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(D ; n) - Cages

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The degree set D of a graph G is the set of degrees of the vertices of G. The girth g(G), is the length of a shortest cycle in G. For a set $D = \{a_1, a_2, \ldots, a_k\}$ of positive integers with $2 \le a_1 < a_2 < \ldots < a_k$ and for an integer $n \ge 3$, define $f(D ; n) = f(a_1, a_2, \ldots, a_k ; n)$ to be the minimum order of a graph having degree set D and girth n. A graph with degree set D, girth n, and order f(D ; n) is termed a (D ; n)-cage. If $D = \{r\}$, the (D ; n)-cages coincide with the (r ; n)-cages which have been extensively studied (for example, see [1] or [5]). The existence of f(D ; n) was shown in [2].

Recently, Harary and Kovacs introduced another generalization of the standard cage question (see [3], [4]). They consider regular graphs with given girth pair (length of the shortest odd and shortest even cycle). Both generalizations offer possible applications to the standard cage question.

The object of this paper is to establish a lower bound for f(D ; n) and, for certain sets D and integers n, to determine the values of f(D ; n). We also indicate possible applications of f(D ; n) to the (r ; n)-cage problem. We begin with the first of these objectives.

Theorem 1: If $D = \{a_1, a_2, \ldots, a_k\}$ is a set of positive integers with $2 \le a_1 < a_2 < \ldots < a_k$ and n is an integer, $n \ge 3$, then

$$f(D ; n) \ge \begin{cases} 1 + \sum_{i=1}^{t} a_k(a_1-1)^{i-1} & \text{if } n = 2t+1 \\ 1 + \sum_{i=1}^{t-1} a_k(a_1-1)^{i-1} + (a_1-1)^{t-1} & \text{if } n = 2t. \end{cases}$$

Proof: Let G be a graph with degree set D and girth $n = 2t + 1$. Then G contains a vertex v_0 of degree a_k, adjacent to vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,a_k}$. If $n = 3$, edges may exist between $v_{1,i}$ and $v_{1,j}$ $(i \neq j)$, however, if $n \geq 5$, no such edge is possible. In this case, each vertex $v_{1,\ell}$ $(1 \leq \ell \leq a_k)$ must be adjacent to at least $a_1 - 1$ distinct new vertices, call them $v_{2,1}, v_{2,2}, \ldots, v_{2,a_k(a_1-1)}$, where $v_{1,w}$ is adjacent with $v_{2,r}$, $(w-1)(a_1-1) + 1 \leq r \leq w(a_1-1)$. If $n=5$, edges of the form $v_{2,i}v_{2,j}$ $(i \neq j)$ are possible, but if $n \geq 7$, each vertex $v_{2,\ell}$ $(1 \leq \ell \leq a_k(a_1-1))$ must be adjacent to at least $a_1 - 1$ distinct new vertices, call them $v_{3,1}, v_{3,2}, \ldots, v_{3,a_k(a_1-1)^2}$. For $g = 2t + 1$, this process must continue until the vertices $v_{t,\ell}$ $(1 \leq \ell \leq a_k(a_1-1)^{t-1})$ have been added, where $v_{t-1,w}$ is adjacent with $v_{t,r}$, $(w-1)(a_1-1) + 1 \leq r \leq w(a_1-1)$. Denote the tree thus constructed by $T(D\ ;\ g)$. Since $T(D\ ;\ g)$ is a subgraph of G, we see that $|V(G)| \geq 1 + \sum_{i=1}^{t} a_k(a_1-1)^{i-1}$.

If $g = 2t$, construct the tree $T(D\ ;\ g-1)$. However, no new edge can be added to $T(D\ ;\ g-1)$ without forming a cycle of length less than g. Thus, new vertices are necessary. Since $\delta(G) = a_1$, at least $(a_1-1)^{t-1}$ new vertices must be added (if each has degree a_k) and the result follows.

When $D = \{d\}$ $(d \geq 3)$ and $a_1 \geq 3$, Theorem 1 may be reduced to the well known lower bound for $f(d\ ;\ n)$ (for example see [1]).

Bounds on the order of an $(r\ ;\ n)$-cage may be obtained from the order of certain $(D\ ;\ n)$-cages.

Proposition 2: (a) If $r \geq 3$ and t is the number of vertices of degree 2r in some $(r, 2r\ ;\ n)$-cage then

$$f(r-1, r\ ;\ n) \leq f(r\ ;\ n) \leq f(r, 2r\ ;\ n) + t.$$

(b) If $f(r\ ;\ n) = m$ then $f(r, kr\ ;\ n) \leq k(m-1) + 1$ and equality

holds when n is odd and m equals the low

Proof: (a) We establish the lower boun cage to obtain a graph with degree {r-1, Clearly the edge may be chosen so that a bound is established by splitting each v degree r.

(b) Identify one vertex from e

Recently, various formulas have be (with at least two elements) and girths additional formulas for $f(D\ ;\ n)$.

Theorem 3: For integers $n \geq 1$ and $m \geq$

Proof: Construct the tree $T(D\ ;\ 2n+1)$ by inserting the edges $v_{n,\ell}v_{n,\ell+1}$ $(1 \leq$ and degree set D, thus, $f(D\ ;\ 2n+1) \leq$ $\geq 1 + mn$, hence, $f(D\ ;\ 2n+1) = 1 + mn$.

Corollary 4: For integers $n \geq 1$ and r

$f(2, 3, r, s\ ;\ 2n +$

Proof: Form the $(2, 3, s\ ;\ 2n + 1)$-c H from G by removing the edges $v_{n,\ell}v_n$ $v_{n,1}v_{n,j}$ $(3 \leq j \leq r - 1)$. Then, H has Thus, $f(2, 3, r, s\ ;\ 2n + 1) \leq 1 + sn$ $\geq 1 + sn$, hence the result follows.

Before proceeding, some terminol is the set of vertices v for which t containing v_0. The level of a verte holds when n is odd and m equals the lower bound in Theorem 1.

Proof: (a) We establish the lower bound by removing an edge of an $(r ; n)$-cage to obtain a graph with degree $\{r-1,r\}$, thus $f(r-1, r ; n) \le f(r ; n)$. Clearly the edge may be chosen so that an n-cycle is maintained. The upper bound is established by splitting each vertex of degree 2r into vertices of degree r.

(b) Identify one vertex from each of k copies of an $(r ; n)$-cage.

Recently, various formulas have been determined for certain degree sets D (with at least two elements) and girths $n \ge 3$ (see [2]). We present some additional formulas for $f(D ; n)$.

Theorem 3: For integers $n \ge 1$ and $m \ge 4$, $f(2, 3, m; 2n+1) = 1 + mn$.

Proof: Construct the tree $T(D ; 2n+1)$. Now form the graph G from $T(D ; 2n+1)$ by inserting the edges $v_{n,\ell}v_{n,\ell+1} (1 \le \ell \le n - 1)$. The graph G has girth $2n + 1$ and degree set D, thus, $f(D ; 2n+1) \le 1 + mn$. By Theorem 1, $f(D ; 2n+1) \ge 1 + mn$, hence, $f(D ; 2n+1) = 1 + mn$.

Corollary 4: For integers $n \ge 1$ and r, $4 \le r \le s - 1$,

$$f(2, 3, r, s ; 2n + 1) = 1 + sn.$$

Proof: Form the $(2, 3, s ; 2n + 1)$-cage G as in Theorem 3. Now form the graph H from G by removing the edges $v_{n,\ell}v_{n,\ell+1} (2 \le \ell \le r - 1)$ and inserting the edges $v_{n,1}v_{n,j} (3 \le j \le r - 1)$. Then, H has degree set D and clearly has girth $2n + 1$. Thus, $f(2, 3, r, s ; 2n + 1) \le 1 + sn$. By Theorem 1, $f(2, 3, r, s ; 2n + 1) \ge 1 + sn$, hence the result follows.

Before proceeding, some terminology will be useful. The branch i of $T(D,n)$ is the set of vertices v for which there exists a path from v to $v_{1,i}$ not containing v_0. The level of a vertex of $T(D,n)$ is given by its first subscript.

If $j = \lceil\frac{g(G)-1}{2}\rceil$, then a j-path in a graph containing T(D,n) is a path composed entirely of j level vertices, while a j-cycle is a cycle composed entirely of level j vertices, while a j-cycle is a cycle composed entirely of level j vertices. An interior vertex is a vertex of T(D,n) which is not a j level vertex. Two j level vertices joined by a path of length 2i through a vertex in level j-1 are called i-conjugates and $\bar{v}^i$ denotes the i-conjugate of v. For simplicity let $\bar{v} = \bar{v}^1$.

Theorem 5: If $m \geq 4$, then $f(3,m;5) = 1 + 3m$.

Proof: By Theorem 1, $f(3,m;5) \geq 1 + 3m$, so consider T(D,5). We will join vertices in the second level in order to form the required graph.

Case 1: Suppose $m \geq 5$. We form two j-cycles, each of length m. Considering only the second subscript of each vertex, the first j-cycle is 1, 3, 5, ..., 2m-1, 1 and the second j-cycle is 2, 4, 6, ..., 2m, 2. The graph G thus formed has degree set {3,m}. Any edge connecting second level vertices creates a cycle of length five containing v_o. Thus, we need only show there are no smaller cycles in G.

It is clear from the construction that v_o lies on no smaller cycles. Thus, suppose there exists a small cycle containing more than one vertex on the first level. Then that cycle must also contain four level 2 vertices and hence has length at least five. Now, suppose there exists a small cycle containing exactly one vertex on the first level, say $v_{1,x}$. Then there exists a j-path of length at most 2 connecting the level 2 vertices adjacent to $v_{1,x}$. But these vertices are on completely different j-cycles, hence there is no j-path connecting them. Thus, any small cycle of G cannot contain vertices on the first level. Finally, since all vertices in the second level were joined in j-cycles of

length $m \geq 5$ no small cycle consists only

Case 2: Suppose m = 4. The arguments of
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Corollary 6: If $m \geq 5$, then f(3,4,m;5)

Proof: Construct the (3,m;5)-cage H as
T(D,5), D = {3,m}, $m \geq 5$. Now, in leve
graph G. It is straightforward to show

Theorem 7: If $m \geq 4$, then f(3,m;7) = 1

Proof: From Theorem 1, $f(3,m;7) \geq 1$ +
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Z_1: 1, 5, 9, ..., 4m - 3, 3, 7, 11, ..
4m - 6, 4m, 4, 6, 12, 14, ..., 4m - 4,

Again no small cycle can contain
two first level vertices. Now assume
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are on different j-cycles. Hence, an
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$v_{3,i}$, $4x - 3 \leq i \leq 4x$, through $v_{1,x}$,
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Case 2: Suppose $m = 4$. The arguments of the preceeding case show no small cycle can contain v_0 or more than one vertex of the first level. Since $m = 4$, we clearly cannot form two j-cycles as before. It is sufficient to connect the 8 vertices on level two in one cycle. Such a cycle is: 1, 6, 7, 4, 5, 2, 3, 8, 1 (second subscripts). It is straightforward to verify the graph contains no small cycles. Therefore, this case is completed and $f(3,m;5) = 1 + 3m$.

Corollary 6: If $m \geq 5$, then $f(3,4,m;5) = f(3,m;5) = 1 + 3m$.

Proof: Construct the (3,m;5)-cage H as in Theorem 1 and note it contains T(D,5), D = {3,m}, $m \geq 5$. Now, in level 2, add the edge $v_{2,1}\, v_{2,8}$, forming the graph G. It is straightforward to show this graph has the necessary properties.

Theorem 7: If $m \geq 4$, then $f(3,m;7) = 1 + 7m$.

Proof: From Theorem 1, $f(3,m;7) \geq 1 + 7m$. We construct T(3,m;7), and join vertices in level 3 in order to form the required graph.

Case 1: Suppose $m = 2t$, $t \geq 2$. We form 2 j-cycles, Z_1 and Z_2, each of length 2m. Considering only the second subscripts of the level 3 vertices, let Z_1: 1, 5, 9, ..., 4m - 3, 3, 7, 11, ..., 4m - 1, 1 and Z_2 2, 8, 10, 16, ..., 4m - 6, 4m, 4, 6, 12, 14, ..., 4m - 4, 4m - 2, 2.

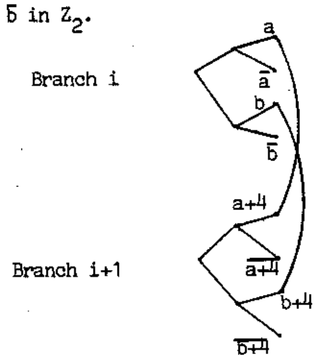
Again no small cycle can contain v_0; moreover, no small cycle can contain two first level vertices. Now assume there exists a small cycle containing exactly one first level vertex. By the way Z_1 and Z_2 were formed, conjugates are on different j-cycles. Hence, any small cycle containing a first level vertex $v_{1,x}$ must have a path of length 4 connecting two vertices of the form $v_{3,i}$, $4x - 3 \leq i \leq 4x$, through $v_{1,x}$, and these two level 3 vertices must be joined by a j-path of length 1 or 2. But recall that there are m vertices joined in a j-cycle before a branch is revisited, and $m \geq 4$. Therefore, the needed j-path of length 1 or 2 does not exist. So there are no small cycles containing exactly one first level vertex.

Suppose now that there is a small cycle containing exactly one second level vertex, $v_{2,x}$. Then there would be a j-path of length at most four joining the members of the conjugate pair adjacent to $v_{2,x}$. But these two vertices are on two different j-cycles, so there is no j-path joining this pair. Thus there cannot be a small cycle containing exactly one second level vertex. A cycle containing more than two second level vertices would have length at least nine, so this presents no danger. So, we must show that given any edge connecting third level vertices $v_{3,x}$ and $v_{3,y}$, that $\bar{v}_{3,x}$ and $\bar{v}_{3,y}$ are not joined by an edge, as this would produce a cycle of length six.

Note that in both Z_1 and Z_2 the branches are visited sequentially in a symmetric pattern. That is branch i is joined by edges to branch i-1 (modulo m) and branch i+1 (modulo m) and to no other branch. Assume there is a small cycle containing two conjugate pairs; it will involve branch i and branch i+1 for some i (modulo m).

<u>Subcase a</u>: Suppose $i \neq m$. By the construction, consecutive vertices in Z_1 have a difference of four. Moreover, vertices in Z_1 all have odd labels. Consider Figure 1. We have a,b in Z_1 and $\bar{a}$, $\bar{b}$ in Z_2.

<u>FIGURE 1</u>



Either $\bar{a}$ must be joined to $\overline{a+4}$ or $\bar{b}$ must be
is formed by adding two or six to $\bar{a}$ and $\bar{b}$,
not adjacent to $\overline{a+4}$ and $\bar{b}$ is not adjacent t
subcase a.

<u>Subcase b</u>: Suppose $i = m$. In Z_1, vertex
vertex $4m - 1$ is adjacent to vertex 1. Si
as are $4m - 2$ and 2. Thus, again no small
is no small cycle containing exactly two s

Note that there are no small j-cycle
each of length 2m. Since $m \geq 4$, $2m > 7$,

<u>Case 2</u>: Suppose $m = 2t + 1$, $t \geq 2$. Form
2, 8, 10, 16, 18, ..., 4m-4, 4, 4m, 12, 1

Recall that if a is connected to b
denote this by $\bar{a} - \bar{b}$. So, in Figure 2,
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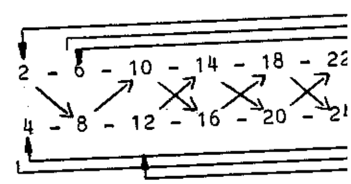

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Now that G has been constructed, it
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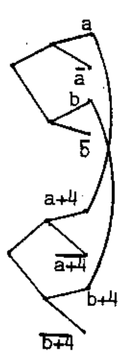

Either $\bar{a}$ must be joined to $\overline{a+4}$ or $\bar{b}$ must be joined to $\overline{b+4}$. But, in fact, Z_2 is formed by adding two or six to $\bar{a}$ and $\bar{b}$, never by adding four. Hence $\bar{a}$ is not adjacent to $\overline{a+4}$ and $\bar{b}$ is not adjacent to $\overline{b+4}$. This contradiction completes subcase a.

<u>Subcase b</u>: Suppose $i = m$. In Z_1, vertex $4m - 3$ is joined to vertex 3 and vertex $4m - 1$ is adjacent to vertex 1. Similarly, in Z_2, $4m$ and 4 are adjacent as are $4m - 2$ and 2. Thus, again no small cycle is formed. Therefore, there is no small cycle containing exactly two second level vertices.

Note that there are no small j-cycles. We constructed two disjoint j-cycles each of length $2m$. Since $m \geq 4$, $2m > 7$, this completes Subcase b and Case 1.

<u>Case 2</u>: Suppose $m = 2t + 1$, $t \geq 2$. Form Z_1 as before and let Z_2 be:
2, 8, 10, 16, 18, ..., 4m-4, 4, 4m, 12, 14, 20, 22, ..., 4m-6, 6, 4m-2, 2.

Recall that if a is connected to b then $\bar{a}$ cannot be connected to $\bar{b}$. We shall denote this by $\bar{a} - \bar{b}$. So, in Figure 2, the hyphens indicate vertices which cannot be adjacent in Z_2 and the arrows represent edges of Z_2.

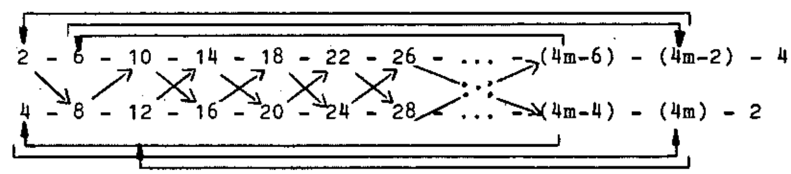

FIGURE 2

Now that G has been constructed, it is left to show that G contains no small cycles. By the same arguments used in Case 1, G does not have a small cycle containing v_o, exactly one vertex on the second level, more than one vertex on the first level, or a small j-cycle. Thus C contains exactly one first level vertex or it contains exactly two vertices on the second level. Assume C contains exactly one vertex, $v_{1,x}$, on the first level. That is, there exists a path of length at most two between two third level vertices of the branch containing $v_{1,x}$. Since the first cycle is identical to the first cycle of Case 1, then the small cycle must include a j-path contained in Z_2.

However, by examining Z_2 one sees that there is a path of length at least 3 between any two level j vertices of the same branch. Therefore, there is no small cycle containing exactly one first level vertex. A small cycle, then, would have to contain exactly two second level vertices. However, Figure 2 shows that it is precisely this property which was avoided in constructing Z_2. Therefore, there are no small cycles containing exactly two second level vertices, and so $f(3,m;7) = 1 + 7m$.

Corollary 8: If $m \geq 8$, then $f(3,4,m;7) = f(3,m;7) = 1 + 7m$.

Proof: Construct the (3,m;7)-cage H as in Theorem 3. Now add the edge 1, 18. The degree set of the graph G, so constructed, is $\{3,4,m\}$. Again by Theorem A, $f(3,4,m;7) \geq 1 + 7m$. It is straightforward to show that G has girth 7.

Theorem 9: If m = 6, 7, 8 then $f(3,m;9) = 1 + 15m$.

Proof: By Theorem 1, $f(3,m;9) \geq 1 + 15m$. Construct $T(\{3,m\};9)$, m = 6, 7, or 8. From the level 4 vertices we form disjoint cycles Z_1 and Z_2, each of length 4m to obtain the required graph G. Let Z_1 be 1, 9, 17, ..., 8m-7, 5, 13, 21, ..., 8m-3, 3, 11, 19, ..., 8m-5, 7, 15, 23, ..., 8m-1, 1 where only the second subscript of the level 4 vertices are used.

We first consider the possible small cycles which could occur when Z_1 and Z_2 are formed. There is no small cycle C containing more than one vertex on level 1 as this implies C has length at least 14. Also, C cannot contain two level 2 vertices, without containing a level 1 vertex or else C would have

length at least 10. Any cycle containing
level 2 vertices is also excluded.

To facilitate the checking for small
Z_1 was formed without creating small cycl
placed on Z_2. Since Z_1 has the form:

..., a, b, c, d, e, f

..., a+2, b+2, c+2, d+2, e

we can use the conjugates of these numbe
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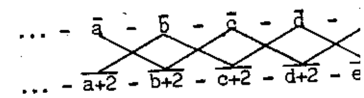

FIGUR

In Figure 3, $\bar{b}$ - $\bar{c}$ if the only if
the edge $\bar{b},\bar{c}$ forbidden, but d_{Z_2} $(\bar{b},\bar{c})$ m
created. Similarly, since $\bar{b}$ and $\bar{d}$ are
d_{Z_1} (b,d) = 1 and thus, $d_{Z_2}(\bar{b},\bar{d})$ must b
$\bar{b}$ and $\bar{e}$, in Figure 3 are separated by
$\bar{b}$ and $\bar{e}$ cannot be adjacent. These rest
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There are vertices which, if join
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edge is $\bar{c}$, $\overline{d+2}$ (see Figure 4). Since
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length at least 10. Any cycle containing 3 or more level 3 vertices, but no level 2 vertices is also excluded.

To facilitate the checking for small cycles of other types we note that Z_1 was formed without creating small cycles. Thus certain restrictions are placed on Z_2. Since Z_1 has the form:

..., a, b, c, d, e, f, ...

..., a+2, b+2, c+2, d+2, e+2, f+2, ...

we can use the conjugates of these numbers to form a diagram (see Figure 3) to describe many edges not allowed in Z_2, denoted u - v.

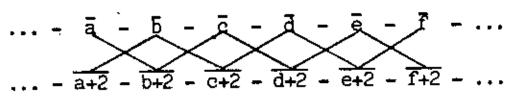

FIGURE 3

In Figure 3, $\bar{b} - \bar{c}$ if the only if b and c are adjacent in Z_1. Not only is the edge $\bar{b},\bar{c}$ forbidden, but $d_{Z_2}(\bar{b},\bar{c})$ must be at least 3, else a small cycle is created. Similarly, since $\bar{b}$ and $\bar{d}$ are separated by one vertex in Figure 1, then $d_{Z_1}(b,d) = 1$ and thus, $d_{Z_2}(\bar{b},\bar{d})$ must be at least 2, Also, if two vertices, say $\bar{b}$ and $\bar{e}$, in Figure 3 are separated by 2 vertices, then $d_{Z_1}(b,e) = 2$ and hence $\bar{b}$ and $\bar{e}$ cannot be adjacent. These restrictions prohibit small cycles containing two level 3 vertices and no level 2 vertex. These rules are all given in (4) below.

There are vertices which, if joined, would create a small cycle containing a level 2 vertex and a level 3 vertex from a different branch. One such forbidden edge is $\bar{c}$, $\overline{d+2}$ (see Figure 4). Since in Figure 3, $\bar{c} - \bar{d}$, then c and d are adjacent in Z_1. Thus if $\bar{c}$ is adjacent to $\overline{d+2}$, the cycle of length 8 in Figure 4 is formed.

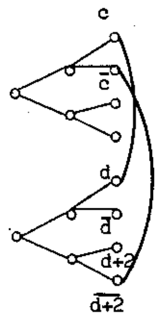

FIGURE 4

Such forbidden edges are described by the slanted lines of Figure 3 and also by (5) below.

Given Z_1, then Z_2 must meet the following requirements.

(1) $d_{Z_2}(a,\bar{a}) \geq 6$

(2) $d_{Z_2}(a,\bar{\bar{a}}) \geq 4$

(3) $d_{Z_2}(a,\bar{a}^3) \geq 2$

(4) If $d_{Z_1}(a,b) = 3 - s$ $(1 \leq s \leq 3)$ then $d_{Z_2}(\bar{a},\bar{b})$ must be at least s.

(5) If $d_{Z_1}(a,b) = 0$ then $d_{Z_2}(\bar{a},\bar{b})$ must be at least 1.

Case 1: If m = 6, the diagram will be:

2-10-18-26-34-42-6-14-22-30-38-46-4-...
4-12-20-28-36-44-8-16-24-32-40-48-2-...

Using the diagram, it is easy to verify that conditions (1)-(5) are satisfied by the cycle Z_2: 2, 14, 20, 32, 34, 46, 8, 10, 22, 28, 40, 42, 4, 16, 18, 30, 36, 48, 6, 12, 24, 26, 38, 44, 2.

Case 2: If m = 7, the diagram is:

2-10-18-26-34-42-50-6-14-22-30-3
4-12-20-28-36-44-52-8-16-24-32-4

Then Z_2 is: 2, 14, 20, 52, 32, 34, 46, 8
50, 30, 36, 48, 6, 12, 24, 56, 26, 38, 4

Case 3: If m = 8, the diagram is:

2-10-18-26-34-42-50-58-6-14-22-
4-12-20-28-36-44-52-60-8-16-24-

Z_2 is: 2, 14, 20, 32, 34, 46, 52, 64,
30, 36, 48, 50, 62, 8, 10, 22, 28, 40,

This completes the proof.

Theorem 10: If m ≥ 9, f(3,m;9) = 1 +

Proof: We use induction on m. For m
We will join vertices in the fourth le
we will form two cycles, Z_1 and Z_2, ea
Z_1 will consist of all vertices in le
That is, conjugates will be on the sa

We must again consider the possi
one vertex on the first, second, or
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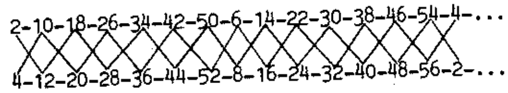
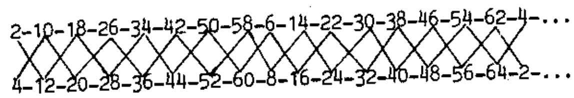
level vertices with no second level
Further, note that there cannot be

Case 2: If $m = 7$, the diagram is:



Then Z_2 is: 2, 14, 20, 52, 32, 34, 46, 8, 10, 22, 54, 28, 40, 42, 4, 16, 18, 50, 30, 36, 48, 6, 12, 24, 56, 26, 38, 44, 2.

Case 3: If $m = 8$, the diagram is:



Z_2 is: 2, 14, 20, 32, 34, 46, 52, 64, 6, 12, 24, 26, 38, 44, 56, 58, 4, 16, 18, 30, 36, 48, 50, 62, 8, 10, 22, 28, 40, 42, 54, 60, 2.

This completes the proof.

Theorem 10: If $m \geq 9$, $f(3,m;9) = 1 + 15m$.

Proof: We use induction on m. For $m = 9$, construct $T(\{3,m\};9)$. We will join vertices in the fourth level to form the required graph. As before, we will form two cycles, Z_1 and Z_2, each of length $4m$. However, in this case Z_1 will consist of all vertices in level 4 which are congruent to 1 or 2 (mod 4). That is, conjugates will be on the same Z_i $(i=1,2)$.

We must again consider the possibility of a small cycle containing exactly one vertex on the first, second, or third level. Also, there is a possibility of a small cycle containing exactly two third level vertices. Note that as before, there cannot be a small cycle containing two first level vertices or two second level vertices with no first level vertex. Also, three or more third level vertices with no second level vertex yields a cycle of length at least 9. Further, note that there cannot be a small cycle entirely contained in level 4.

Assume that a cycle C_n, $n \le 8$, containing at least one vertex from each Z_i exists. The path joining vertices in different Z_i has at least 3 intermediate vertices. Thus, $n = 8$, and C_n contains exactly one vertex v_i from Z_i $(i=1,2)$. But then v_1 and v_2 are joined by two distinct paths of interior vertices, a contradiction. Thus, no small cycle contains vertices from both Z_1 and Z_2. Therefore, if the vertices in Z_1 can be successfully joined to avoid small cycles, it will suffice to form Z_2 in a similar manner.

In forming Z_1 it is necessary and sufficient to obey the following restrictions.

(1) $d_{Z_1}(a,\bar{a}) \ge 6$

(2) $d_{Z_1}(a,\bar{a}^3) \ge 2$

(3) If $d_{Z_1}(a,b) = 3 - s$ $(1 \le s \le 3)$ then $d_{Z_1}(\bar{a},\bar{b})$ must be at least s.

Rule (1) insures that no small cycle containing exactly one third level vertex is formed. Rule (2) guarantees that no small cycle containing a first level vertex is formed. No level 2 vertex can lie on a small cycle as this would imply that the cycle contains vertices from both Z_i $(i=1,2)$. Rule (3) prohibits small cycles containing exactly two third level vertices with no level 2 vertex.

We now give Z_1, found by ad hoc methods. The reader can verify by the procedure above that all the conditions have been satisfied.

Z_1: 1, 9, 17, 5, 65, 49, 57, 2, 45, 54, 25, 10, 21, 50, 33, 18, 14, 37, 58, 41, 29, 34, 53, 6, 61, 38, 69, 30, 46, 66, 22, 62, 42, 26, 13, 70, 1.

We obtain Z_2 by adding two to each number. Since each Z_i $(i=1,2)$ by itself forms no small cycles, the proof is completed for $m = 9$.

Now, assume the theorem is true for
the graph G_1 has order $1 + 15m$ and has no
The fourth level vertices induce two cycl
that Z_1 consists of all vertices in leve
(mod 4). We will form the graph G_2 whic
C_n, $n \le 8$. Further, the fourth level ve
Z_2' of order $4(m+1)$ and Z_1' will consist c
congruent to 1 or 2 (mod 4).

Let Z_1 be: $a_1, a_2, \ldots, a_{4m}$. We w
$8m + 5$, and $8m + 6$ into Z_1.

Subdivide edges a_3,a_4 and a_5,a_6 w
Note that $8m + 1$ and $8m + 5$ are separa
There exist $4m - 13$ edges between a_{11}
between a_{11} and a_{4m-2}, then $8m + 1$ an
by at least six vertices, as required
separated by at least two vertices.

Now since $8m + 1$ is adjacent to
by at least three vertices from both
twelve edges made unavailable for 8m
is separated by one vertex from bot
made unavailable for $8m + 2$ by $\bar{a}_2$ a
tices from both a_1 and $8m + 5$. So
able for $8m + 2$ by $\bar{a}_1$. Note that
since it hasn't been placed yet it

Thus, at most 22 edges are un

containing at least one vertex from each
s in different Z_i has at least 3 inter-
C_n contains exactly one vertex v_i from
joined by two distinct paths of interior
small cycle contains vertices from both
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$_{Z_1}(\bar{a},\bar{b})$ must be at least s.

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25, 10, 21, 50, 33, 18, 14, 37, 58,
6, 66, 22, 62, 42, 26, 13, 70, 1.

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ted for m = 9.

Now, assume the theorem is true for $f(3,m;9)$, $m \geq 9$. Assume also that the graph G_1 has order $1 + 15m$ and has no cycles of length less than nine. The fourth level vertices induce two cycles, Z_1 and Z_2, each of order $4m$, and that Z_1 consists of all vertices in level 4 which are congruent to 1 or 2 (mod 4). We will form the graph G_2 which will have $1 + 15m$ vertices and no C_n, $n \leq 8$. Further, the fourth level vertices will induce two cycles Z_1' and Z_2' of order $4(m+1)$ and Z_1' will consist of all fourth level vertices which are congruent to 1 or 2 (mod 4).

Let Z_1 be: $a_1, a_2, \ldots, a_{4m}$. We will satisfactorily place $8m + 1$, $8m + 2$, $8m + 5$, and $8m + 6$ into Z_1.

Subdivide edges a_3,a_4 and a_5,a_6 with vertices $8m + 1$ and $8m + 5$ respectively. Note that $8m + 1$ and $8m + 5$ are separated by two vertices, as required by (2). There exist $4m - 13$ edges between a_{11} and a_{4m-2}. If $8m + 2$ is similarly placed between a_{11} and a_{4m-2}, then $8m + 1$ and $8m + 2$ are separated on the fourth level by at least six vertices, as required by (1). Also, $8m + 2$ and $8m + 5$ are separated by at least two vertices.

Now since $8m + 1$ is adjacent to both a_3 and a_4, then $8m + 2$ must be separated by at least three vertices from both $\bar{a}_3$ and $\bar{a}_4$. That is, there are at most twelve edges made unavailable for $8m + 2$ by $\bar{a}_3$ and $\bar{a}_4$. Similarly, since $8m + 1$ is separated by one vertex from both a_2 and a_5, there are at most eight edges made unavailable for $8m + 2$ by $\bar{a}_2$ and $\bar{a}_5$. Also, $8m + 1$ is separated by two vertices from both a_1 and $8m + 5$. So there are at most two more edges made unavailable for $8m + 2$ by $\bar{a}_1$. Note that $\overline{8m+5}$ does not affect the placement of $8m + 2$ since it hasn't been placed yet itself.

Thus, at most 22 edges are unavailable for $8m + 2$ because of its proximity to $\bar{a}_i$ $(1 \leq i \leq 5)$. Since $m \geq 9$, $4m - 13 \geq 23$. Therefore, there exists at least one available edge for $8m + 2$. Subdivide one such edge with the vertex $8m + 2$. There are now at least 27 edges between a_{11} and a_1 since the above subdivision created a new edge. There are at most 12 edges unavailable for $8m + 6$ due to vertices $\bar{a}_5$ and $\bar{a}_6$. Similarly, there are at most 8 edges unavailable for $8m + 6$ due to vertices $\bar{a}_4$ and $\bar{a}_7$. There are 2 edges made unavailable by $\bar{a}_8$. Now consider $8m + 2 = \overline{8m + 1}$. Since it is in the same branch as $8m + 6$, there must be at least 2 vertices separating them; thus 4 edges are made unavailable by $8m + 2$. This condition also satisfies the weaker requirement that $8m + 2$ and $8m + 6$ cannot be adjacent since there are 2 vertices separating $8m + 1$ and $8m + 5$.

Thus, at most 26 edges are unavailable for $8m + 6$. Since there are at least 27 edges, there is an edge between a_{11} and a_1 which can be satisfactorily subdivided with vertex $8m + 6$, completing Z_1'. Z_2' is completed by adding 2 to each number in Z_1' as described before. This completes the proof.

Little is known about $(D\ ;\ n)$-cages when n is even. However, when some additional restrictions are placed on the degree set, some conclusions are possible.

<u>Theorem 11</u>: Let $D = \{2, r, s\}$ where $r \geq 3$ and $s = r + 2$, $s = 2r - 2$, or $s = 2r$. Then $f(D\ ;\ 2n) = s(n - 1) + 3$, when $n \geq 2$.

<u>Proof</u>: Construct the tree $T(D\ ;\ 2n - 1)$ and let v be the additional vertex called for in Theorem 1. We know v can only be adjacent to vertices from the set $\{v_{n-1,\ell} \mid 1 \leq \ell \leq s\}$. If deg $v = s$, then no additional edges are possible and the graph so constructed has no vertex of degree r. If deg $v < s$, then some $v_{n-1,j}$ has degree 1. However, no additional edges between vertices of

$T(D\ ;\ 2n\text{-}1)$ are possible, so that th
degree set D. Thus, at least s(n-1)

To the tree $T(D\ ;\ 2n\text{-}1)$, add
insert the edges $w_1\ v_{n-1,j}$ $(1 \leq j \leq r)$ a

If $s = 2r$, insert the edges w_1
while if $s = 2r - 2$, insert the edg
$(r \leq k \leq 2r-2)$ along with the edge w_1w_2

In each case the graph formed
$s(n-1) + 3$.

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1. Capobianco, M. and Molluzzo, J
   Theory. North Holland, New Yo
2. Chartrand, G.; Gould, R. J. an
   degree set and girth. (Prepri
3. Harary, F. and Kovacs, P., Sma
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4. Harary, F. and Kovacs, P., Reg
5. Wong, P., A survey on cages.

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≥ 3 and s = r + 2, s = 2r − 2, or s = 2r.
2.

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T(D ; 2n-1) are possible, so that the graph thus constructed does not have degree set D. Thus, at least s(n-1) + 3 vertices are required.

To the tree T(D ; 2n-1), add the vertices w_1 and w_2. If s = r + 2, insert the edges $w_1 v_{n-1,j}$ $(1 \le j \le r)$ and $w_2 v_{n-1,k}$ $(r+1 \le k \le r+2)$.

If s = 2r, insert the edges $w_1 v_{n-1,j}$ $(1 \le j \le r)$ and $w_2 v_{n-1,k}$ $(r+1 \le k \le 2r)$. While if s = 2r − 2, insert the edges $w_1 v_{n-1,j}$ $(1 \le j \le r-1)$ and $w_2 v_{n-1,k}$ $(r \le k \le 2r-2)$ along with the edge $w_1 w_2$.

In each case the graph formed has degree set D, girth 2n, and order s(n-1) + 3.

REFERENCES

1. Capobianco, M. and Molluzzo, J., Examples and Counterexamples in Graph Theory. North Holland, New York (1978).
2. Chartrand, G.; Gould, R. J. and Kapoor, S. F., Graphs with prescribed degree set and girth. (Preprint.)
3. Harary, F. and Kovacs, P., Smallest graph with given girth pair. Caribbean J. Computational Math. (To appear.)
4. Harary, F. and Kovacs, P., Regular graphs with given girth pair. (Preprint.)
5. Wong, P., A survey on cages. (To appear in J. Graph Theory.)