Discrete Mathematics 309 (2009) 6299–6311

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A look at cycles containing specified elements of a graph

Ronald J. Gould

Emory University, Atlanta GA 30322, United States

ARTICLE INFO

Article history:
Received 11 September 2006
Accepted 3 April 2008
Available online 15 May 2008

Keywords:
Cycle
Hamiltonian cycle
Path
Ordered cycle

ABSTRACT

This article is intended as a brief survey of problems and results dealing with cycles containing specified elements of a graph. It is hoped that this will help researchers in the area to identify problems and areas of concentration.

1. Introduction

One of the most heavily studied areas in graph theory deals with questions concerning cycles. The development of this area has undergone a natural growth and evolution in the questions studied and results obtained. One particular subarea involves questions about cycles containing specified elements of a graph; either sets of vertices, sets of independent edges (possibly even disjoint paths) or a combination of paths and vertices (general linear forests).

This article is intended as a brief survey of this subarea. As such, it will be impossible to include all results of this kind. But I do hope to highlight many of the types of questions that have been asked and the results obtained to date. I also hope to include enough open problems to stimulate more work in this area. In part, this will be done by providing a simple and natural framework for questions of this kind. It is also expected that the reader is somewhat familiar with both graph theory in general and cycle problems in particular.

Throughout this article we consider finite simple graphs $G = (V, E)$, unless otherwise indicated. We reserve n to denote the order ($|V|$) and q the size ($|E|$) of G. Let $G[S]$ be the graph induced on the set S. We use $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degrees of G, respectively, and let $N(x)$ and $N(S)$ denote the neighborhood of the vertex x and set S, respectively. We use $\alpha(X)$ for the independence number of $G[X]$ and $\kappa(X)$ will denote the minimum cardinality of a set of vertices in G separating two vertices of X. For convenience and custom we will let $\alpha(G) = \alpha(V(G))$ and $\kappa(G) = \kappa(V(G))$.

Further, let $c(G)$ denote the circumference of G, that is, the length of a longest cycle, and $g(G)$ be the girth, that is, the length of a shortest cycle. Also, let

$$\sigma_k(G) = \min\{\deg x_1 + \cdots + \deg x_k \mid x_1, \ldots, x_k \text{ are independent in } G\}.$$

Graphs satisfying lower bounds on σ_k with $k = 2$ are often called Ore-type graphs, while if $k = 1$, Dirac-type graphs. If G is a bipartite graph with partite sets V_1 and V_2, we define

$$\sigma_{1,1} = \min\{\deg x + \deg y \mid x \in V_1, y \in V_2, xy \notin E(G)\}.$$

If G contains no induced subgraph isomorphic to any graph in the set $F = \{H_1, \ldots, H_k\}$, we say G is F-free, or H_1-free if F contains only H_1. For terms not defined here see [8].

E-mail address: rg@mathcs.emory.edu.

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.04.017

There is a natural framework for asking questions about cycles containing specified elements of a graph. This framework consists of four parts:

1. The type of cycle (or cycles) of interest.
   Here the natural choices include hamiltonian cycles, shortest cycles, long cycles (relative to the order), arbitrary cycles, and cycle systems such as sets of disjoint cycles, 2-factors or pancyclic-type set systems.
2. The conditions that allow formation of our cycle(s).
   Here lower bounds on connectivity, minimum degree, degree sum, cardinality of neighborhood unions and many others have all been used. Also, combinations of these conditions can provide different results.
   Widening the type of conditions that have been used to place elements on cycles will be a recurring question in this paper. Of special interest is the interplay between connectivity and other conditions such as degree conditions. Increasing connectivity beyond some absolute minimum value (often 2-connectedness) may allow us to relax other conditions.
3. The type of specified elements to be included in the cycle.
   Here the natural choices are a set of vertices of specified size, a set of vertices satisfying a property (like all vertices of maximum degree or all vertices of degree at least k), a set of independent edges of specified size, a set of independent edges satisfying a specified property, a set of disjoint paths containing a fixed number of edges, a combination of vertices and edges, or a linear forest with certain parameters.
4. Additional properties we wish our specified elements to possess.
   Here some natural choices include their order of appearance on the cycle(s), strong ordering (that is, assuming an orientation to the edges), distance distributions between elements, or any natural graph property that these elements can convey. Of course, sometimes we include no special properties.

One possibility is to ask any of these questions in a fixed special class of graphs like planar graphs or hypercubes. For lack of space, we shall not concentrate on these questions here.

2. Connectivity-based results

In this section we concentrate on results of graphs containing specified sets of elements, where some form of connectivity condition plays the major role. Connectivity is at the heart of all path and cycle questions, and hence is a perfect place to begin. We will see examples of results from various categories of our framework and we will use this framework to suggest possible future work.

In a classic paper, Dirac [17] showed several important results that have since then stimulated a great deal of further work. Dirac's results are summarized in the following.

Theorem 1. *In a k-connected graph G,*

1. *given any k vertices, there is a cycle containing the k vertices;*
2. *given any two edges and any $k-2$ vertices ($k \geq 2$), there is a cycle containing all of them.*

Watkins and Mesner [69] described the k-connected graphs which do not contain cycles through $k+1$ given vertices. Tsikopoulos [68] characterized the k-connected graphs in which every $k+1$ vertex set lies on a cycle. Other related work is due to Holton [43].

These ideas are further expanded by Holton and Plummer [44] and later Kaneko and Saito [46], who made the following definition.

Definition 1. A graph is said to satisfy property $P(m, l)$ ($m \geq l$) if for any $X \subseteq V(G)$ with $|X| = m$, there is a cycle C such that $|X \cap V(C)| = l$.

Thus, Dirac's Theorem shows that every k-connected graph satisfies $P(k, k)$. Using Dirac's Theorem it is easy to see that for any $2 \leq l \leq k$, every k-connected graph has property $P(k, l)$. Kaneko and Saito [46] further showed the following.

Theorem 2. 1. *If $k \geq 3$, every k-connected graph satisfies $P(k+1, k)$.*
2. *If $k \geq 3$, every k-connected graph of order at least $k+2$ satisfies $P(k+2, k)$.*
3. *If k, r are nonnegative integers such that $k \geq \max\{3, (2r-1)(r+1)\}$, then every k-connected graph of order at least $k+r$ satisfies $P(k+r, k)$.*

This last bound has recently been improved by Kawarabayashi [48].

Theorem 3. *Let k, r be nonnegative integers. If $k \geq \max\{3, 2r\}$, then every k-connected graph of order at least $k+r$ satisfies $P(k+r, k)$.*

Kawarabayashi [48] also conjectured the following.

Conjecture 1. *Let k, r be nonnegative integers. If $k \geq \max\{3, r+1\}$, then every k-connected graph of order at least $k+r$ satisfies $P(k+r, k)$.*

Although Watkins and Mesner [69] described the k-connected graphs that fail to satisfy $P(k+1,k)$, more can be said in certain special cases. Häggkvist and Mader [39] define the function $h(k)$ for $k \geq 3$ as the largest integer such that every k-connected k-regular graph satisfies $P(k+h(k), k+h(k))$. Examples of nonhamiltonian graphs from [59] show that $h(k) \leq 9k-11$ for $k \geq 3$ and a construction in [58] gives $h(k) \leq 5k-4$ for all $k = 4m$ with m a positive integer. Häggkvist and Mader [39] show the following:

Theorem 4. $h(k) \geq \lfloor \frac{\sqrt{k}}{3} \rfloor$.

Further, they conjecture that the true magnitude of $h(k)$ is ck where c is a constant greater than zero (and perhaps 1 or even 2).

Häggkvist and Thomassen [40] also generalized Dirac's Theorem.

Theorem 5. *Let G be a k-connected graph ($k \geq 2$).*

1. *For any set S of independent edges of size $k-1$, there is a cycle in G containing all edges of S;*
2. *For any set S of independent paths (except single vertices) of total length $k-1$, there is a cycle in G containing all elements of S.*

Denley and Wu [18] were able to further generalize this work. Here a cocircuit of a graph is a minimal edge-cut.

Theorem 6. *Let G be a k-connected graph ($k \geq 2$).*

1. *Let S be a set of independent paths (except single vertices) with a total of s edges and T be a set of t vertices, where $s+t=k$ and $t \geq 1$. Then there is a cycle of G containing each path of S and each vertex of T.*
2. *Given any two edges e and f and $k-2$ cocircuits $C_1^*, \ldots, C_{k-2}^*$, there is a cycle of G containing $\{e, f\}$ and meeting all of $C_1^*, \ldots, C_{k-2}^*$.*

Fournier [33] established a Chvátal–Erdös [15] type result with the following:

Theorem 7. *Let G be a 2-connected graph and let $X \subseteq V(G)$. If $\alpha(X) \leq \kappa(G)$, then G is X-cyclable, that is, there is a cycle containing all the vertices of X.*

In the following result from [6], Theorem 7 is generalized in part (1) while part (2) generalizes earlier work from [2].

Theorem 8. *Let G be a 2-connected graph of order n and $X \subseteq V(G)$.*

1. *If $\alpha(X) \leq \kappa(X)$, then G is X-cyclable.*
2. *If $\sigma_3(X) \geq n + \min\{\kappa(X), \delta(X)\}$, then G is X-cyclable.*

Using the local connectedness of X, Harant [41] showed the following result. Here $K_{X,Y}$ denotes the complete bipartite graph with partite sets X and Y.

Theorem 9. *Let G be a graph and $X \subseteq V(G)$:*

1. *such that $|X| = \kappa(X)+1 \geq 3$. If G has no cycle containing X, then $|X| = \alpha(X)$ and there is a set $Y \subseteq V(G) - X$ with $|Y| = |X|-1$ such that G contains a subdivision of $K_{X,Y}$ as a subgraph.*
2. *with an integer $p \geq 1$ and $\kappa(X) \geq 2$ and $|X| \leq \kappa(X)+p$, and a set P of p pairwise disjoint paths of $G[X]$ each containing at least one edge. If there is a cycle of G containing all paths of P as subpaths, then there is even a cycle of G containing X and all paths of P as subpaths.*

Corollary 10. *Given a graph G and $X \subseteq V(G)$:*

1. *with $3 \leq |X| \leq \kappa(X)+1$ and an edge e connecting two vertices of X, there is a cycle of G containing X and e.*
2. *with $|X| \leq \kappa(X)+2$ and two independent edges e and e' both connecting vertices of X, there is a cycle of G containing X, e and e'.*

Continuing with connectivity-based results, Bondy and Lovasz [5] provided the following.

Theorem 11. *If S is a set of k vertices in a $(k+1)$-connected graph G, then*

1. *the set of cycles of G through S generates the cycle space of G;*
2. *if G is nonbipartite, then G has an odd cycle through S.*

Finally, by considering a generalized Fan-type [23] degree condition (as the distance between u and v may not be two), Sakai [64] improved on an earlier result due to Egawa, Glas and Locke [21].

Theorem 12. *Let k, d be integers with $d \geq k \geq 3$. Let G be a k-connected graph of order at least $2d$ and let $X \subseteq V(G)$ with $|X| \leq k$. If $\max\{\deg u, \deg v\} \geq d$ for any nonadjacent distinct vertices u, v of G, then G is X-cyclable and the cycle has length at least $2d$.*

An alternate connectedness measure is toughness and, of course, there are results relating toughness with cycles containing prescribed vertices. However, this does seem to be an area where more results should be possible. In [55], the following result was given.

Theorem 13. *If G is a 1-tough graph of order n and $X \subseteq V(G)$ such that $\sigma_3(G) \geq n$ and, for all $x, y \in X$, dist$(x, y) = 2$ implies* $\max\{\deg x, \deg y\} \geq \frac{n-4}{2}$*, then G is X-cyclable.*

Stacho [66] showed that the degree requirement in Theorem 21 could be relaxed for 1-tough graphs. In order to understand this result, several definitions are needed. Let u and v be nonadjacent vertices in G. Let $\psi(u, v)$ be the number of components of $G[N(u)]$ containing no neighbor of v. Let $\alpha(u, v) = |N(u) \cap N(v)|$ and let $\beta(u, v) = |\{x| \ x \notin N(u) \cup N(v), \mathrm{dist}(u, x) = 2 \text{ or } \mathrm{dist}(v, x) = 2\}|$. Let $\mathrm{pos}(x) = \max\{0, x\}$ and finally let

$$\chi(u, v) = \mathrm{pos}(\min\{\psi(u, v), \psi(v, u) - 1\}) + \mathrm{pos}(\alpha(u, v) - \beta(u, v) - 1).$$

Theorem 14 (*[66]*). *Let G be a 1-tough graph of order n and let $W \subseteq V(G)$. If for each pair of nonadjacent vertices $x, y \in W$ we have*

$$\deg x + \deg y + \chi(u, v) \geq n,$$

then G contains a cycle through all vertices of W.

In discussions with Saito (personnel communication) the following problem emerged.

Problem 1. For each real number r, $0 < r \leq 1$, does there exist a function $f(r)$ so that any $\lfloor rn \rfloor$ vertices of an $f(r)$-tough graph lie on a common cycle?

Clearly, when $r = 1$, this is the well-known toughness problem of Chvátal [14] for hamiltonian graphs. It is also natural to wonder about an edge analogue to this problem.

We should also mention here that Gerlach and Harant [35] impose a local connectedness condition on a set X of vertices of G to show G is X-cyclable. While Kojima and Ando [51] have used the k-wide diameter to place k vertices on a cycle. Göring, Harant, Hexel and Tuza [37] investigate the set $M(k, c)$ of all pairs (n, l) of integers with $n \geq k$ and $l > \max\{2, k\}$ such that for every c-connected graph G of order n and for every $X \subseteq V(G)$ with $|X| = k$, there is a cycle C of G containing X and such that $|V(C)| < l$. Also, McCuaig [57] considered placing k independent edges on a cycle in cyclically $(k + 1)$-connected cubic graphs. Finally, Kelmans and Lomonosov [49] used T-separators to show that any set T of $k + 2$ vertices in a k-connected graph is on a cycle unless G has a T-separator.

3. Forbidden subgraphs

The use of forbidden subgraphs is a well-established tool in the study of cycles. But, until recently, little had been done in this area concerning our topic. Recently, Fujisawa, Ota, Sugiyama and Tsugaki [34] found a generalization of the results in [19] on {*claw*, *net*}-free graphs. Recall a claw is the complete bipartite graph $K_{1,3}$ and a net is a triangle with three additional vertices where each additional vertex is adjacent to a distinct vertex of the triangle. To understand the work in [34] we need the following definitions.

Let G be a graph and $S \subseteq V(G)$. An induced subgraph F is called an *S-claw* if F satisfies the following properties:

1. F consists of three paths P_1, P_2, P_3 such that they have only one common vertex x and $V(F) = V(P_1) \cup V(P_2) \cup V(P_3)$.
2. For any $i \in \{1, 2, 3\}$, the end vertex of P_i which is not x is contained in S.
3. For any $i \in \{1, 2, 3\}$, the internal vertices of P_i are contained in $V(G) - S$.
4. $E(F) = E(P_1) \cup E(P_2) \cup E(P_3)$.

Similarly, an induced subgraph $F' \subseteq G$ is called an *S-net* if F' satisfies the following properties:

1. F' contains a triangle T with $V(T) = \{x_1, x_2, x_3\}$.
2. There exist three vertex disjoint paths P_1, P_2, P_3 such that x_i is an end vertex of P_i and $V(F') = V(P_1) \cup V(P_2) \cup V(P_3)$.
3. For any $i \in \{1, 2, 3\}$, the end vertex of P_i which is not x_i is contained in S.
4. For any $i \in \{1, 2, 3\}$, the internal vertices of P_i are contained in $V(G) - S$.
5. $E(F) = E(P_1) \cup E(P_2) \cup E(P_3) \cup E(T)$.

Under these definitions, it is clear that a $\{V(G)$-claw, $V(G)$-net$\}$-free graph is a {claw, net}-free graph.

Theorem 15 (*[34]*). *Let G be an $\{$ S-claw, S-net$\}$-free graph, $S \subseteq V(G)$.*

1. *If G is connected, then G contains a path P such that $S \subseteq V(P)$.*
2. *If G is 2-connected, then G contains a cycle C such that $S \subseteq V(C)$.*

It is natural to wonder if other forbidden subgraph results can also be extended in this manner. The above authors have some results in this direction, in particular, involving P_6. Also, can an edge version also be found by considering the end edges of S-claws (defined in a similar manner) and S-nets, where now S is a set of disjoint edges?

A closure-type approach has also been used to find cycles containing specific elements. We say a vertex x is $*$-*eligible* if

1. x is not the center of a claw,
2. $G[N(x)]$ is not a complete graph,
3. there is a tree T such that,
   (a) $N(x) \subseteq V(T) \subseteq N^2(x)$,
   (b) for any $s \in S(T) = \{s \in V(T) \mid \deg_T(s) \geq 2\}$, the set $N(s) - N[x]$ induces a clique (possibly empty),
   (c) $V(T) - S(T) \subseteq N(x)$.

The local completion of G at x, denoted G_x^*, is the graph obtained from G by adding to $G[N(x)]$ all missing edges.

Now, let $H \subseteq V(G)$ be an arbitrary set of vertices and let $cl_H^*(G)$ be the graph obtained from G by recursively performing the local completion at the $*$-eligible vertices in H (this is clearly a local specialization of the standard closure of claw-free graphs). Using the above, Čada, Flandrin, Li and Ryjáček [7] obtained the following stability result and subsequent theorem.

Theorem 16. *Let G be a graph, $S \subseteq V(G)$, $S \neq \emptyset$, and let k be an integer, $1 \leq k \leq |S|$. Let $H \subseteq V(G)$ be an arbitrary set of vertices. Then G contains a cycle C with $|V(C) \cap S| \geq k$ if and only if $cl_H^*(G)$ contains a cycle C' with $|V(C') \cap S| \geq k$. Hence, for $H, S \subseteq V(G)$, S is cyclable in G if and only if S is cyclable in $cl_H^*(G)$.*

Theorem 17. *Let G be a 2-connected graph of order $n \geq 33$ and let $S \subseteq V(G)$, $S \neq \emptyset$ be such that*

1. *no vertex in $S \cup N(S)$ is a claw center,*
2. *$\sigma_3(G) \geq n - 2$.*

Then, S is cyclable in G.

4. The classic density approach

The most common conditions in cycle results are edge density conditions. These often take the form of degree conditions, or neighborhood union conditions, or some combination of the two. But powers, line graphs, closures and other graph operations that add edges to graphs can also be viewed here. Certainly it is no surprise that these conditions have been extensively used in creating results on cycles containing specified elements.

One of the oldest results of this kind is due to Pósa [63] (for minimum degree), and it was extended to σ_2 by Kronk [53]. Here a (k, t, s)-linear forest is a forest with exactly k edges, t total paths, with s of the paths being single vertices. When the number of single vertex paths in F is not critical, we will denote F simply as a (k, t)-linear forest. Further, a graph G is (k, t)-hamiltonian if for each (k, t)-linear forest F of G, there is a hamiltonian cycle of G containing the linear forest F.

Theorem 18 (*[63,53]*). *Let $0 \leq t \leq k$ be integers and G a graph of order n. If $\sigma_2(G) \geq n + k$, then for any $(k, t, 0)$-linear forest F, there is a hamiltonian cycle of G that contains the linear forest F. Also, the σ_2 bound is sharp with respect to general n and general $(k, t, 0)$-linear forests.*

Sugiyama [67] generalized this result as follows:

Theorem 19. *Let G be a graph on $n \geq 5$ vertices and S a set of $m \geq 0$ edges inducing a linear forest in G. If $\sigma_2(G) \geq n + m$, then for every $t = 0, 1, 2, \ldots, m$ there is a hamiltonian cycle C_t in G such that $|E(C_t) \cap S| = t$.*

The graph of Fig. 1 shows that Theorem 18 is sharp in some sense. The forest F is a single path of length k in the K_{k+1}. However, for forests other than a single path, it may not be sharp as was shown in [28].

Theorem 20 (*[28]*). *Let G be a graph of order n. Let k, t and n be positive integers with $2 \leq k + t \leq n$ and let F be a (k, t)-linear forest. If*

1. *$\sigma_2(G) \geq n + k$ when $F = P_{k+1} \cup (t - 1)K_1$, and*
2. *$\sigma_2(G) \geq n + k - \epsilon(k, n)$ otherwise,*

then G is (k, t)-hamiltonian, where $\epsilon(n, k) = 1$ if $2|(n - k)$ and $\epsilon(n, k) = 0$ otherwise. Furthermore, the condition on σ_2 is sharp.

In this result, the sharpness of part 2 is demonstrated by the graph obtained from the join of F and the complete bipartite graph $H = K_{(n-k+1+\epsilon)/2,(n-k-2t-1-\epsilon)/2}$. The bipartite graph has path cover number in excess of t and hence with the unbalanced nature of H, there is no hamiltonian cycle containing the forest F.

The next result is due independently to Bollobás and Brightwell [3] (as a corollary to a more general result) and Shi [65]. It uses the classic Dirac-type density condition for the subset S of $V(G)$. Recall, $\delta(S)$ is the minimum degree in G of a vertex of S.

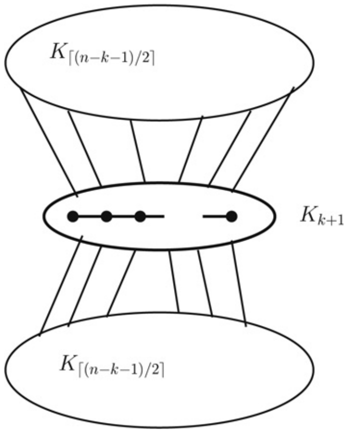

Fig. 1. The sharpness example for Theorem 18.

Theorem 21. *Let G be a 2-connected graph and S a subset of $V(G)$. If $\delta(S) \geq n/2$ then S is cyclable in G.*

Ota [62], as a corollary to a more general result, made the natural extension to degree sums of pairs of nonadjacent vertices in S, denoted $\sigma_2(S, G)$.

Theorem 22. *Let G be a 2-connected graph of order n and S a subset of $V(G)$. If $\sigma_2(S, G) \geq n$, then S is cyclable in G.*

This was further pushed to sums of three vertices in [31], extending an earlier result of Flandrin, Jung and Li [32].

Theorem 23 *([31]). Let G be a 2 connected graph of order n and S a subset of $V(G)$. If $\deg x + \deg y + \deg z \geq n + |N(x) \cap N(y) \cap N(z)|$ for any three independent vertices $x, y, z \in S$, then S is cyclable in G.*

The next result of Broersma, Li, Li, Tian and Veldman [6] extends the hamiltonian work in [2]. Here $\sigma_t(S, G)$ is the degree sum in G of any t pairwise nonadjacent vertices of S and $\delta(S)$ is the minimum degree in G of a vertex in S.

Theorem 24. *Let G be a 2-connected graph of order n and S a subset of $V(G)$. If $\sigma_3(S, G) \geq n + \min\{\kappa(S), \delta(S)\}$, then S is cyclable in G.*

While in [42] 3-connected graphs were studied.

Theorem 25. *Let G be a 3-connected graph of order n and S a subset of $V(G)$.*

1. *If $\sigma_4(S, G) \geq n + 2\alpha(S) - 2$, then S is cyclable.*
2. *If $\sigma_4(S, G) \geq n + \delta(S)$ and $\deg v \geq n/2$ for every $v \in S - (N(w) \cup \{w\})$, where $w \in S$ and $\deg w = \delta(S)$, then S is cyclable in G.*
3. *If $\sigma_2(G) \geq n/2 + \delta(G)$, then G is hamiltonian.*

Recently, a bipartite version for cyclable sets was also found.

Theorem 26 *([1]). Let $G = (X \cup Y, E)$ be a 2-connected balanced bipartite graph of order $2n$ and S is a subset of X. If $\deg x + \deg y \geq n + 1$ for every nonadjacent pair $x \in S$, $y \in Y$, then S is cyclable in G.*

Standard edge density conditions have also been used to place edges on cycles. If F is a 1-factor of G and there exists a hamiltonian cycle in G containing all edges of F, then we say G is F-hamiltonian. An early result of this type is due to Häggkvist [38].

Theorem 27. *Let G be a graph, $|V(G)| = n \geq 4$, n even. If $\sigma_2(G) \geq n + 1$, then for any 1-factor F, G is F-hamiltonian.*

Las Vergnas [54] made the natural transition to bipartite graphs.

Theorem 28. *Let $G = (A \cup B, E)$ be a bipartite with $|A| = |B| = n \geq 2$. If for each pair u, v of nonadjacent vertices with $u \in A$ and $v \in B$ we have $\deg u + \deg v \geq n + 1$, then for any 1-factor F of G, G is F-hamiltonian.*

Yang [70] provided a true edge density result as well. Suppose K_6 has vertex set $\{y_1, \ldots, y_6\}$. Let $S_1 = K_6$ minus the edges $\{y_1y_2, y_1y_4, y_2y_3, y_3y_4\}$. It is easy to see that if F is the matching $\{y_1y_3, y_2y_4, y_5y_6\}$, then S_1 is not F-hamiltonian.

Theorem 29. *Let G be a graph on n vertices ($n \geq 4$, n even). If $\delta(G) \geq 2$ and $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 1$, then for any 1-factor F of G, G is F-hamiltonian if and only if $G \ncong S_1$.*

Yang [70] also provided a bipartite version of the above result.

An extension of the idea of cyclable sets is the following. A graph G is said to be *S-pancyclable* if for every integer l, $3 \le l \le |S|$, there is a cycle in G that contains exactly l vertices of S. An Ore-type result is then:

Theorem 30 (*[30]*). *If G is a graph of order n and $\sigma_2(G) \ge n$, then either G is S-pancyclable or else n is even, $S = V(G)$ and $G = K_{n/2,n/2}$, or $|S| = 4$, $G[S] = K_{2,2}$ and the structure of G is well characterized.*

While in [1] bipartite graphs are considered.

Theorem 31. *Let G be a 2-connected balanced bipartite graph of order $2n$ and bipartition (X, Y). Let S be a subset of X of cardinality at least 3. Then if the degree sum of every pair of nonadjacent vertices $x \in S$ and $y \in Y$ is at least $n + 3$, then G is S-pancyclable.*

It is also natural to expect that a closure property would apply to problems of our type. Again in [7] this was considered. Of course, their first interest was in the stability of S-cyclability and S-pancyclability.

Theorem 32. *Let G be a graph of order n, let $S \subseteq V(G)$, $S \ne \emptyset$, and let k be an integer, $1 \le k \le |S|$. Let $u, v \in V(G)$ be such that $uv \notin E(G)$ and $\deg u + \deg v \ge n$. Then G contains a cycle C with $|V(C) \cap S| \ge k$ if and only if $G' = G + uv$ contains a cycle C' with $|V(C') \cap S| \ge k$.*

Now the *k-closure* of G is that graph obtained from G by recursively joining pairs of nonadjacent vertices x, y satisfying $\deg x + \deg y \ge k$ until no such pair remains. We denoted the resulting graph $C_k(G)$.

Theorem 33 (*[7]*). *Let G be a graph of order n, let $S \subseteq V(G)$, $|S| \ge 3$, and let $u, v \in V(G)$ be such that $uv \notin E(G)$ and*

$$\deg u + \deg v \ge n + |S| - 3.$$

Then S is pancyclable in G if and only if S is pancyclable in $G + uv$. Hence, S is pancyclable in G if and only if S is pancyclable in $C_{n+|S|-3}(G)$.

In [7], they then localize the closure as follows. For $S \subseteq V(G)$ and any integer k, define the (k, S)-closure of G denoted $C_k^S(G)$, as the graph obtained by recursively adding all missing edges uv with $\deg u + \deg v \ge k$, $u, v \in S$. The closure $C_k^S(G)$ is uniquely determined and if G is large while S is small, it is somewhat easier to handle. For $S \subset V(G)$, we say the *S-length* of a cycle in G is the number of vertices of S that the cycle contains. Then the *S-circumference* of G is the maximum S-length.

Theorem 34 (*[7]*). *Let G be a graph of order n and let $S \subseteq V(G)$, $|S| \ge 3$. Then*

1. *the S-circumference of G equals the circumference of $C_n^S(G)$,*
2. *S is cyclable in G if and only if S is cyclable in $C_n^S(G)$,*
3. *S is pancyclable in G if and only if S is pancyclable in $C_{n+|S|-3}^S(G)$.*

Next we consider one of the oldest ideas. A graph G of order n is said to be *vertex pancyclic* if for any vertex x, there is a cycle in G containing x of length l, for each l, $3 \le l \le n$. Bondy [4] initiated the study of pancyclic and vertex pancyclic graphs and he showed that if $\delta(G) \ge (n + 1)/2$, then G is vertex pancyclic. To date there are over 50 papers with results on vertex pancyclic graphs. Many of these results are density conditions. We shall not address them here. Instead, we wish to consider the next natural question: What about sets of more than one vertex?

Clearly, we cannot place k vertices on a 3-cycle when $k > 3$. Thus, we must adjust our idea of what pancyclic means. Recently, two approaches to this question appeared. The first we consider is due to Goddard [36].

Definition 2. For $k \ge 2$ we say G is *k-vertex pancyclic* if every set S of k vertices is in a cycle of every possible length. Further, G is *set-pancyclic* if G is k-vertex pancyclic for all $k \ge 2$.

Now by "possible length", Goddard means at least $k +$ path cover number of $G[S]$, where the path cover number of $G[S]$ is the least number of paths that cover all the vertices of $G[S]$. This is easily seen to be a reasonable range, since if $G[S]$ has path cover number t, then at least t new vertices will be needed to link the paths (containing our k vertices) into a cycle. Goddard [36] was able to show the following.

Theorem 35. *If G has order n and $\delta(G) \ge (n + 1)/2$, then G is set pancyclic.*

At the same time a second approach was developed in [26]. To understand this result, we develop some notation that will be useful later as well.

Definition 3. Let $k \ge 0$, $s \ge 0$, and $t \ge 1$ be fixed integers with $s \le t$ and G a graph of order n. For an integer m with $k + t \le m \le n$, a graph G is (k, t, s, m)-*pancyclic* if for each (k, t, s)-linear forest F, there is a cycle C_r of length r in G containing F for each $m \le r \le n$.

With this idea, the following was shown in [26].

Theorem 36. *Let $1 \leq t \leq m \leq n$ be integers, and G be a graph of order n. The graph G is $(0, t, t, m)$-pancyclic if $\sigma_2(G)$ satisfies any of the following conditions:*

1. $\sigma_2(G) \geq n$ *if* $m = n$,
2. $\sigma_2(G) \geq \lfloor (4n+1)/3 \rfloor$ *if* $t = 1$ *and* $m = 3$,
3. $\sigma_2(G) \geq 2n - 3$ *if* $t = 2$ *or* 3 *and* $m = 3$,
4. $\sigma_2(G) \geq 2n - m$ *if* $t = 3$ *and* $m = 4$ *or* 5,
5. $\sigma_2(G) \geq 2n - 2\lceil (m-1)/2 \rceil - 1$ *if* $4 \leq t \leq m < 2t$, $n > m$,
6. $\sigma_2(G) \geq n + 1$ *if* $t \geq 1$, $m \geq \max\{4, 2t\}$, $n > m$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

There is a corresponding δ result as well.

5. Disjoint cycles

In this section we consider the results where elements are distributed over disjoint cycles. We begin with a result of Ishigami and Jiang [45] which partially answered a conjecture of Enomoto (see [45]).

Theorem 37. *Let k a positive integer. If G is a graph of order $n \geq ck^2$ with c a large enough absolute constant, and $\delta(G) \geq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$, then for any k vertices, G contains k vertex disjoint cycles each of which contains one of the specified vertices. Furthermore, the cycles can be chosen so that each has length at most six.*

In [12], perfect matchings were placed on 2-factors of bipartite graphs, extending the F-hamiltonian result of Las Vergnas.

Theorem 38. *Let G be a balanced bipartite graph of order $2n$, $n \geq 9k$, and $\delta(G) \geq \frac{n+1}{2}$. Then for each perfect matching M of G, there exists a 2-factor with exactly k cycles, which contains all the edges of M.*

It should be noted that $n \geq 9k$ in the result may not be sharp, but examples are shown to indicate $n > 3k$ is necessary. Egawa et al. [22] considered partitions into cycles passing through specified edges. They obtained the following.

Theorem 39. *Let G be a graph. Suppose $k \geq 2$, $|V(G)| = n \geq 3k$ and either*

$$\sigma_2(G) \geq \max\left\{n + 2k - 2, \left\lfloor \frac{n}{2} \right\rfloor + 4k - 2\right\}$$

or

$$\delta(G) \geq \max\left\{\left\lceil \frac{n}{2} \right\rceil + k - 1, \left\lceil \frac{n+5k}{3} \right\rceil - 1\right\}.$$

Then for any independent edges $e_1, e_2, \ldots, e_k$, G can be partitioned into cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$.

Partitions into cycles containing vertices also have been considered in [20].

Theorem 40. *Let G be a graph of order n and suppose one of the following conditions is satisfied:*

1. $n = 3k$ *and* $\delta(G) \geq \frac{7k-2}{3}$,
2. $3k + 1 \leq n \leq 4k$ *and* $\delta(G) \geq \frac{2n+k-3}{3}$,
3. $4k \leq n \leq 6k - 2$ *and* $\delta(G) \geq 3k - 1$,
4. $n \geq 6k - 2$ *and* $\delta(G) \geq \frac{n}{2}$.

Then for any distinct vertices $v_1, v_2, \ldots, v_k$, the graph G can be partitioned into cycles $C_1, C_2, \ldots, C_k$ such that $v_i \in V(C_i)$.

Chen, Enomoto, Kawarabayashi, Ota, Lou and Saito [9] consider the bipartite analogue.

Theorem 41. *Suppose $G = (V_1, V_2, E)$ is a balanced bipartite graph with $|V_1| = |V_2| = n$, $k \geq 1$, $n \geq 4k - 2$, $\delta(G) \geq \frac{n+1}{2}$ and $v_1, v_2, \ldots, v_k$ are distinct vertices of G. Then either*

1. *G can be partitioned into k cycles $C_1, C_2, \ldots, C_k$ such that $v_i \in V(C_i)$, $1 \leq i \leq k$; or*
2. $k = 2$ *and* $G - \{v_1, v_2\} = 2K_{(n-1)/2,(n-1)/2}$.

In particular, v_1 and v_2 belong to different partite sets. Moreover, if $v_1 \in V_1$, then $V_2 - \{v_2\} \subseteq N_G(v_1)$ and $V_1 - \{v_1\} \subseteq N_G(v_2)$.

Matsumura [56] placed edges of a bipartite graph on short cycles.

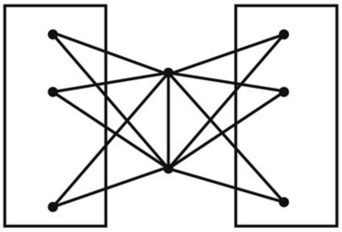

Fig. 2. A sharpness example for Theorem 46.

Theorem 42. *Let G be a bipartite graph of order $2n$. Suppose $k \geq 1$, $1 \leq s \leq k$, $n \geq 2k$ and*

$$\sigma_{1,1}(G) \geq \max\left\{\left\lceil \frac{4n+2s-1}{3} \right\rceil, \left\lceil \frac{2n-1}{3} \right\rceil + 2k\right\},$$

then for any independent edges $e_1, e_2, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$, $|C_i| \leq 6$, and there are at least s 4-cycles in $C_1, \ldots, C_k$.

And for minimum degree, Matsumura [56] obtained another conclusion.

Theorem 43. *Let G be a balanced bipartite graph of order $2n$. Suppose $k \geq 1$, $0 \leq s \leq k$, $n \geq 2k$, and*

$$\delta(G) \geq \max\left\{\left\lceil \frac{2n+2k+s}{4} \right\rceil, \left\lceil \frac{2n+4k}{5} \right\rceil\right\}.$$

Then for any independent edges $e_1, e_2, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$, $|C_i| = 4$, $1 \leq i \leq s$, and $|C_i| \leq 6$ for $s+1 \leq i \leq k$.

Then, as a corollary of these two results we obtain:

Theorem 44 (*[56]*). *Let G be a balanced bipartite graph of order $2n$. Let $k \geq 1$, $n \geq 2k$, and either*

$$\sigma_{1,1}(G) \geq \left\lceil \frac{4n+2k-1}{3} \right\rceil$$

or

$$\delta(G) \geq \left\lceil \frac{2n+3k}{4} \right\rceil.$$

Then for any independent edges $e_1, e_2, \ldots, e_k$, G contains k disjoint 4-cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$.

Finally, in [10], the following bipartite analogue was obtained.

Theorem 45. *Let G be a balanced bipartite graph of order $2n$. Let $k \geq 2$, $n \geq 2k$ and either $\sigma_{1,1}(G) \geq \max\{n+k, \lceil \frac{2n-1}{3} \rceil\}$ or $\delta(G) \geq \max\{\frac{n+k}{2}, \lceil \frac{2n+4k}{5} \rceil\}$, then for any independent edges $e_1, e_2, \ldots, e_k$, G can be partitioned into cycles $C_1, C_2, \ldots, C_k$ such that $e_i \in C_i$.*

6. Sets with extra properties

Next we turn our attention to some results that impose an extra property on the elements. The first is an interesting result due to Kaneko and Yoshimoto [47] concerning distributing the vertices of a specified set on a hamiltonian cycle. Here $\mathrm{dist}_C(u, v)$ denotes the distance between vertices u and v along the cycle C.

Theorem 46. *Let G be a graph with n vertices and $\delta(G) \geq n/2$, and let d be a positive integer such that $d \leq n/4$. Then, for any vertex subset A with $|A| \leq n/2d$, there is a hamiltonian cycle C such that $\mathrm{dist}_C(u, v) \geq d$ for any $u, v \in A$.*

This result is sharp as can be seen from the graph $(K_{n/2-1} \cup K_{n/2-1}) + K_2$ of Fig. 2. When all vertices of A are placed on one side, the bound on $|A|$ becomes clear. This example makes one wonder if a stronger result (in the sense of distribution distance) can be obtained with a higher connectivity assumption?

Now we turn to another strong hamiltonian property introduced by Chartrand (see [60]). A graph is *k-ordered* (*hamiltonian*) if for every ordered sequence of k vertices there is a (hamiltonian) cycle that encounters the vertices of the sequence in the given order. Clearly, every hamiltonian graph is 3-ordered hamiltonian.

Ng and Schultz [60] were the first to investigate such graphs.

Theorem 47 (*[60]*). *Let G be a graph of order n and let k be an integer with $3 \le k \le n$. If $\deg u + \deg v \ge n + 2k - 6$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

Corollary 48 (*[60]*). *Let G be a graph of order n and let k be an integer with $3 \le k \le n$. If $\deg u \ge n/2 + k - 3$ for every vertex u of G, then G is k-ordered hamiltonian.*

Clearly, this theorem and corollary are analogs of Ore's [61] and Dirac's [16] fundamental results, respectively. Both bounds for k-ordered hamiltonicity were improved for small k with respect to n. Theorem 47 was improved by Faudree, Faudree, Gould, Jacobson and Lesniak [25].

Theorem 49 (*[25]*). *Let $k \ge 3$ be an integer and let G be a graph of order $n \ge 53k^2$. If $\deg u + \deg v \ge n + (3k - 9)/2$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

Corollary 48 was improved by Kierstead, Sárközy and Selkow [50] as follows.

Theorem 50 (*[50]*). *Let $k \ge 2$ be an integer and let G be a graph of order $n \ge 11k - 3$. If $\deg u \ge \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$ for every vertex u of G, then G is k-ordered hamiltonian.*

The bound on n above was improved to $n \ge 5k + 6$ (for a more general linkage problem) in [52]. Note that the bound on $\delta(G)$ above is sharp. Unexpectedly, for small k, the Dirac-type bound does not follow from the Ore-type bound. In [29], this was further investigated and the following shown:

Theorem 51 (*[29]*). *Let k be an integer with $3 \le k \le n/2$ and let G be a graph of order n. If $\deg u + \deg v \ge n + (3k - 9)/2$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

The bound in Theorem 51 is sharp and for large k it implies the bound of Dirac-type. Thus,

(a) for large k, the Ore-type bound yields the Dirac-type bound;
(b) for small k, the Ore-type bound is more than twice the Dirac-type bound; and
(c) for moderate k, the situation is still not clear.

We summarize the above more precisely as follows. Let $\delta(n, k)$ be the smallest integer m for which any graph of order n with minimum degree at least m is k-ordered hamiltonian. The following theorem is from [29].

Theorem 52. *For positive integers k, n with $3 \le k \le n$ we have*

1. $\delta(n, k) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$, *for* $k \le (n + 3)/11$;
2. $\delta(n, k) > \frac{n}{2} + \frac{k}{2} - 2$, *for* $(n + 3)/11 < k \le n/3$;
3. $\delta(n, k) \ge 2k - 2$, *for* $n/3 < k < 2(n + 2)/5$;
4. $\delta(n, k) = \left\lceil \frac{n}{2} + \frac{3k-9}{4} \right\rceil$, *for* $2(n + 2)/5 \le k \le n/2$;
5. $\delta(n, k) = n - 2$, *for* $n/2 < k \le 2n/3$; *and*
6. $\delta(n, k) = n - 1$, *for* $2n/3 < k \le n$.

A slight improvement in the minimum degree condition is possible when considering graphs with larger connectivity.

Theorem 53 (*[13]*). *Let G be a graph on n vertices with minimum degree $\delta(G) \ge n/2$. Let $k \le n/176$ be an integer. If G is $3\lfloor k/2 \rfloor$-connected, then G is k-ordered hamiltonian.*

The connectivity bound is best possible, as illustrated by the following graph G. Let L, M, R be complete graphs with $|R| = \lfloor k/2 \rfloor$, $|M| = 2\lfloor k/2 \rfloor - 1$, $|L| = n - |M| - |R|$. Let G' be the union of these three graphs, adding all possible edges containing vertices of M. Let $x_i \in L$ if i is odd, and let $x_i \in R$ otherwise. Add all edges $x_i x_j$ whenever $|i - j| \notin \{0, 1, k - 1\}$, and the resulting graph is G. The degree sum condition is satisfied and G is $(\lfloor 3k/2 \rfloor - 1)$-connected. But there is no cycle containing the x_i in the proper order, since such a cycle would contain $2\lfloor k/2 \rfloor$ paths through M.

Order properties can be applied to more than vertex sets. For $k \ge 0$ and $0 \le s \le t$ fixed integers, a graph G of order n is (k, s, t)-*ordered hamiltonian* if there is a hamiltonian cycle C that contains any linear forest with k edges, t paths and with s of the paths being single vertices and respecting the order of the paths. The graph is *strongly* (k, s, t)-*ordered* if both the order of the paths and orientation of the paths are respected.

Theorem 54 (*[11]*). *If $0 \le s \le t \le k$ are fixed integers, and G is a graph of order $n \ge \max\{178t + k, 8t^2 + k\}$ with*

1. $\sigma_2(G) \ge n + k - 3$ *if* $s = 0, t \ge 3$,
2. $\sigma_2(G) \ge n + k + s - 4$ *if* $0 < 2s \le t, t \ge 3$,
3. $\sigma_2(G) \ge n + k + (t - 9)/2$ *if* $2s > t \ge 3$,
4. $\sigma_2(G) \ge n + k - 2$ *if* $s \le 1, t = 2$,
5. $\sigma_2(G) \ge n + k - 1$ *if* $s = 0, t = 1$,

6. $\sigma_2(G) \geq n$ *if* $s = t \leq 2$,

then G *is strongly* (k, t, s)*-ordered hamiltonian.*

The sharpness of this result for case (1) is shown by the following graph. Let G consist of three complete graphs: $A = K_{\frac{n-k+2}{2}}$, $K = K_{k-2}$, $B = K_{\frac{n-k+2}{2}}$. Add all edges between A and K and all edges between K and B. The degree sum condition is just missed and G is not $(k, t, 0)$-ordered. The ordered linear forest F is placed so that x_1, the first vertex of the first path, is in A and y_k, the last vertex of the last path, is in B and $k-2$ intermediate vertices of F are placed in K (recall F has $k+t$ vertices and $t \geq 3$ here). Similar graphs exist for the other cases.

The situation for minimum degree was considered by Faudree and Faudree [24].

Theorem 55. *Let* $k \geq 1$ *and* $0 \leq s < t$ *be integers, and* G *a graph of sufficiently large order* n*. The graph* G *is strongly* (k, t, s)*-ordered hamiltonian if* $\delta(G)$ *satisfies any of the following conditions:*

1. $\delta(G) \geq (n + k + t - 3)/2$ *when* $t \geq 3$,
2. $\delta(G) \geq (n + k)/2$ *when* $t \leq 2$.

Also, all the conditions on $\delta(G)$ *are sharp.*

Faudree and Faudree [24] also considered the not strong case, where the bounds are slightly different.

Theorem 56. *Let* $k \geq 1$ *and* $0 \leq s \leq t$ *be integers and* G *a graph of sufficiently large order* n*. The graph* G *is* (k, s, t)*-ordered hamiltonian if* $\sigma_2(G)$ *satisfies any of the following conditions:*

1. $\sigma_2(G) \geq n + k + t - 5$ *when* $s = 0$ *and* $t \geq 5$,
2. $\sigma_2(G) \geq n + k + t - 4$ *when* $s = 0$ *and* $t = 4$,
3. $\sigma_2(G) \geq n + k + t + s - 6$ *when* $0 < 2s \leq t$, $s \geq 3$, *and* $t \geq 6$,
4. $\sigma_2(G) \geq n + k + t - 3$ *when* $0 < 2s \leq t$, $s = 1, 2$, $t \geq 3$ *or* $s = 0$, $t = 3$,
5. $\sigma_2(G) \geq n + k + (3t - 9)/2 - \lceil 4(1 - s/t) \rceil$ *when* $3 \leq t < 2s$ *and* $(s, t) \neq (3, 5)$ *or* $(2, 3)$,
6. $\sigma_2(G) \geq n + k + t - 3$ *when* $s = 3$ *and* $t = 5$ *or* $s = 2$ *and* $t = 3$,
7. $\sigma_2(G) \geq n + k$ *when* $t \leq 2$.

Also, all of the conditions on $\sigma_2(G)$ *are sharp.*

Their corresponding minimum degree result is the following:

Theorem 57. *Let* $k \geq 1$ *and* $0 \leq s \leq t$ *be integers and* G *a graph of sufficiently large order* n*. The graph* G *is* (k, s, t)*-ordered hamiltonian if* $\delta(G)$ *satisfies any of the following conditions:*

1. $\delta(G) \geq (n + k + t - 5)/2$ *when* $s = 0$ *and* $t \geq 5$,
2. $\delta(G) \geq (n + k + t - 4)/2$ *when* $s = 1$ *and* $t \geq 4$ *or* $s = 0$ *and* $t = 4$,
3. $\delta(G) \geq (n + k + t - 3)/2$ *when* $1 < s < t$ *and* $t \geq 3$ *or* $s = 0, 1$ *and* $t = 3$.
4. $\delta(G) \geq (n + k)/2$ *when* $t \leq 2$.

Also, all of the conditions on $\delta(G)$ *are sharp.*

This work was extended to the generalized pancyclic case in [26]. Here pancyclic ordered means the set is ordered and placed on cycles for a range of lengths, namely for cycles of lengths m to n.

Theorem 58. *Let* $4 \leq t \leq m \leq n$ *be positive integers and let* G *be a graph of order* n*. Then* G *is* $(0, t, t, m)$*-pancyclic ordered if* $\sigma_2(G)$ *satisfies any of the following conditions:*

1. $\sigma_2(G) \geq 2n - 3$ *when* $t \leq m < \lfloor 3t/2 \rfloor$,
2. $\sigma_2(G) \geq 2n - 4$ *when* $\lfloor 3t/2 \rfloor \leq m < \lceil (5t - 2)/3 \rceil$,
3. $\sigma_2(G) \geq 2n - 5$ *when* $\lceil (5t - 2)/3 \rceil \leq m < 2t$,
4. $\sigma_2(G) \geq n + 4t - m - 6$ *when* $2t \leq m \leq (5t - 3)/2$,
5. $\sigma_2(G) \geq n + (3t - 9)/2$ *when* $m > (5t - 3)/2$.

Also, all of the conditions on $\sigma_2(G)$ *are sharp.*

Minimum degree conditions vary here and are found in [27].

Theorem 59 (*[27]*). *Let* $4 \leq t \leq m \leq n$ *be positive integers, and let* G *be a graph of sufficiently large order* n*. The graph* G *is* $(0, t, t, m)$*-pancyclic ordered if* $\delta(G)$ *satisfies any of the following conditions (where* $\epsilon_n = n - 2\lfloor n/2 \rfloor$*):*

1. $\delta(G) = n - 1$ *when* $t \leq m < \lfloor 3t/2 \rfloor$,
2. $\delta(G) \geq n - 2$ *when* $\lfloor 3t/2 \rfloor \leq m < 2t$,
3. $\delta(G) \geq n/2 + 2$, *when* $m = 10$ *or* 11, $t = 5$ *and* n *even.*
4. $\delta(G) \geq n/2 + 7/2$, *when* $m = 12$, $t = 6$ *and* n *odd.*
5. $\delta(G) \geq \lceil n/2 \rceil + \lfloor t/2 \rfloor + p$ *when* $m = 3t - 2p - 6 - \epsilon_n$ *for* $-1 < p \leq (t - 6 - \epsilon_n)/2$
6. $\delta(G) \geq \lceil n/2 \rceil + \lfloor t/2 \rfloor - 1$ *when* $m \geq \max\{2t, 3t - 4 - \epsilon_n\}$, *unless* $m = 11$, $t = 5$ *and* n *even.*

7. Conclusions

From our framework of questions and the results we have seen, it is obvious that many variations remain unexplored. Further, one must wonder if there are other, as yet unexplored, properties that we may place on our elements? Finally, we again wish to stress the relationship between connectivity and other conditions which appear in these results. Much remain to explore as to how higher connectivity conditions can change the other requirements in these results.

Acknowledgements

The author would like to thank the referees for their very helpful comments.

References

[1] M.E.K. Abderrezzak, E. Flandrin, D. Amar, Cyclability and pancyclability in bipartite graphs, Discrete Math. 236 (1–3) (2001) 3–11.
[2] D. Bauer, H.J. Broersma, R. Li, H.J. Veldman, A generalization of a result of Häggkvist and Nicoghossian, J. Combin. Theory B 47 (1989) 237–243.
[3] B. Bollobás, G. Brightwell, Cycles through specified vertices, Combinatorica 13 (1993) 147–155.
[4] J.A. Bondy, Pancyclic Graphs I, J. Combin. Theory Ser. B 11 (1971) 80–84.
[5] J.A. Bondy, L. Lovasz, Cycles through specified vertices of a graph, Combinatorica 1 (1981) 117–140.
[6] H. Broersma, H. Li, J.P. Li, F. Tian, H.J. Veldman, Cycles through subsets with large degree sums, Discrete Math. 171 (1997) 43–54.
[7] R. Čada, E. Flandrin, H. Li, Z. Ryjáček, Cycles through given vertices and closures, Discrete Math. 276 (2004) 65–80.
[8] G. Chartrand, L. Lesniak, Graphs & Digraphs, third edition, Chapman & Hall, London, 1996.
[9] G. Chen, H. Enomoto, K. Kawarabayashi, D. Lou, K. Ota, A. Saito, Vertex-disjoint cycles containing specified vertices in a bipartite graph, J. Graph Theory 46 (3) (2004) 145–166.
[10] G. Chen, H. Enomoto, K. Kawarabayashi, D. Lou, K. Ota, A. Saito, Vertex-disjoint cycles containing specified edges in a bipartite graph, Australas. J. Combin. 23 (2001) 37–48.
[11] G. Chen, R.J. Faudree, R.J. Gould, M.S. Jacobson, L. Lesniak, F. Pfender, Linear forests and ordered cycles, Discuss. Math. Graph Theory 24 (2004) 359–372.
[12] G. Chen, R.J. Gould, M.S. Jacobson, On 2-factors containing 1-factors in bipartite graphs, Discrete Math. 197–198 (1999) 185–194.
[13] G. Chen, R.J. Gould, F. Pfender, New conditions for k-ordered hamiltonian graphs, Ars Combin. 70 (2004) 245–255.
[14] V. Chvátal, Tough graphs and Hamiltonian circuits, Discrete Math. 5 (1973) 215–223.
[15] V. Chvátal, P. Erdös, A note on hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
[16] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.
[17] G.A. Dirac, In abstraken Graphen vorhandene vollstiändige 4-Graphen und ihre unterteilungen, Math. Nachr. 22 (1960) 61–85.
[18] T. Denley, H. Wu, A generalization of a Theorem of Dirac, J. Combin. Theory B 82 (2001) 322–326.
[19] D. Duffus, R.J. Gould, M.S. Jacobson, Forbidden subgraphs and the hamiltonian theme, in: Chartrand, Alavi, Goldsmith, Lesniak, Lick (Eds.), The Theory and Applications of Graphs, 1981, pp. 297–316.
[20] Y. Egawa, H. Enomoto, R.J. Faudree, H. Li, I. Schiermeyer, Two-factors each component of which contains a specified vertex, J. Graph Theory 43 (2003) 188–198.
[21] Y. Egawa, R. Glas, S. Locke, Cycles and paths through specified vertices in k-connected graphs, J. Combin. Theory Ser. B 52 (1) (1991) 20–29.
[22] Y. Egawa, R.J. Faudree, E. Gyori, Y. Ishigami, R.H. Schelp, H. Wang, Vertex-disjoint cycles containing specified edges, Graphs Combin. 16 (2000) 81–92.
[23] G.H. Fan, New sufficient condition for cycles in graphs, J. Combin. Theory B 37 (1984) 221–227.
[24] J.R. Faudree, R.J. Faudree, Hamiltonian cycles containing ordered linear forests, preprint.
[25] J.R. Faudree, R.J. Faudree, R.J. Gould, M.S. Jacobson, L. Lesniak, On k-ordered graphs, J. Graph Theory 35 (2) (2000) 69–82.
[26] R.J. Faudree, R.J. Gould, M.S. Jacobson, L. Lesniak, Generalizing pancyclic and k-ordered graphs, Graphs Combin. 20 (2004) 291–309.
[27] R.J. Faudree, R.J. Gould, M.S. Jacobson, L. Lesniak, Minimum degree and (k, m)-pancyclic ordered graphs, Graphs Combin. 2 (2005) 197–211.
[28] R.J. Faudree, R.J. Gould, M.S. Jacobson, Pancyclic graphs and linear forests, Discrete Math. (in press).
[29] R.J. Faudree, R.J. Gould, A. Kostochka, L. Lesniak, I. Schiermeyer, A. Saito, Degree Conditions for k-ordered hamiltonian graphs, J. Graph Theory 42 (3) (2003) 199–210.
[30] O. Favaron, E. Flandrin, H. Li, F. Tian, An Ore-type condition for pancyclability, Discrete Math. 206 (1–3) (1999) 139–144.
[31] E. Favaron, E. Flandrin, H. Li, Y.P. Liu, F. Tian, Sequences, claws and cyclability, J. Graph Theory 21 (1996) 357–369.
[32] E. Flandrin, H.A. Jung, H. Li, Hamiltonism, degree sum and neighborhood intersection, Discrete Math. 90 (1991) 41–52.
[33] I. Fournier, Thése d'Etat, LRI, Université de Paris-Sud, France, 1985.
[34] J. Fujisawa, K. Ota, T. Sugiyama, M. Tsugaki, Forbidden subgraphs and the existence of paths and cycles through specified vertices, preprint.
[35] T. Gerlach, J. Harant, On cycles through a set of specified vertices, Stud. Univ. Žilina Math. Ser. 16 (1) (2003) 35–46.
[36] W. Goddard, Minimum degree conditions for cycles including specified sets of vertices, Graphs Combin. 20 (4) (2004) 467–483.
[37] F. Göring, J. Harant, E. Hexel, Zs. Tuza, On short cycles through prescribed vertices of a graph, Discrete Math. 286 (2004) 67–74.
[38] R. Häggkvist, On F-hamiltonian graphs, in: J.A. Bondy, U.S.R. Murty (Eds.), Graphs and Related Topics, Academic Press, New York, 1979, pp. 219–231.
[39] R. Häggkvist, W. Mader, Circuits through prescribed vertices, J. Graph Theory 39 (2002) 145–163.
[40] R. Häggkvist, C. Thomassen, Circuits through specified edges, Discrete Math. 41 (1982) 29–34.
[41] J. Harant, On paths and cycles through specified vertices, Discrete Math. 286 (2004) 95–98.
[42] A. Harkat-Benhamdine, H. Li, F. Tian, Cyclability of 3-connected Graphs, J. Graph Theory 34 (3) (2000) 191–203.
[43] D.A. Holton, Cycles through specified vertices in k-connected regular graphs, Ars Combin. 13 (1982) 129–143.
[44] D. Holton, M.D. Plummer, Cycles through prescribed and forbidden point sets, Ann. Discrete Math. 16 (1982) 53–62.
[45] Y. Ishigami, T. Jiang, Vertex-disjoint cycles containing prescribed vertices, J. Graph Theory 42 (4) (2003) 276–296.
[46] A. Kaneko, A. Saito, Cycles intersecting a prescribed vertex set, J. Graph Theory 15 (6) (1991) 655–664.
[47] A. Kaneko, K. Yoshimoto, On a Hamiltonian cycle in which specified vertices are uniformly distributed, J. Combin. Theory B 81 (2001) 100–109.
[48] K. Kawarabayashi, Cycles through a prescribed vertex set in N-connected graphs, J. Combin. Theory B 90 (2004) 315–323.
[49] A.K. Kelmans, M.V. Lomonosov, On cycles through prescribed vertices in weakly separable graphs, Discrete Math. 46 (2) (1983) 183–189.
[50] H. Kierstead, G. Sárközy, S. Selkow, On k-Ordered Hamiltonian Graphs, J. Graph Theory 32 (1999) 17–25.
[51] T. Kojima, K. Ando, Minimum length of cycles through specified vertices in graphs with wide-diameter at most d, Ars Combin. 58 (2001) 245–256.
[52] A. Kostochka, G. Yu, An extremal problem for H-linked graphs, J. Graph Theory 50 (4) (2005) 321–339.
[53] H. Kronk, A generalization of a Theorem of Pósa, Proc. Amer. Math. Soc. 21 (1969) 77–78.
[54] M. Las Vergnas, Thesis, University of Paris, Paris, 1972.
[55] J. Li, R. Shen, F. Tian, Cycles containing given subsets in 1-tough graphs, Ars Combin. 58 (2001) 193–204.
[56] H. Matsumura, Vertex disjoint 4-cycles containing specified edges in a bipartite graph, Discrete Math. 297 (2005) 78–90.
[57] W. McCuaig, Cycles through edges in cyclically k-connected cubic graphs, Discrete Math. 103 (1) (1992) 95–98.
[58] W.D. McCuaig, M. Rosenfeld, Cyclicability or r-regular r-connected graphs, Bull. Austral Math. Soc. 29 (1984) 1–11.

[59] G.H.J. Meredith, Regular n-valent n-connected nonhamiltonian non-n-edge-colorable graphs, J. Combin. Theory B 14 (1973) 55–60.
[60] L. Ng, M. Schultz, k-Ordered hamiltonian graphs, J. Graph Theory 24 (1997) 45–57.
[61] O. Ore, A note on hamiltonian circuits, Amer. Math. Monthly 67 (1960) 55.
[62] K. Ota, Cycles through prescribed vertices with large degree sum, Discrete Math. 145 (1995) 201–210.
[63] L. Pósa, On the circuits of finite graphs, Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963) 355–361.
[64] T. Sakai, Long paths and cycles through specified vertices in k-connected graphs, Ars Combin. 58 (2001) 33–65.
[65] R. Shi, 2-neighborhoods and hamiltonian conditions, J. Graph Theory 16 (1992) 267–271.
[66] L. Stacho, Cycles through specified vertices in 1-tough graphs, Ars. Combin. 56 (2000) 263–269.
[67] T. Sugiyama, Hamiltonian cycles through a linear forest, SUT J. Math. 40 (2) (2004) 103–109.
[68] N. Tsikopoulos, Cycles through $k+2$ vertices in k-connected graphs, Discrete Math. 50 (1) (1984) 113–114.
[69] M.E. Watkins, D.M. Mesner, Cycles and connectivity in graphs, Canad. J. Math. 19 (1967) 1319–1328.
[70] Z. Yang, Note on F-hamiltonian graphs, Discrete Math. 196 (1999) 281–286.