$ . Define the irregularity strength  $s(G)$  of a simple graph  $G$  to be

$$\min_{f \in I(G)} \max_{e \in E(G)} f(e) = s(G)$$

if  $I(G)$  is nonempty and  $s(G) = \infty$  otherwise.

Graph irregularity strength was introduced in [4] by Chartrand et al. where, amongst other results, it was shown that  $s(G) < \infty$  if and only if  $G$  contains no isolated edges and at most one isolated vertex. Upper bounds are known for general graphs of order  $n$  (Nierhoff [8] shows the sharp bound  $s(G) \leq n - 1$ ), and  $d$ -regular graphs (Frieze et al. [5] show a bound of  $s(G) \leq c \cdot n/d$  for  $d \leq n^{1/2}$ , and  $s(G) \leq c \cdot n \log n/d$  for general  $d$ , which was recently improved to  $s(G) \leq c \cdot n/d$  for all  $d$  by Przybyło [9]). The exact irregularity strength is known only for very few classes of graphs.

Let  $n_i$  denote the number of vertices of degree  $i$  in a graph  $G$ . Then a simple counting argument shows that

$$s(G) \geq \lambda(G) := \max_k \left[ \frac{1}{k} \sum_{i=1}^k n_i \right].$$

Kinch and Lehel [6] demonstrated, by considering the irregularity strength of  $tP_3$ , that  $\lambda(G)$  and  $s(G)$  may differ asymptotically. It was subsequently conjectured (see [7]) that if  $G$  is connected graph, then  $\lambda(G)$  and  $s(G)$  differ by at most an additive constant.

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It can be shown that for trees,  $\frac{1}{k} \sum_{i=1}^k n_i$  attains its maximum for  $k = 1$  or for  $k = 2$ . Cammack et al. [3] show that  $s(T) = \lambda(T)$  for full  $d$ -ary trees, and Amar and Togni [1] show that  $s(T) = \lambda(T) = n_1$  for all trees with  $n_2 = 0$  and  $n_1 \geq 3$ . For general trees, it is not even the case that  $s(T)$  is within an additive constant of  $n_1$ . Bohman and Kravitz [2] present an infinite sequence of trees with irregularity strength converging to  $\frac{11-\sqrt{5}}{8}n_1 > n_1 > \frac{n_1+n_2}{2}$ .

In this paper, we present an iterative algorithm showing that  $s(T) = \lambda(T)$  for another class of trees, but this time  $n_1 < n_2$ , i.e.  $s(T) = \lceil \frac{n_1+n_2}{2} \rceil$ . We believe that the methods developed here have the potential to be modified and used to determine the irregularity strength of a broader class of trees or more general graphs. The following is the main result of this paper.

**Theorem 1.** *Let  $T$  be a tree in which every two vertices of degree not equal to two are at distance at least 8, and with  $n_1 \geq 3$ . Then  $s(T) = \lambda(T) = \lceil \frac{n_1+n_2}{2} \rceil$ .*

The reader should note that we may obtain  $T$  from a tree containing no vertices of degree 2 by subdividing each edge at least 7 times.

## 2. Proof of Theorem 1

### 2.1. A helpful lemma

Repeated application of the following lemma is at the heart of our algorithm.

**Lemma 2.** *Let  $P = v_0v_1 \cdots v_{\ell+1}$ ,  $\ell \geq 1$  be a path, and let  $w_1, w_2, \dots, w_{\ell}$  be a strictly increasing sequence of natural numbers greater than one, so that all even numbers between  $w_1$  and  $w_{\ell}$  are part of the sequence. Then there exists a weighting  $w$  of the edges of  $P$  such that*

- (1)  $w(v_0)$  is odd,
- (2)  $w(v_i) = w_i$  for  $1 \leq i \leq \ell$ ,
- (3)  $-1 \leq w(v_iv_{i+1}) - w(v_{i-1}v_i) \leq 2$  for  $1 \leq i \leq \ell$ .

**Proof.** We proceed by induction on  $\ell$ . To begin, let  $\ell = 1$ . Depending on  $w_1$ , we assign edge weights as follows:

$w_1$	$w(v_0v_1)$	$w(v_1v_2)$
$4k$	$2k - 1$	$2k + 1$
$4k + 1$	$2k + 1$	$2k$
$4k + 2$	$2k + 1$	$2k + 1$
$4k + 3$	$2k + 1$	$2k + 2$

Assume now that  $\ell \geq 2$  and we are given  $w_1, \dots, w_{\ell}$ . By the induction hypothesis, we can assign weights to the edges of  $P' = v_0v_1 \cdots v_{\ell}$  so that  $w(v_0)$  is odd,  $w(v_i) = w_i$ , and  $-1 \leq w(v_iv_{i+1}) - w(v_{i-1}v_i) \leq 2$  for  $1 \leq i \leq \ell - 1$ .

If  $w_{\ell} - w_{\ell-1} = 1$ , let  $w(v_{\ell}v_{\ell+1}) = w(v_{\ell-2}v_{\ell-1}) + 1$ . Then  $w(v_{\ell}) = w_{\ell}$  and

$$-1 \leq \underbrace{w(v_{\ell}v_{\ell+1}) - w(v_{\ell-1}v_{\ell})}_{=w(v_{\ell-2}v_{\ell-1})-w(v_{\ell-1}v_{\ell})+1} \leq 2.$$

If  $w_{\ell} - w_{\ell-1} = 2$ , and thus  $w_{\ell-1}$  is an even number, let  $w(v_{\ell}v_{\ell+1}) = w(v_{\ell-2}v_{\ell-1}) + 2$ . Then  $w(v_{\ell}) = w_{\ell}$  and

$$0 \leq \underbrace{w(v_{\ell}v_{\ell+1}) - w(v_{\ell-1}v_{\ell})}_{=w(v_{\ell-2}v_{\ell-1})-w(v_{\ell-1}v_{\ell})+2} \leq 2. \quad \square$$

### 2.2. Setting up the weighting

We are given a tree  $T$  in which any two vertices of degree not equal to two are at distance at least 8. We decompose  $E(T)$  into edge disjoint paths such that the end vertices of the paths correspond to the vertices in  $T$  with degree not equal to 2. If one thinks of  $T$  as a subdivision of a tree  $T'$  with  $n_2(T') = 0$ , then each path corresponds to an edge of  $T'$ . A bottom vertex in a path is called a pendant vertex if it is a leaf of  $T$  and we will call any of these paths a *pendant path* if it contains a pendant edge.

We will root  $T$  at a vertex root of maximum degree, giving each path a top-to-bottom orientation. We then order the paths in our decomposition of  $T$  in the following manner. Select any  $d_T(\text{root})$  pendant paths to be the final, or bottom, paths in the ordering. We then order the remaining paths such that any path having bottom vertex  $v$  with  $d_T(v) \geq 3$  will have exactly  $d_T(v) - 2$  pendant paths directly above it in the path ordering. For an example, see Fig. 1.

Let  $P_1, \dots, P_t$  denote the paths under this ordering, where  $P_1$  is the topmost path. We will also allow this path ordering to induce an order on the vertices of the paths, where  $x$  in  $P_i$  is below  $y$  in  $P_j$  if either  $i > j$  or  $i = j$  and  $x$  is below  $y$  on  $P_i$ .

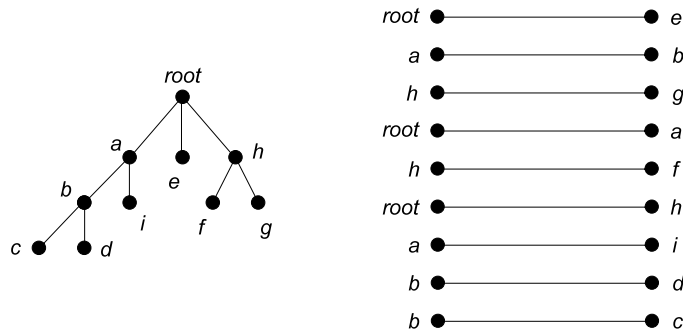


Fig. 1. A tree and its associated path ordering.

Let  $\bar{M}$  be the set consisting of the topmost vertex of degree two from each path, and let  $M$  be the set of all other vertices of degree two. Our initial weighting uses Lemma 2 to assign the weights  $2, \dots, |M| + 1$  to the vertices in  $M$  starting with the bottommost vertex. In applying the lemma, we will require that the lowest internal vertex in each path receive the lowest weight and so on. Finally, for  $\lambda = \lambda(T) = \lceil \frac{n_1+n_2}{2} \rceil$ , we will label the top edge on each path with either  $\lambda$  or  $\lambda - 1$  in a way that assures each vertex in  $\bar{M}$  has odd weight. We call this initial weighting  $w_0$ .

Observe that  $w_0$  is not an irregular weighting, as each pendant vertex  $p$  will have the same weight as some vertex in  $x$  in  $M$  preceding  $p$  in the ordering. Our general approach is to attempt to improve our weighting iteratively, each time utilizing Lemma 2. Let  $H_0$  denote the set of weights of pendant and  $\bar{M}$ -vertices under  $w_0$ . Note that by construction these weights are distinct, and each is odd.

Starting with the bottommost vertex, we assign to each vertex in  $M$  the smallest weight that is neither in  $H_0$  nor assigned to a lower vertex in  $M$ . Given that each pendant vertex and each vertex in  $\bar{M}$  has odd weight, the vertices in  $V(P_i) \cap M$ , from bottom to top, are assigned weights  $w_1 \leq \dots \leq w_\ell$  such that every even integer between  $w_1$  and  $w_\ell$  appears in the list. Consequently, we may use Lemma 2 to assign edge labels resulting in this weighting of the vertices in  $M$ .

Again, we will conclude this new weighting by labeling each top edge of each path with either  $\lambda$  or  $\lambda - 1$  such that the weight of the corresponding vertex in  $\bar{M}$  is odd. We will call this new weighting  $w_1$ . If this is not an irregular weighting with maximum edge weight at most  $\lambda$ , then we will repeat this process by constructing a weighting that avoids the (odd) weights  $H_1$  of the pendant and  $\bar{M}$ -vertices in  $w_1$ , and so on.

Throughout this process, the following facts hold.

**Fact 3.** Let  $m_j, m_k \in M$  such that  $m_j$  is below  $m_k$ . Then  $w_i(m_j) < w_i(m_k)$  for any  $i > 0$ .

The next Fact follows from our assignment of desired weights to the vertices in  $M$ , the fact that each vertex in  $\bar{M}$  receives an odd weight and condition (3) of Lemma 2.

**Fact 4.** The weights of the pendant and  $\bar{M}$ -vertices depend on the weight of their neighbor in  $M$  as follows:

Neighbor	Pendant	Vertex in $\bar{M}$	
		$\lambda$ even	$\lambda$ odd
$4k$	$2k - 1$	$\lambda + 2k - 1$ or $\lambda + 2k + 1$	$\lambda + 2k$
$4k + 1$	$2k + 1$	$\lambda + 2k - 1$ or $\lambda + 2k + 1$	$\lambda + 2k$
$4k + 2$	$2k + 1$	$\lambda + 2k + 1$	$\lambda + 2k$ or $\lambda + 2k + 2$
$4k + 3$	$2k + 1$	$\lambda + 2k + 1$	$\lambda + 2k$ or $\lambda + 2k + 2$

The following three useful lemmas follow from Fact 4 and Lemma 2. The first is concerned with the weight assigned to the pendant vertices in  $T$ .

**Lemma 5.** Let  $p_j$  and  $p_k$  denote the bottom vertices of  $P_j$  and  $P_k$  respectively, where  $j > k$  and  $d_T(p_j) = d_T(p_k) = 1$ . Then for any  $i \geq 0$ ,  $w_i(p_k) - w_i(p_j) \geq 2$ .

**Proof.** Each of the paths in our decomposition have length at least eight and as such, the second lowest and third highest vertices on each path are distance at least five apart. Consequently when applying Lemma 2 these vertices receive weights at least five apart in any iteration. This observation, implies that the weight assigned to the neighbors of  $p_j$  and  $p_k$  differ by more than four. The lemma then follows from Fact 4.  $\square$

The next lemma considers the weights assigned to vertices in  $T$  having degree higher than two. We omit the proof as it is similar to that of Lemma 5, save that it utilizes the fact that any path having bottom vertex  $v$  with  $d_T(v) \geq 3$  will have exactly  $d_T(v) - 2$  pendant paths directly above it in the path ordering.

**Lemma 6.** Let  $p_j$  and  $p_k$  denote the bottom vertices of  $P_j$  and  $P_k$  respectively, where  $j > k$  and  $d_T(p_j) = d_T(p_k) \geq 3$ . Then for any  $i \geq 0$ ,

$$w_i(p_k) - w_i(p_j) \geq 2d_T(p_j) - 2.$$

Finally, Lemma 7 demonstrates that the weights assigned to the vertices in  $\bar{M}$  must differ by at least two. Again we omit the proof as it is similar to that of Lemma 5.

**Lemma 7.** Let  $\{p_j\} = \bar{M} \cap V(P_j)$  and  $\{p_k\} = \bar{M} \cap V(P_k)$ , where  $j > k$ . Then for any  $i \geq 0$ ,  $w_i(p_k) - w_i(p_j) \geq 2$ .

As a corollary of Lemma 6, we get the following statement.

**Lemma 8.** If  $x$  and  $y$  are distinct vertices of degree three or more in  $T$ , then  $w_i(x) \neq w_i(y)$  for all  $i \geq 0$ .

**Proof.** All that is left to show is the inequality for vertices  $x, y$  with  $d_T(x) > d_T(y) \geq 3$ . As root is a vertex of maximum degree in  $T$  and  $x$  must be the bottom vertex on  $P_k$  for some  $k \geq d_T(\text{root}) + 1 > d_T(x)$ , we have

$$w_i(y) \leq \lambda d_T(y) < (\lambda - 1)(d_T(x) - 1) + 3d_T(x) \leq w_i(x). \quad \square$$

Note that  $w_{i+1}$  is completely determined by the set  $H_i \subset \{1, 2, \dots, 2\lambda\}$ . As there are only finitely many such sets, this process will eventually stabilize in a loop with some period  $p$ , i.e.  $H_i = H_{i+p}$  for  $i \geq i_0$  and minimal  $p \geq 1$ . If  $p = 1$ , then  $w_{i_0+1}$  is an irregular weighting with maximum edge weight at most  $\lambda$  and we are done, so we assume in the following that  $p > 1$ .

For  $0 \leq i < p$ , let  $\hat{w}_i = w_{i_0+1+i}$ , where all index calculations regarding  $\hat{w}$  will be modulo  $p$ . Define  $\hat{H}_i$  in a similar manner. Let

$$m = \max_{\substack{i,k \\ x \in M}} \hat{w}_i(x) - \hat{w}_{i-k}(x).$$

The parameter  $m$  is the maximum amount that the weight of an  $M$ -vertex can vary as we iteratively modify the edge labels throughout one period.

The following lemma is crucial for the proof of Theorem 1.

**Lemma 9.**  $m = 1$ .

**Proof.** If  $m = 0$  then  $p = 1$ , which we already excluded, so  $m \geq 1$  and we need to show that  $m \leq 1$ . Let  $x \in M$  be the lowest vertex in the ordering for which we can find  $i$  and  $k$  such that  $\hat{w}_i(x) - \hat{w}_{i-k}(x) = m$ .

The fact that the weight of  $x$  has increased by  $m$  implies that there are  $m$  pendant vertices or  $\bar{M}$ -vertices  $x_1, x_2, \dots, x_m$ , such that for all  $t \geq 0$

$$\hat{w}_t(x_1) \leq \hat{w}_t(x_2) - 2 \leq \dots \leq \hat{w}_t(x_m) - 2m + 2, \tag{1}$$

$$\hat{w}_{i-k-1}(x_1) > \hat{w}_{i-k}(x), \tag{2}$$

and

$$\hat{w}_{i-1}(x_m) < \hat{w}_i(x). \tag{3}$$

Here (1) follows from the fact pendants and  $\bar{M}$  vertices receive distinct odd weights, while (2) and (3) follow from the assumption that the weight of  $x$  has changed by exactly  $m$ . For  $m \leq 3$ , this implies that

$$\begin{aligned} \hat{w}_{i-1}(x_1) &\stackrel{(1)}{\leq} \hat{w}_{i-1}(x_m) - 2m + 2 \\ &\stackrel{(3)}{\leq} \hat{w}_i(x) - 2m + 1 = \hat{w}_{i-k}(x) - m + 1 \\ &\stackrel{(2)}{\leq} \hat{w}_{i-k-1}(x_1) - m. \end{aligned}$$

Both  $\hat{w}_{i-1}(x_1)$  and  $\hat{w}_{i-k-1}(x_1)$  are odd, so in fact

$$\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1) \geq 2 \left\lceil \frac{m}{2} \right\rceil. \tag{4}$$

For  $m \geq 4$ , we get

$$\begin{aligned} \hat{w}_{i-1}(x_1) &\stackrel{(1)}{\leq} \hat{w}_{i-1}(x_m) - 2m \\ &\stackrel{(3)}{\leq} \hat{w}_i(x) - 2m - 1 = \hat{w}_{i-k}(x) - m - 1 \\ &\stackrel{(2)}{\leq} \hat{w}_{i-k-1}(x_1) - m - 2. \end{aligned}$$

Both  $\hat{w}_{i-1}(x_1)$  and  $\hat{w}_{i-k-1}(x_1)$  are odd, so in fact

$$\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1) \geq 2 \left\lceil \frac{m}{2} \right\rceil + 2.$$

Let  $y_1 \in M$  be a neighbor of  $x_1$ , and suppose that  $m \geq 4$ . If  $d_T(x_1) = 1$ , i.e.  $\hat{w}_i(x_1) < \lambda$ . **Fact 4** then yields that

$$\hat{w}_{i-1}(y_1) \leq 2\hat{w}_{i-1}(x_1) + 2$$

and

$$\hat{w}_{i-k-1}(y_1) \geq 2\hat{w}_{i-k-1}(x_1) - 1.$$

Thus,

$$\hat{w}_{i-k-1}(y_1) - \hat{w}_{i-1}(y_1) \geq 2(\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1)) - 3 \geq 4 \left\lceil \frac{m}{2} \right\rceil + 1 > m,$$

a contradiction to (2.2).

Now suppose that  $d_T(x_1) = 2$ . Here, **Fact 4** yields that

$$\hat{w}_{i-1}(y_1) \leq 2(\hat{w}_{i-1}(x_1) - \lambda + 1) + 1$$

and

$$\hat{w}_{i-k-1}(y_1) \geq 2(\hat{w}_{i-k-1}(x_1) - \lambda) - 2.$$

Thus,

$$\hat{w}_{i-k-1}(y_1) - \hat{w}_{i-1}(y_1) \geq 4 \left\lceil \frac{m}{2} \right\rceil - 1 > m,$$

a contradiction. This implies that  $m \leq 3$ .

It then follows that between  $\hat{w}_i$  and  $\hat{w}_{i-k}$ , the weight of a vertex in  $\bar{M}$  can change by at most four, and the weight of a pendant vertex can change by at most two. This is a consequence of **Fact 4**, which shows that if the weight of some pendant vertex were to change by three or more, then the weight of its neighbor in  $M$  would change by at least four. A similar analysis shows that no vertex in  $\bar{M}$  can have its weight change by more than four.

Next assume that  $m = 3$ . From (4), we know that

$$\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1) \geq 4,$$

thus  $x_1, x_2, x_3 \in \bar{M}$ . Further,

$$\hat{w}_{i-k-1}(y_1) - \hat{w}_{i-1}(y_1) \geq 3,$$

as otherwise  $\hat{w}(x_1)$  could not decrease by four. But since  $\hat{w}_{i-1}(x_1) < \hat{w}_{i-1}(x)$ , we know that  $\hat{w}_{i-1}(y_1) < \hat{w}_{i-1}(x)$ , contradicting the choice of  $x$ . This shows that  $m \leq 2$  and in turn that weights of vertices in  $\bar{M}$  can change by at most two, again by **Fact 4**.

Finally assume that  $m = 2$ . All the inequalities in (1) are in fact equalities and we get that for some  $t$ ,

$$\hat{w}_{i-k-1}(x_1) = \hat{w}_{i-1}(x_2) = 2t + 3,$$

$$\hat{w}_{i-k-1}(x_2) = 2t + 5,$$

$$w_{i-1}(x_1) = 2t + 1.$$

Let  $y_2$  be the neighbor of  $x_2$  in  $M$ . If  $x_1$  is a pendant vertex, then **Fact 4** shows that  $\hat{w}_{i-1}(y) \geq 4t + 1$  and

$$\hat{w}_{i-1}(y_2) \leq \hat{w}_{i-k-1}(y_2) + 2 \leq 4t + 6,$$

a contradiction as at least five vertices from  $M$  lie between  $y_1$  and  $y_2$  in the path-ordering, so  $\hat{w}_{i-1}(y_2) - \hat{w}_{i-1}(y_1) \geq 6$ .

Thus,  $x_1 \in \bar{M}$ . As  $y_1$  and  $y_2$  come before  $x$  in the ordering,

$$|\hat{w}_{i-1}(y_1) - \hat{w}_{i-k-1}(y_1)| \leq 1 \quad \text{and} \quad |\hat{w}_{i-1}(y_2) - \hat{w}_{i-k-1}(y_2)| \leq 1.$$

**Fact 4** shows that  $\hat{w}_{i-1}(y_2) - \hat{w}_{i-1}(y_1) \leq 6$  with equality only if

$$|\hat{w}_{i-1}(y_1) - \hat{w}_{i-k-1}(y_1)| = |\hat{w}_{i-1}(y_2) - \hat{w}_{i-k-1}(y_2)| = 1.$$

But this last equality implies that  $H_{i-2}$  contains a number between  $\hat{w}_{i-1}(y_1)$  and  $\hat{w}_{i-1}(y_2)$ , and therefore  $\hat{w}_{i-1}(y_2) - \hat{w}_{i-1}(y_1) \geq 7$ , the final contradiction proving the lemma.  $\square$

The vertices outside of  $M$  are sufficiently far apart in  $T$  to immediately yield the following corollary to **Lemma 9**.

**Lemma 10.** For two vertices  $x, y \notin M$ ,  $\hat{w}_i(x) > \hat{w}_i(y)$  implies  $\hat{w}_j(x) > \hat{w}_k(y)$  for all  $i, j, k$ .

### 2.3. Clean up

We will now modify the weighting  $\hat{w}_1$  to get an irregular weighting  $\hat{w}$  with  $\hat{H} = \hat{H}_1$ . Let  $x \in M$  such that  $\hat{w}_1(x) \in \hat{H}_1$ . By Lemma 9, either  $\hat{w}_2(x) = \hat{w}_1(x) + 1$  or  $\hat{w}_2(x) = \hat{w}_1(x) - 1$ . Let  $y \in M$  be the neighbor of  $x$  with  $\hat{w}_1(y) = \hat{w}_2(x)$ . Note that there is no vertex  $z$  with  $\hat{w}_1(z) = 2\hat{w}_2(x) - \hat{w}_1(x)$  (i.e.  $\hat{w}_1(z) = \hat{w}_1(x) \pm 2$ ), as  $2\hat{w}_2(x) - \hat{w}_1(x) \in \hat{H}_0 \setminus \hat{H}_1$ . We differentiate four cases.

**Case 1.**  $xy \in E$  and  $\hat{w}_2(x) = \hat{w}_1(x) + 1$ .

$$\text{Set } \hat{w}(xy) = \hat{w}_1(xy) + 1.$$

**Case 2.**  $xy \in E$  and  $\hat{w}_2(x) = \hat{w}_1(x) - 1$ .

$$\text{Set } \hat{w}(xy) = \hat{w}_1(xy) - 1.$$

**Case 3.**  $xy \notin E$  and  $\hat{w}_2(x) = \hat{w}_1(x) + 1$ .

Let  $x_1 \in M$  be the neighbor of  $x$  with  $\hat{w}_1(x_1) = \hat{w}_1(x) - 1$ , and  $y_1 \in M$  be the neighbor of  $y$  with  $\hat{w}_1(y_1) = \hat{w}_1(x) + 3$ . Set  $\hat{w}(xx_1) = \hat{w}_1(xx_1) + 3$  and  $\hat{w}(yy_1) = \hat{w}_1(yy_1) - 2$ .

**Case 4.**  $xy \notin E$  and  $\hat{w}_2(x) = \hat{w}_1(x) - 1$ .

Let  $x_1 \in M$  be the neighbor of  $x$  with  $\hat{w}_1(x_1) = \hat{w}_1(x) + 1$ , and  $y_1 \in M$  be the neighbor of  $y$  with  $\hat{w}_1(y_1) = \hat{w}_1(x) - 3$ . Set  $\hat{w}(xx_1) = \hat{w}_1(xx_1) - 3$  and  $\hat{w}(yy_1) = \hat{w}_1(yy_1) + 2$ .

Repeating the above for every  $x \in M$  with  $\hat{w}_1(x) \in \hat{H}_1$  will result in an irregular weighting. Observe that if  $x$  and  $x'$  both fall in Case 3 or 4, then  $|\hat{w}_1(x) - \hat{w}_1(x')| \geq 6$ , and therefore the weight changes used to correct the weighting do not affect each other. For all other cases, Lemma 10 guarantees that the weight changes stemming from different vertices will not interfere. It is easy to check that none of the edge weights in  $\hat{W}$  are below one or above  $\lambda + 1$ .

If there is an edge with  $\hat{w}(xy) = \lambda + 1$ , then  $x$  and  $y$  are the second and third to last vertices of the last path,  $\hat{w}(x) = 2\lambda + 1$  and  $\hat{w}(y) = 2\lambda$ , and there is no vertex  $z$  with  $\hat{w}(z) = 2\lambda - 1$ . Change the weight of  $xy$  to  $\hat{w}'(xy) = \lambda$ , and the resulting weighting  $\hat{W}'$  is irregular and does not use edge weights above  $\lambda$ . This finishes the proof of Theorem 1.

### 3. Conclusions

We have made some new progress towards the problem of determining  $s(T)$  for an arbitrary tree  $T$ . More importantly, however, we have given an explicit algorithm that will generate an irregular weighting for trees in the class under consideration. We are hopeful that this iterative approach will be adaptable to a larger class of trees or more general graphs. For instance, it may be possible to show, via a modification of our algorithm, that there is some absolute constant  $c$  such that if  $T$  is any tree with  $n_2(T) \geq cn_1(T)$ , then  $\lambda(T) = s(T) = \lceil \frac{n_1+n_2}{2} \rceil$ . This would represent marked progress towards the general result.

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