

CHVÁTAL-ERDÖS TYPE THEOREMS

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Abstract

The Chvátal-Erdős theorems imply that if G is a graph of order $n \geq 3$ with $\kappa(G) \geq \alpha(G)$, then G is hamiltonian, and if $\kappa(G) > \alpha(G)$, then G is hamiltonian-connected. We generalize these results by replacing the connectivity and independence number conditions with a weaker minimum degree and independence number condition in the presence of sufficient connectivity. More specifically, it is noted that if G is a graph of order n and $k \geq 2$ is a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - k)/(k + 1)$, and $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian. It is shown that if G is a graph of order n and $k \geq 3$ is a

positive integer such that $\kappa(G) \geq 4k^2 + 1$, $\delta(G) > (n + k^2 - 2k)/k$, and $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected. This result supports the conjecture that if G is a graph of order n and $k \geq 3$ is a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$, and $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected, and the conjecture is verified for $k = 3$ and 4 .

Keywords: Hamiltonian, Hamiltonian-connected, Chvátal-Erdős condition, independence number.

2010 Mathematics Subject Classification: Primary: 05C45; Secondary: 05C35.

1. INTRODUCTION

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. Given a subset of vertices (or subgraph) H of a graph G and a vertex v , let $d_H(v)$ denote the degree of v relative to H , and $N_H(v)$ the neighborhood of v in H . The minimum degree, independence number, and connectivity of G will be denoted by $\delta(G)$, $\alpha(G)$, and $\kappa(G)$ respectively.

Two classical results of Chvátal and Erdős [2] are the following:

Theorem 1. *If G is a graph of order $n \geq 3$ such that $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.*

Theorem 2. *If G is a graph of order $n \geq 3$ such that $\kappa(G) > \alpha(G)$, then G is hamiltonian-connected.*

The following result on the existence of hamiltonian cycles, which is an analogue of Theorem 1, will be proved. Actually, we note that Proposition 1 is an easy consequence of a result of Fraisse [5] and follows from a result of Ota [6] with the appropriate interpretation of the condition on α .

Proposition 1. *Let G be a graph of order n and $k \geq 2$ a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - k - 1)/(k + 1)$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian.*

Corresponding to the hamiltonian result of Theorem 1 and an analogue to Theorem 2, we make the following conjecture.

Conjecture 1. Let G be a graph of order n and $k \geq 3$ a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.

We prove the following two results supporting Conjecture 1. The first result has the same degree and independence number conditions and conclusion as the conjecture, but requires a higher connectivity assumption on the graph. The second result verifies the conjecture for the cases $k = 3$ and 4.

Theorem 3. Let G be a graph of sufficiently large order n and $k \geq 3$ a positive integer such that $\kappa(G) \geq 4k^2 + 1$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.

Theorem 4. Let G be a graph of sufficiently large order n and $k = 3$ or 4 such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.

In Section 2 we will give a short proof of Proposition 1 and present a family of graphs which show that none of the conditions in Proposition 1 and Theorem 4 can be weakened. In Section 3 we prove the main results.

2. PRELIMINARY RESULTS AND SHARPNESS EXAMPLES

We begin by describing graphs H_i for $1 \leq i \leq 5$ which demonstrate the sharpness of the conditions in Proposition 1 and Theorem 4.

For $k \geq 2$, let $H_1(k) = K_k + (k+1)K_{(n-k)/(k+1)}$, where $n \equiv k \pmod{k+1}$. Since there are $k + 1$ components in the graph $H_1(k) - K_k$, the graph $H_1(k)$ is not hamiltonian. Also, $\kappa(H_1(k)) = k$, $\delta(H_1(k)) = (n + k^2 - k - 1)/(k + 1)$, and $\alpha(H_1(k)) = k + 1 \leq \delta(H_1(k))$ for n sufficiently large.

For $k \geq 2$, let $H_2(k) = K_{(n-k)/(k+1)} + ((n + 1)/(k + 1))K_k$ where $n \equiv k \pmod{k+1}$. Since there are strictly more than $(n - k)/(k + 1)$ components in $H_2(k) - K_{(n-k)/(k+1)}$, the graph $H_2(k)$ is not hamiltonian. Also, $\delta(H_2(k)) = (n + k^2 - k - 1)/(k + 1)$, and $\alpha(H_2(k)) = (n + 1)/(k + 1) = \delta(H_2(k)) - (k - 2)$.

For $k \geq 3$, let $H_3(k) = K_k + kK_{(n-k)/k}$, when $n \equiv 0 \pmod{k}$. Since there are as many components in $H_3(k) - K_k$ as there are vertices in K_k , the graph $H_3(k)$ is not hamiltonian-connected. Also, $\kappa(H_3(k)) = k$, $\delta(H_3(k)) = (n + k^2 - 2k)/k$, and $\alpha(H_3(k)) = k \leq \delta(H_3(k))$ for n sufficiently large.

For $k \geq 3$, let $H_4(k) = K_{n/k} + (n/k)K_{k-1}$ where $n \equiv 0 \pmod k$. The graph $H_4(k)$ is not hamiltonian-connected, since there are as many components in $H_4(k) - K_{n/k}$ as there are vertices in $K_{n/k}$. Also, $\delta(H_4(k)) = (n+k^2-2k)/k$, and $\alpha(H_4(k)) = n/k = \delta(H_4(k)) - (k-2)$.

For $(n+1)/3 < \delta(G) < n/2$, the graph $H_5(\delta) = K_\delta + (\overline{K}_\delta \cup K_{n-2\delta})$ is δ -connected, not hamiltonian and not hamiltonian-connected, and $\alpha(H_5(\delta)) = \delta(H_5(\delta)) + 1$.

The graph $H_1(k)$ implies the minimum degree condition of Theorem 1 cannot be decreased for $k \geq 2$. Note that the graph $H_2(k-1)$ satisfies the relationship $\delta(H_2(k-1)) = \alpha(H_2(k-1)) + k - 3$, and all of the other conditions of Theorem 1, so the conditions cannot be decreased in Theorem 1 for $k \geq 3$. The graph $H_5(\delta)$ verifies the sharpness of Theorem 1 when $k = 2$.

The graph $H_3(k)$ implies that the minimum degree conditions of Conjecture 1 and Theorem 4 cannot be decreased. Note that the graph $H_4(k-1)$ satisfies the relationship $\delta(H_4(k-1)) \geq \alpha(H_4(k-1)) + k - 3$ and all of the other conditions of Conjecture 1, so the conditions cannot be decreased in Conjecture 1 and also Theorem 4 for $k \geq 4$. The graph $H_5(\delta)$ verifies the sharpness of Conjecture 1 and Theorem 4 when $k = 3$.

The above examples verify the sharpness of Theorem 1, Conjecture 1 and Theorem 4 when n satisfies the appropriate congruence relative to k . Analogous examples, which are less symmetric, can be described for general n .

Before starting our main proofs, for convenience we describe additional notation. Given a positive integer p , $\sigma_p(G) = \min\{d(v_1) + d(v_2) + \dots + d(v_p) : \{v_1, v_2, \dots, v_p\} \text{ is an independent set of } G\}$. Given a positive integer λ , a cycle C in a graph G is a D_λ -cycle if each component of $G - C$ has fewer than λ vertices. With this notation, we can state the following result of Fraisse in [5].

Theorem A. *If G is a k -connected graph of order $n \geq 3$ with $\sigma_{k+1}(G) \geq n + k(k-1)$, then G contains a D_k -cycle.*

Proof (of Proposition 1). Select a maximal length cycle C of G that is a D_k -cycle. Theorem A implies that such a cycle exists. If C is hamiltonian, then the proof is complete, so assume not. Select a vertex v in one of the components, say H , of $G - C$. Since $|H| \leq k-1$, we know $d_H(v) \leq k-2$, and so $d_C(v) \geq \delta(G) - k + 2$. Let $S = N_C(v)$, and let S^+ denote the successors of S on C for some orientation of C . Since C is a maximal length D_k -cycle,

the set $S^* = S^+ \cup \{v\}$ is an independent set with at least $\delta(G) - k + 3$ vertices. This is a contradiction, since $\alpha(G) \leq \delta(G) - k + 2$. This completes the proof of Proposition 1. ■

Proposition 1 is also a direct consequence of the main result of Ota [6], and is related to the following corollary of the main result of Ota [6].

Theorem B. *Let G be a k -connected graph of order $n(n \geq 3)$ with $\alpha(G) \leq (n + 1)/(k + 1) + 1$. If $\sigma_{k+1}(G) \geq n + k^2 - k$, then G is hamiltonian.*

3. MAIN RESULTS

Theorem 3. *Let G be a graph of sufficiently large order n and $k \geq 3$ a positive integer such that $\kappa(G) \geq 4k^2 + 1$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.*

Proof. Suppose that G is not hamiltonian-connected, and select distinct vertices $x, y \in V(G)$ and let P be a path in G from x to y with the maximum number of vertices, say $m < n$. Let $t = 4k^2 + 1$, so $\kappa(G) \geq t$.

We would like to show that each vertex of $H = G - P$ may be adjacent to at most $\alpha(G)$ vertices of P . Suppose a vertex $v \in H$ has a set S of at least $\alpha(G) + 1$ adjacencies in P . Since, for either orientation of P , at most one vertex of S will not have a successor, we know $|S^+| \geq |S| - 1$. Because P is a path of maximum length, $S^+ \cup \{v\}$ must be an independent set of order at least $\alpha(G) + 1$, which is a contradiction. Thus, $\delta(H) \geq \delta(G) - \alpha(G) \geq k - 2$.

Let s be the cardinality of a maximum length cycle $C \subseteq H$, if a cycle exists. If H has no cycle, let $s = 1$.

Claim 1. $s < 2(m - 1)/(t - 1)$.

Proof. Since the conditions of Theorem 3 imply that G is hamiltonian, so there is a path between x and y that contains at least $(n + 1)/2$ vertices. Thus, we have $m \geq \delta(G)$, since $\delta(G) \geq (n + 1)/2$ would imply hamiltonian-connected by the result of Dirac ([3]). Assume $s < t = 4k^2 + 1$. Then, since $m \geq \delta(G) \geq (n + k^2 + k - 1)/(k + 1)$, for n sufficiently large, $s < 2(m - 1)/(t - 1)$. Thus, assume $s \geq t$. There exist t vertex-disjoint paths between C and P . Two of these paths, say Q_1 and Q_2 , have end vertices in P with at most $(m - 1)/(t - 1) - 1$ vertices of P between them. The end vertices of Q_1 and Q_2 on C have at least $(s - 2)/2$ vertices between

them in one direction around the cycle C . The maximality of P implies that $(s - 2)/2 + 2 \leq (m - 1)/(t - 1) - 1$. Therefore, $s < 2(m - 1)/(t - 1)$. This completes the proof of the claim. ■

Claim 2. The order of the path P is given by $m \geq kn/(k + 1)$.

Proof. Our proof is by contradiction. Since the longest cycle in H has length s , the endvertices of a longest path in H have degree less than s . Therefore, there exist two vertices, say u and v , such that $d_H(u), d_H(v) \leq s$. Furthermore, since $\delta(H) \geq k - 2$, this path, denoted by Q , from u to v in H has at least $k - 1$ vertices.

Since P is a maximum length path, no vertex of H can be adjacent to two consecutive vertices of P . Let $U = N_P(u), V = N_P(v), W = U \cap V$ and, without loss of generality, assume that $|U| \leq |V|$. Thus, there are $|U| + |V| - |W|$ vertices of P adjacent to either u or v , and there are the same number, or possibly one more or one less of “open” intervals of P with no adjacencies to either u or v . Since the path P is of maximum length, each “open” interval contains at least one vertex, and those intervals between a vertex in U and a vertex in V , which will we call “long” intervals, contain at least $k - 1$ vertices. There are at least $|W| - 1$ such “long” intervals. Thus,

$$(1) \quad \begin{aligned} m &\geq |U| + |V| - |W| + (k - 1)(|W| - 1) + (|U| + |V| - 2|W|) \\ &= 2|U| + 2|V| + (k - 4)|W| - k + 1. \end{aligned}$$

All of the vertices in $(U \cup V)^+ \cup \{u\}$ are independent, since any edge between vertices in this set would contradict the fact that P was chosen of maximum length. This implies that $|U| + |V| - |W| \leq \alpha(G) \leq \delta(G) - k + 2$. Therefore $(\delta(G) - d_H(u)) + (\delta(G) - d_H(v)) - \delta(G) + k - 2 \leq |W|$, and so

$$|W| \geq \delta(G) - 4(m - 1)/(t - 1) + k.$$

Hence, by Equation 1, we get

$$\begin{aligned} m &\geq 4(\delta(G) - s + 1) + (k - 4)|W| - k + 1 \\ &\geq 4(\delta(G) - 2(m - 1)/(t - 1) + 1) \\ &\quad + (k - 4)(\delta(G) - 4(m - 1)/(t - 1) + k) - k + 1. \end{aligned}$$

Using the bounds $m < kn/(k + 1)$ and $\delta(G) \geq (n + k^2 - 2k)/k$ in the previous

equation gives

$$(t + 4k - 9)kn/(k + 1) > (t - 1)(n + k^2 - 2k) + (t - 1)(k^2 - 5k + 5) + 4k - 8.$$

However, this implies $t \leq 4k^2 - 8k + 1$ which is a contradiction. Finally we may conclude that $|P| = m \geq kn/(k + 1)$ completing the proof of the claim. ■

Assume that P is not a hamiltonian path. Select a longest path Q in H , say with q vertices and with end vertices u and v . Note that since $\delta(H) \geq k - 2$, we get $q \geq k - 1$. Recall that each vertex of H has at most $\alpha(G)$ adjacencies in P . In fact, $|N_P(Q)| \leq \alpha(G)$ for the same reason. Assume that $s = d_H(u) \geq d_H(v)$, thus u has $s \geq k - 2$ adjacencies on Q . Denote the predecessors of these s vertices by $\{u = u_1, u_2, \dots, u_s\}$. Between v and any of the vertices u_i for $1 \leq i \leq s$, there is a path with q vertices, and between u_i and u_j for $i \neq j$ there is a path with at least $(s + 1)/2$ vertices. There is no loss of generality in assuming that $d_H(u_i) \leq s$ for all i , and so each vertex of $S = \{u_1, u_2, \dots, u_s, v\}$ has at least $\delta(G) - s$ adjacencies in P .

As before, let $U = N_P(u), V = N_P(v), W = U \cap V$, and assume that $|U| \leq |V|$. Since $d_P(u), d_P(v) \geq (n/k) + k - 2 - s$ and $\alpha(G) \leq n/k$, we have $|W| = |U| + |V| - |U \cup V| \geq 2((n/k) + k - 2 - s) - n/k = (n/k) + 2k - 4 - 2s$. This implies

$$\begin{aligned} n - s - 1 &\geq n - q \geq (s + 2)|W| + (|U| - |W|) + (|V| - |W|) - 1 \\ &\geq m \geq (s + 2)((n/k) + 2k - 4 - 2s) + 2(s + 2 - k) - 1. \end{aligned}$$

Therefore, we know

$$2s^2 - ((n/k) + 2k - 5)s + n - (2n)/k - 2k + 4 \geq 0.$$

However, for $k - 2 \leq s \leq n/(2k) - 1$ the previous inequality is contradicted, so we can assume that $s > n/(2k) - 1$. We have previously shown in Claim 1 that $s \leq n/(2k^2)$. Therefore, the assumption that the path P was not a hamiltonian path between x and y is contradicted. This completes the proof of Theorem 3. ■

The conditions of Theorem 3 are sharp except for the condition on the connectivity $\kappa(G)$, but for small values of k , we prove Theorem 4 which uses the sharp condition for $\kappa(G)$.

Theorem 4. *Let G be a graph of sufficiently large order n and $k = 3$ or 4 such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.*

Proof. Suppose that G is not hamiltonian-connected, and select distinct vertices x and y and let P be a path of G from x to y with a maximum number of vertices, say $m < n$. Let $H = G - P$. If $\kappa(G) \geq 4k^2 + 1$, then the proof is complete by Theorem 3, so we can assume that $k \leq \kappa(G) \leq 4k^2$.

We would like to show each vertex of H can be adjacent to at most $\alpha(G)$ vertices of P . Otherwise, a vertex $v \in H$ has a set S of at least $\alpha(G) + 1$ adjacencies in P . Note that $|S^+| \geq |S| - 1$, since at most one vertex of S will not have a successor. Since P is a path of maximum length, this implies that $S^+ \cup \{v\}$ is an independent set of order at least $\alpha(G) + 1$, a contradiction. Thus, $\delta(H) \geq \delta(G) - \alpha(G) \geq k - 2$.

We next show that $|P| \geq (k - 1)n/k + k - 1$, so assume not. Select a minimal cutset S of G , and let C_1, C_2, \dots, C_t be the components of $G - S$. Thus, $k \leq |S| = s \leq 4k^2$, and we can assume that $|C_1| \geq |C_2| \geq \dots \geq |C_t|$.

First consider the case $k = 3$, and so $\delta(G) \geq n/3 + 2$, $\alpha(G) \leq \delta(G) - 1$, and $\kappa(G) \leq 36$. If $t \geq 4$, then $|C_t| \leq (n - s)/t$, and for any vertex $v \in C_t$, $d(v) \leq (n - s)/t + s - 1 < n/3$, a contradiction. Therefore, $t \leq 3$. If $t = 3$, then $n/3 + s - 6 \geq |C_1| \geq |C_2| \geq |C_3| \geq n/3 + 3 - s$, and $\delta(C_i) \geq n/3 - s + 2$ for each i . If $s = 3$, then for any vertex v in C_3 , $d(v) \leq (n - 3)/3 + 2$, a contradiction. Thus, $s \geq 4$. Each of the graphs C_i are nearly complete, and are hamiltonian-connected even after the deletion of any small number of vertices. Also, there is an s -matching between S and each of the components C_i , since S is a minimal cut set. Since $s \geq 4$, independent of the location of the vertices x and y , it is an easy and straightforward case analysis to show that there is a path P from x to y containing all of the vertices of $G - S$ and either 2, 3 or 4 vertices of S . Thus, in this case, $|P| \geq n - s + 2 > 2n/3 + 2$.

We now consider the case when $t = 2$. Hence, $n/3 + 3 - s \leq |C_2| \leq |C_1| \leq 2n/3 - 3$, and $\delta(C_i) \geq n/3 + 2 - s$ for each i . The component C_2 is hamiltonian-connected, but if $|C_1| \leq 2n/3 + 3 - 2s$, the component C_1 is also hamiltonian-connected. Consider the case when C_1 is hamiltonian-connected. If one of x or y is not in C_1 , then it is straightforward to form a path from x to y using all of the vertices of C_1 and C_2 along with 2 or 3 vertices of S . This implies $|P| \geq n - s + 2 > 2n/3 + 2$. If x and y are both in C_1 then there is a hamiltonian path Q in C_1 from x to y . There

is also a matching with s edges between S and Q . Using two of these s edges, whose end vertices in Q are of minimum distance apart on Q , along with a hamiltonian path of C_2 gives a path P from x to y of length at least $n - (|C_1| - s)/(s - 1) - (s - 2) \geq n - (2n/3 - 3 - 3s)/(s - 1) - s + 2 > n/3 + 2$. Thus, we can assume that C_1 is not hamiltonian-connected, and so $|C_1| \geq 2n/3 + 4 - 2s$.

If $\kappa(C_1) \leq 2$, then there is a cut set, say S' , with $|S'| = 1$ or 2 , such that $C_1 - S'$ has two components, say C'_1 and C''_1 . The minimum degree in each component is at least $n/3 - s$, so each of these components, and also C_2 is nearly complete. Also, there are $s - 2$ independent edges between S and each of the components C'_1 and C''_1 and s independent edges between S and C_2 . Hence, just as in the case when there were 3 components of $G - S$, it is an easy and straightforward case analysis to find a path P with at least $n - s + 2$ vertices from x to y , independent of the location of x and y . Therefore, we can assume that $\kappa(C_1) \geq 3$.

Since C_1 is 3-connected, by a result of Dirac [3], there is a cycle C in C_1 of length at least $2n/3 + 4 - 2s$, and there are s independent paths from S to C . Select two end vertices of these s paths that have a minimum distance between them on C . If x and y are not in C_1 , then a path from x to y can be formed using all of the vertices of C_2 (since C_2 is hamiltonian-connected), at least two vertices of S , and all of the vertices of C except for possibly $(|C| - s)/s$. Thus the path P will have at least $(n/3 + 3 - s) + 2 + (s - 1)(2n/3 + 4 - 2s)/s \geq 7n/9 + 7 - 3s > 2n/3 + 2$ vertices. If x and y are in C_1 , then by a result of Enomoto in [4] there is a path between x and y with at least $2n/3 + 4 - 2s$ vertices. Thus, just as in the case of the cycle, a path with at least $7n/9 + 8 - 5s/2 > 2n/3 + 2$ vertices can be formed. If $x \in C_1$ and $y \notin C_1$, then using a path Q from x to some vertex z in C_1 with a least $2n/3 + 4 - 2s$ vertices, a path between x and y can be formed using all of the vertices of Q and C_2 , thereby using more than $2n/3 + 2$ vertices. This completes the proof of the claim that there is a path from x to y with at least $2n/3 + 2$ vertices.

Let P be a path between x and y of maximum length m , and let $H = G - P$. Select a path Q with a maximum number of vertices, say q , in H with end vertices u and v . Without loss of generality, let $s = d_H(u) \geq d_H(v)$, which means u has s adjacencies in Q and $q \geq s + 1$. Denote the predecessors of these s vertices by $S' = \{u = u_1, u_2, \dots, u_s\}$, and let $S = S' \cup \{v\}$. Observe that no vertex of H can be adjacent to two consecutive vertices of P , since P is a maximum length path. Let $U = N_P(u)$, $V = N_P(v)$, $W = U \cap V$, and

so $|U|, |V| \geq \delta(G) - s + 1$. There are $|U| + |V| - |W|$ vertices of P adjacent to either u or v , and there are the same number or possibly one more or one less “open” interval of P with no adjacencies to either u or v . Since the path P is of maximal length, each of the “open” intervals will have at least one vertex, and those intervals between a vertex in U and a vertex in V , which we will call “long” intervals, will have at least q vertices.

If $U = V = W$, then $|W| \geq n/3 + 3 - q$, and there will be at least $|W| - 1$ “long” intervals. Hence,

$$n - q \geq m \geq (q + 1)(n/3 + 2 - q) + 1.$$

This implies the inequality $q^2 - (n/3 + 2)q + 2n/3 - 3 \geq 0$. However, for $2 \leq q \leq n/3$, this gives a contradiction. If $q = 1$, then u has at least $\delta(G)$ adjacencies in P , which implies the existence of an independent set of order $\delta(G) > \alpha(G)$, a contradiction.

In general, if s is small, then u and v will have nearly identical neighborhoods in P . More specifically, $|U \cup V| \leq \alpha(G) \leq \delta(G) - 1$ to avoid an independent set with more than $\alpha(G)$ vertices. Since $|U|, |V| \geq \delta(G) - s$, this implies that $|U \cap V| \geq \delta(G) - 2s + 1$. An immediate consequence of this is that there are $\delta(G) - 2s$ vertex disjoint intervals of P (between the common adjacencies of u and v on P) each with at least $s + 2$ vertices. This implies

$$n \geq (s + 1) + (\delta(G) - 2s)(s + 2) + 1 \geq (s + 1) + (n/3 - 2s + 2)(s + 2) + 1,$$

which is a contradiction for $s \leq 3$. Thus, we assume $q \geq 5$ and $s \geq 4$. When $U \neq V$, we get

$$\begin{aligned} n - q \geq m &\geq |U| + |V| - |W| + q|W| + (|U| + |V| - 2|W|) - 1 \\ &= 2|U| + 2|V| + (q - 3)|W| - 1. \end{aligned}$$

Since $n > 2|U| + 2|V|$, we know $|U| < n/4$ and $s \geq n/3 + 2 - n/4 > n/12$, for otherwise this gives a contradiction.

Let R be the set of r vertices of P with at least two adjacencies in S . The interval between two adjacencies of distinct vertices of S will have at least $(s+1)/2$ vertices. If $r \geq \delta(G) - s$, then there will be at least $\delta(G) - s - 1$ distinct intervals of P with at least $(s+1)/2$ vertices with no adjacencies in Q , and one of the intervals will have at least q such vertices. This implies that

$$n - q \geq m \geq (\delta(G) - s - 2)(s + 1)/2 + q + \delta(G) - s.$$

The previous equation implies that $s^2 - (\delta(G) - 5)s - 3\delta(G) + 2 + 2n - 4q \geq 0$. However, this is a contradiction for $4 \leq s < n/3 - 3$, so we may assume that $r < \delta(G) - s$.

There are at least $r - 1$ distinct intervals of P with at least $(s + 1)/2$ vertices with no adjacencies in S and also an additional s intervals with this same property because the predecessor and the successor of the interval are from distinct vertices of S . There are at least $s(\delta(G) - s - r - 1)$ additional intervals with at least one vertex with no adjacencies in S . This implies that

$$n - s - 1 \geq m \geq (r - 1)(s + 3)/2 + s(s + 3)/2 + s(\delta(G) - s - r - 1)2.$$

Since the lower bound on m in the previous equation is a decreasing function of r , this implies that the extreme value when $r = \delta(G) - s - 1$ is also a lower bound, and so $n \geq (\delta(G) - 2)(s + 3)/2 + s + 1 > (n/3)(7/2) + 4$, a contradiction. This completes the proof of the case $k = 3$. The proof of the case $k = 4$ is identical, except the analysis to show that there is a path between x and y with at least $3n/4 + 3$ vertices is much more tedious. This completes the proof of Theorem 4. ■

4. QUESTIONS

The obvious problem is to extend Theorem 4 to all values of $k \geq 5$ and verify Conjecture 1 when the order n of the graph is sufficiently large. It is also desirable to be able to drop the n sufficiently large condition.

Many degree conditions that imply a graph is hamiltonian have analogues that imply much more, such as panconnected, hamiltonian ordered, etc. Are there similar analogues for the Chvátal-Erdős type conditions?

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Received 20 January 2009

Revised 25 June 2009

Accepted 25 June 2009