Discrete Mathematics 309 (2009) 3811–3820

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Disjoint hamiltonian cycles in bipartite graphs

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ARTICLE INFO

Article history:
Received 4 May 2005
Received in revised form 28 July 2008
Accepted 20 October 2008
Available online 21 December 2008

Keywords:
Graph
Degree sum
Bipartite
Disjoint hamiltonian cycles

ABSTRACT

Let $G = (X, Y)$ be a bipartite graph and define $\sigma_2^2(G) = \min\{d(x) + d(y) : xy \notin E(G), x \in X, y \in Y\}$. Moon and Moser [J. Moon, L. Moser, On Hamiltonian bipartite graphs, Israel J. Math. 1 (1963) 163–165. MR 28 # 4540] showed that if G is a bipartite graph on $2n$ vertices such that $\sigma_2^2(G) \geq n + 1$, then G is hamiltonian, sharpening a classical result of Ore [O. Ore, A note on Hamilton circuits, Amer. Math. Monthly 67 (1960) 55] for bipartite graphs. Here we prove that if G is a bipartite graph on $2n$ vertices such that $\sigma_2^2(G) \geq n + 2k - 1$, then G contains k edge-disjoint hamiltonian cycles. This extends the result of Moon and Moser and a result of R. Faudree et al. [R. Faudree, C. Rousseau, R. Schelp, Edge-disjoint Hamiltonian cycles, Graph Theory Appl. Algorithms Comput. Sci. (1984) 231–249].

1. Introduction and terminology

For any graph G, let $V(G)$ and $E(G) \subseteq V(G) \times V(G)$ denote the sets of vertices and edges of G respectively. An edge between two vertices x and y in $V(G)$ shall be denoted xy. Furthermore, let $\delta(G)$ denote the minimum degree of a vertex in G. For a given subgraph H of G and vertices x and y in H, we will let $\text{dist}_H(x, y)$ denote the distance from x to y in H. Also, for convenience, given a path P in G and u, v in $V(P)$, let uPv denote the subpath of P that starts at the vertex u and ends at the vertex v. Given two disjoint sets of vertices X and Y in $V(G)$, we let $E_G(X, Y)$ denote the set of edges in G with one endpoint in X and one endpoint in Y. Similarly, we will let $\delta(X, Y)$ denote the minimum degree between vertices of X and Y. A useful reference for any undefined terms is [1].

We assume that all cycles have an implicit clockwise orientation and, for convenience, given a vertex v on a cycle C we will let v^+ denote the successor of v along C. Along the same lines, given an x–y path P, we will let v^+ denote the successor of a vertex v in $V(P)$ as we traverse P from x to y. Analogously, we define v^- to be the predecessor of a vertex v on C or P. Given a set of vertices $S \subseteq C$ $(\subseteq P)$, we let S^+ denote the set $\{s^+ \mid s \in S\}$. The set S^- is defined analogously.

If G is bipartite with bipartition (X, Y) we will write $G = (X, Y)$. If $|X| = |Y|$, then we will say that G is *balanced*. A *proper pair* in G is a pair of non-adjacent vertices (x, y) with x in X and y in Y.

We shall denote a cycle on t vertices by C_t. A *hamiltonian cycle* in a graph G is a cycle that spans $V(G)$ and, if such a cycle exists, G is said to be *hamiltonian*. Hamiltonian graphs and their properties have been widely studied. A good reference for recent developments and open problems is [3].

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doi:10.1016/j.disc.2008.10.026

In general, we are interested in degree conditions that ensure hamiltonian cycles in a graph. For an arbitrary graph G, we define $\sigma_2(G)$ to be the minimum degree sum of non-adjacent vertices in G. Of interest for our work here is Ore's Theorem [5], which uses this parameter.

Theorem 1.1 ([5]). *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then G is hamiltonian.*

In a bipartite graph G, we are interested instead in the parameter $\sigma_2^2(G)$, defined to be the minimum degree sum of a proper pair. Moon and Moser [4] extended Ore's theorem to bipartite graphs as follows.

Theorem 1.2 (Moon, Moser 1960). *If $G = (X, Y)$ is a balanced bipartite graph on $2n$ vertices such that $\sigma_2^2(G) \geq n + 1$, then G is hamiltonian.*

Faudree, Rousseau and Schelp [2] were able to give Ore-type degree-sum conditions that ensured the existence of many disjoint hamiltonian cycles in an arbitrary graph.

Theorem 1.3 ([2]). *If G is a graph on n vertices such that $\sigma_2(G) \geq n + 2k - 2$ then for n sufficiently large, G contains k edge-disjoint hamiltonian cycles.*

In this paper we will extend the previous two results by proving the following.

Theorem 1.4. *If $G = (X, Y)$ is a balanced bipartite graph of order $2n$, with $n \geq 128k^2$ such that $\sigma_2^2(G) \geq n + 2k - 1$, then G contains k edge-disjoint hamiltonian cycles.*

2. Veneering numbers and *k*-extendibility

To prove our main theorem, we need some results on path systems in bipartite graphs. Our strategy is to develop k systems of edge-disjoint paths and show that they can be extended to k edge-disjoint hamiltonian cycles. The following definitions and theorems can be found in [6].

Let $\mathbf{W}_k(G)$ be the family of all k-sets $\{(w_1, z_1), \ldots, (w_k, z_k)\}$ of pairs of vertices of G where $w_1, \ldots, w_k, z_1, \ldots, z_k$ are all distinct. Let $\mathcal{S}_k(G)$ denote the collection of edge-disjoint path systems in G that have exactly k paths. If $W \in \mathbf{W}_k(G)$ lists the end-points of a path system $\mathcal{P}$ in $\mathcal{S}_k(G)$, we say that $\mathcal{P}$ is a *W-linkage*. A graph G is said to be *k-linked* if there is a W-linkage for every $W \in \mathbf{W}_k(G)$. A graph G is said to be *k-extendible* if any W-linkage of maximal order is spanning.

In order to tailor the idea of extendible path systems to bipartite graphs, the notion of a veneering path system was introduced in [6].

Definition 1. A path system $\mathcal{P}$ veneers a bipartite graph G if it covers all the vertices of one of the partite sets.

Let $G = (X, Y)$ be a bipartite graph. Given a $W \in \mathbf{W}_k(G)$, we denote by W^X those pairs of W that are in X^2, by W^Y those that are in Y^2, and by W^1 the set of bipartite pairs of W. Also, with a slight abuse of notation, we will let W_X (resp. W_Y) be the set of vertices of X (resp. Y) that are used in the pairs of W.

Definition 2. Let G be a bipartite graph and $W \in \mathbf{W}_k(G)$. The veneering number $\vartheta_Y^X(W)$ of W is defined to be

$$\begin{aligned}\vartheta_Y^X(W) &= (|X| - |Y|) - (|W^X| - |W^Y|),\\ &= (|X| - |Y|) - \frac{|W_X| - |W_Y|}{2}.\end{aligned}$$

Note that one consequence of the definition is that $\vartheta_Y^X = -\vartheta_X^Y$. For a given path system $\mathcal{P}$, let $\partial(\mathcal{P})$ denote the set of pairs of endpoints of paths in $\mathcal{P}$ and let $\mathring{\mathcal{P}}$ denote $\mathcal{P} - \partial(\mathcal{P})$. We define the veneering number of such a $\mathcal{P}$ to be the veneering number of $\partial\mathcal{P}$. The veneering number of a given set of endpoints is of interest, because it represents the minimum possible number of vertices left uncovered by a path system with those endpoints.

As an example, consider $G = K_{6,7}$ and let X denote the partite set of order six. Furthermore, let (x_1, x_2) be a pair of distinct vertices in X. Clearly, any $x_1 - x_2$ path in G has order at most eleven and omits at least two vertices from Y. Let $W = \{(x_1, x_2)\}$. Then

$$\begin{aligned}\vartheta_Y^X(W) &= (|X| - |Y|) - (|W^X| - |W^Y|)\\ &= (6 - 7) - (1 - 0) = -2.\end{aligned}$$

This indicates that the minimum possible number of uncovered vertices in a path system with endpoints in W is two. The fact that $\vartheta_Y^X(W) < 0$ indicates that those vertices would be in Y. Similarly, $\vartheta_X^Y(W) = 2$, yielding the same information.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two path systems of G, we write $\mathcal{P}_1 \leq \mathcal{P}_2$ when every path of $\mathcal{P}_1$ is contained in a path of $\mathcal{P}_2$. The following fact will prove most useful.

Proposition 2.1. *Let $G = (X \cup Y, E)$ be a bipartite graph and $\mathcal{P}_1, \mathcal{P}_2 \in S(G)$ be such that $\mathcal{P}_1 \leq \mathcal{P}_2$. Let*

$$G_1 = (X_1 \cup Y_1, E_1) = G - \mathring{\mathcal{P}}_1,$$

$$G_2 = (X_2 \cup Y_2, E_2) = G - \mathring{\mathcal{P}}_2,$$

then

$$\vartheta_{Y_1}^{X_1}(\mathcal{P}_1) = \vartheta_{Y_2}^{X_2}(\mathcal{P}_2).$$

Proof (*Sketch*). Suppose that $\mathcal{P}_1$ consists entirely of the paths $\mathcal{R}_1$ and $\mathcal{R}_2$ and that $\mathcal{P}_2$ consists entirely of the path $\mathcal{S}$ with $\mathcal{R} \subset \mathcal{S}$. We consider the case where $\mathcal{R}_1$ has endpoints y_1 and y_2 in Y, $\mathcal{R}_2$ has endpoints y_1' and y_2' in Y and that $\mathcal{S}$ has endpoints x in X and y in Y. All of the other cases, both when the systems contain multiple paths or have different endpoints follow by a nearly identical analysis.

By definition,

$$\vartheta_{Y_1}^{X_1}(\mathcal{P}_1) = (|X_1| - |Y_1|) - (|W^{X_1}| - |W^{Y_1}|).$$

Since both paths in $\mathcal{R}$ have their endpoints in Y $|W^{X_1}| = 0$ and $|W^{Y_1}| = 2$. Additionally, since each path in $\mathcal{R}$ has one more internal vertex in X than in Y, we observe that $(|X_1| - |Y_1|) = -2$ and hence we conclude that $\vartheta_{Y_1}^{X_1}(\mathcal{R}) = |X| - |Y|$.

Similarly,

$$\vartheta_{Y_2}^{X_2}(\mathcal{S}) = (|X_2| - |Y_2|) - (|W^{X_2}| - |W^{Y_2}|).$$

Since $\mathcal{S}$ has an endpoint in each partite set, it follows that $|W^{X_2}| = |W^{Y_2}| = 0$ and for some integer ℓ, $|X_2| = |X| - \ell$ and $|Y_2| = |Y| - \ell$. Consequently, $\vartheta_{Y_2}^{X_2}(\mathcal{S}) = |X| - |Y|$. □

We are now ready to give our definition of a k-extendible bipartite graph.

Definition 3. Let G be a bipartite graph. Then G is said to be *k-extendible* if for any path system $\mathcal{P}$ in $\mathcal{S}_k(G)$ there exists some veneering path system $\mathcal{P}'$ in $\mathcal{S}_k(G)$ that preserves the endpoints of $\mathcal{P}$.

We will utilize the following in the proof of our main theorem.

Theorem 2.2 ([6]). *If $k \geq 2$ and $G = (X, Y)$ is a bipartite graph of order n such that $|X|, |Y| > 3k$ and $\sigma_2^2(G) \geq \lceil \frac{n+3k}{2} \rceil$, then G is k-extendible.*

It is important to note that a maximal path system with veneering number zero is spanning. Thus, if a graph G that meets the σ_2^2 bound for k-extendibility has some path system $\mathcal{P}$ in $\mathcal{S}_k(G)$ such that $\vartheta(\mathcal{P}) = 0$, then G must have a spanning path system.

We give two more results from [6] that will be very useful. The first is relatively straightforward to prove, and the second is a weaker version of a result in [4].

Theorem 2.3. *If $G = (X \cup Y, E)$ is a balanced bipartite graph of order $2n$ with $\sigma_2(G) \geq n + 2k - 2$ then for any set $\mathbf{W}$ in $\mathbf{W}_k(G)$ comprised entirely of proper pairs of G, there exists a system of k edge-disjoint paths whose endpoints are exactly the pairs in $\mathbf{W}$.*

Theorem 2.4. *If $G = (X \cup Y, E)$ is a balanced bipartite graph of order $2n$ such that for any $x \in X$ and any $y \in Y$, $d(x) + d(y) \geq n + 2$, then for any pair (x, y) of vertices of G, there exists a hamiltonian path between x and y. The degree-sum condition is the best possible.*

3. Proof of Theorem 1.4

Suppose the theorem is not true, and let G be a counterexample of order $2n$ with a maximum number of edges. The maximality of G implies that for any proper pair (x, y), $G + xy$ contains k edge-disjoint hamiltonian cycles, one of these containing the edge xy. Thus, with any proper pair (x, y) we will associate $k-1$ edge-disjoint hamiltonian cycles $H_1, \ldots, H_{k-1}$ and an (x, y)-hamiltonian path $P = (x = z_1, z_2, \ldots, z_{2n} = y)$.

Let H denote the union of subgraphs $H_1, \ldots, H_{k-1}$, and $L = L(x, y)$ denote the subgraph obtained from G by removing the edges of H. Before we go on proving our theorem we will state a few facts about L. Throughout these proofs, we must keep in mind that

$$n \geq 128k^2, \qquad (1)$$

and for any vertex w of G, we have

$$d_L(w) = d_G(w) - 2(k-1). \qquad (2)$$

Thus, the degree-sum condition on any proper pair (x, y) of G is

$$d_G(x) + d_G(y) \geq n + 2k - 1. \tag{3}$$

This yields the following:

Fact 1. *For any proper pair* (x, y) *of* G, *we have*

$$d_L(x) + d_L(y) \geq n - 2k + 3. \tag{4}$$

Fact 2. *If there is a proper pair* (x, y) *of* G, *with*

$$d_G(x) + d_G(y) \geq n + 4k - 3,$$

or equivalently

$$d_L(x) + d_L(y) \geq n + 1,$$

then L *contains a hamiltonian cycle.*

Proof. If there were a proper pair (x, y) of G such that $d_G(x) + d_G(y) \geq n + 4k - 3$, then by (2), $d_L(x) + d_L(y) \geq n + 1$, hence if we consider the (x, y)-path P in L, we see that there must be a vertex $z \in V(P)$ such that z is in $N(y)$ and z^+. Then $xz^+ \cup [z^+, y]_P \cup yz \cup [x, z]$ is a hamiltonian cycle in L. □

Note that the existence of P shows that L is connected. In fact, L must be 2-connected.

Lemma 3.1. *If* L *has a cut-vertex, then there are* k *edge-disjoint hamiltonian cycles in* G.

Proof. Suppose w is a cut-vertex of L; we assume, without loss of generality, that $w \in X$. Since L admits a hamiltonian path, $L - w$ can only have two components, one of them being balanced. Let B be the subgraph of G induced by the balanced component of $L - w$ and $A = G - B$. Note that $w \in A$, and $E_L(A_X - w, B) = E_L(A_Y, B) = \emptyset$. Let $a = |A_X| = |A_Y|$ and $b = |B_X| = |B_Y|$.

Claim 1. $a, b > \frac{n}{2k}$.

Proof. Assume $a \leq \frac{n}{2k}$. Then $a(2k-2) + a < 2ak \leq n$, implying $a(2k-2) < n - a = b$, so $|E_H(A_Y, B_X)| < |B_X| = b$. Thus there is a vertex $u \in B_X$ such that $E_H(u, A_Y) = \emptyset$, so $E_G(u, A_Y) = \emptyset$. Take any $v \in A_Y$. We have $uv \notin E(G)$, so

$$\begin{aligned} d(u) + d(v) &\leq |A_X| + d_H(v, B_X) + |B_Y| + d_H(u, A_Y) \\ &\leq a + 2(k-1) + b \\ &< n + 2k - 1, \end{aligned}$$

which contradicts the condition of our theorem. $\square_{\text{Claim 1}}$

The following two claims give lower bounds on the degrees of the vertices in L.

Claim 2. *For any* $z \in A - w$, $d_L(z) \geq \frac{|A|}{2} - 2k + 3$ *and for any* $z \in B$, $d_L(z) \geq \frac{|B|}{2} - 2k + 3$.

Proof. Assume $z \in B_Y$ (the cases $z \in B_X$, $z \in A_X$, $z \in A_Y$ are similar). By Claim 1 and the fact that $n \geq 128k^2$, we have $|A_X - w| = a - 1 > \frac{n}{2k} - 1 > 2(k-1)$, so there is a $z' \in A_X - w$ such that $zz' \notin E(H)$, thus $zz' \notin E(G)$, so that $d_L(z) + d_L(z') \geq n - 2k + 3$. Then since $d_L(z') \leq |A_Y| = a$, we get $d_L(z) \geq n - 2k + 3 - a = b - 2k + 3$. $\square_{\text{Claim 2}}$

Claim 3. $d_L(w) \geq \frac{n}{2k} - 2k + 3$.

Proof. If w is adjacent, in G, to all the vertices of A_Y, then the Claim is obviously true. If not, there is a $v \in A_Y$ with $wv \notin E(G)$, so that $d_L(w) + d_L(v) \geq n - 2k + 3$. Since $d_L(v) \leq a = n - b < n - \frac{n}{2k}$, we get

$$\begin{aligned} d_L(w) &\geq n - 2k + 3 - d_L(v) \\ &> n - 2k + 3 - \left(n - \frac{n}{2k}\right) \\ &= \frac{n}{2k} - 2k + 3. \end{aligned}$$ $\square_{\text{Claim 3}}$

Finally:

Claim 4. $|E_G(A_X, B_Y)|, |E_G(A_Y, B_X)| \geq 2k - 1$.

Proof. If $G[(A_X, B_Y)]$ is complete, the result is obvious. If not, there is a pair of non-adjacent vertices $u \in A_X$ and $v \in B_Y$, so $d(u) + d(v) \geq n + 2k - 1$. Yet $d(u, A_Y) \leq a$ and $d(v, B_X) \leq b$, so

$$\begin{aligned} d(u, B_Y) + d(v, A_X) &\geq n + 2k - 1 - a - b \\ &= 2k - 1. \end{aligned}$$

The proof is identical for (A_Y, B_X). $\square_{\text{Claim 4}}$

By Claims 2 and 3, and the fact that $n \geq 128k^2$ we have, for any pair of vertices $(u, v) \in A_X \times A_Y$

$$\begin{aligned} d_A(u) + d_A(v) &\geq |A| - 2k + 3 + \frac{n}{2k} - 2k + 3 \\ &> |A| + 2k \\ &= 2a + 2k > a + 66k. \end{aligned}$$

Thus, A, and by a similar computation B, satisfies the conditions of Theorem 2.4. Hence take k pairs (e_i, e_i') of edges such that the e_i are distinct edges of $E_G(A_X, B_Y)$ and the e_i' are distinct edges of $E_G(A_Y, B_X)$. These edges exist by Claim 4.

Let $u_i \in A_X$ and $v_i \in B_Y$ be the end vertices of e_i, and $u_i' \in A_Y$ and $v_i' \in B_X$ be the end vertices of e_i'. Since pairs of vertices from A and B satisfy the conditions of Theorem 2.4 and removing a hamiltonian path reduces the degree sum of any pair of vertices by at most 4, there are k edge-disjoint hamiltonian paths $U_1, \ldots, U_k$ in A such that u_i and u_i' are the end-vertices of U_i, and there are k edge-disjoint hamiltonian paths $V_1, \ldots, V_k$ in B such that v_i and v_i' are the end-vertices of V_i. Together with the e_i and e_i' edges we get k edge-disjoint hamiltonian cycles in G, which contradicts the assumption that no such collection of cycles exists in G. Hence the lemma is proven. $\square$

Now we show that the 2-connectedness of L ensures that L contains a relatively large cycle.

Lemma 3.2. *If L is 2-connected, then it contains a cycle of order at least $2n - 4k + 4$.*

Proof. Recall that the maximality of G implies that L is traceable, so let $P = x_1, y_1, \ldots, x_n, y_n$ be a hamiltonian path in L. The path P induces a natural ordering of the vertices in G, specifically, $z \prec z'$ if we encounter z before z' while traversing P from x_1 to y_n. For convenience, we will say that a vertex w is the *minimum* (respectively *maximum* vertex with respect to a given property if $w \prec w'$ (resp. $w' \prec w$) for each other w' in $V(L)$ satisfying this property.

Since, by assumption, L is 2-connected, each of x_1 and y_n have at least two adjacencies on P. Let x^* be the minimum vertex of $N(y_n)$ and let y^* be the maximum vertex of $N(x_1)$. We consider two cases.

Case 1: Suppose $x^* \prec y^*$. Amongst all $x_i \in N(y_n)$ and $y_j \in N(x_1)$ such that $x_i \prec y_j$, pick the pair, call them x and y such that $\text{dist}_P(x, y)$ is minimum. By this choice of x and y note that there are no neighbors of x_1 or y_n between x and y on P. Note that the subpath P' of P that goes from x^+ to y^- cannot contain more than $4k - 2$ vertices since otherwise, as x_1 and y_n are not adjacent, we would have

$$d_L(y_n) \leq (n - 1) - (2k - 2) - d_L(x_1),$$

or

$$d_L(x_1) + d_L(y_n) \leq n - 2k + 1.$$

This contradicts Fact 1.

However, if P' has at most $4k - 4$ vertices then the cycle

$$x_1 y_1 \ldots x y_n x_n \ldots y x_1,$$

which excludes only the vertices in P' has length at least $2n - 4k + 4$, as desired.

Case 2: Suppose $y^* \prec x^*$. Since L is 2-connected, there exists a sequence of $\ell \geq 1$ adjacent pairs of vertices (u_i, v_i) with the following properties. First, $u_1 \prec y^*$, $y^* \prec v_1$ and $x^* \prec v_\ell$. Then, for each $1 \leq i \leq \ell - 1$, $u_i \prec u_{i+1}$, $v_i \prec v_{i+1}$ and $u_{i+1} \prec v_i$. We will also choose these vertices so that $v_{\ell-1} \prec x^*$ and $y^* \prec u_2$, as this will simplify things going forward.

Next, choose y in $N(x_i)$ such that $u_1 \prec y$ and $\text{dist}_P(u_i, y)$ is minimum. Similarly, select x in $N(y_n)$ such that $x \prec v_\ell$ and $\text{dist}_P(x, v_\ell)$ is minimum. Now we consider the cycle

$$C' = x_1 y P u_2 v_2 P u_4 \ldots u_\ell v_\ell P y_n x P v_{\ell-1} u_{\ell-1} P v_{\ell-3} \ldots v_1 u_1 P x_1.$$

In C', we omit several vertices from P. Specifically, we omit $u_1^+ P y^-$, $x^+ P v_\ell^-$ and segments of the form $u_i^+ P v_{i-1}^-$ for $2 \leq i \leq \ell$. Note that by our choice of x and y, neither x_1 nor y_n have any adjacencies in these subpaths of P. Counting as above, if these subpaths contain $4k - 3$ or more vertices, we violate Fact 1, while if these subpaths total $4k - 4$ or fewer vertices, C' will be the desired cycle. $\square$

3.1. Path systems

In order to prove an important technical lemma, we must first establish some facts about extending paths and path systems.

Lemma 3.3. *Let $G = (X \cup Y, E)$ be a bipartite graph, and let $\mathcal{P}$ be a path system of G. Let X' be a subset of $(\partial\mathcal{P})_X$, and let Y' be a subset of $Y - (\mathring{\mathcal{P}})_Y$. Suppose that $|X'| = s + t$, where s is the number of vertices in X' arising from paths of $\mathcal{P}$ consisting of a single vertex. Furthermore let ℓ denote the number of vertices of Y' that are endpoints of some non-trivial path in $\mathcal{P}$. If*

$$\delta(X', Y') > \frac{t+\ell}{2} + s$$

then there exists another path system, $\mathcal{P}'$, of G such that $\mathcal{P} \le \mathcal{P}'$ and $(\partial\mathcal{P}')_{X'} = \emptyset$.

Proof. We will first show that s may be assumed to be 0. If $s > 0$, let $P_1, P_2, \ldots, P_s$ be the trivial paths of $\mathcal{P}$ contained in X'. Now, for every $i \in [s]$, replace $P_i = \{x_i\}$ with a path P_i' on three vertices such that the endvertices of P_i' are new vertices added to X' and the middle vertex of P_i' is a new vertex added to Y. In addition, let the endvertices of P_i' be adjacent to the neighbors of x_i. Let $\mathcal{P}_1$ be the new path system, and let X_1', consisting of X' and the vertices added to X', be the new set of endvertices we wish to eliminate.

The new system $\mathcal{P}_1$ now contains no trivial paths, and $|X_1'| = t + 2s$. Thus, if our lemma were true for systems with no trivial paths, then the condition

$$\delta(X', Y') > \frac{t+2s}{2} = \frac{t}{2} + s$$

ensures the existence of a path system $\mathcal{P}_1'$ such that $\mathcal{P}_1 \le \mathcal{P}_1'$ and $(\partial\mathcal{P}_1')_{X_1'} = \emptyset$. By replacing every P_i' by P_i within the appropriate paths of $\mathcal{P}_1'$, we obtain the desired path system of G.

So assume that $X' = \{x_1, \ldots, x_t\}$. Note that the result clearly holds if $t = 1$, so assume that $t \ge 2$. Our goal is to find edges from each x_i to vertices in Y', allowing us to create a new path system in which no x_i is an endpoint.

Given some x_i in X', let P_i be the path in $\mathcal{P}$ containing x_i, let w_i be the other endpoint of P_i. Our goal is to select an element y_i in $NY'(x_i)$ that will allow us to extend P_i to a path with one fewer endpoint in X'. We will extend the P_i, $1 \le i \le t$, in order and at the time we consider x_i, let Z_i denote the set of internal vertices in the current (updated) path system. It remains to show that $N_{Y'}(x_i) - w_i - Z_i$ is non-empty.

Initially, no vertex of Y' was interior to a path in $\mathcal{P}$. Each vertex in Y' that was already an endpoint of some non-trivial path in $\mathcal{P}$ can be selected once to extend a path and each other vertex in Y' can be selected twice. If exactly j vertices in $Y' \cap Z_i$ were endpoints of some non-trivial path in $\mathcal{P}$, then

$$|Z_i| \le \max\left\{0, \frac{i-j-2}{2} + j\right\} \le \frac{t-j-2}{2} + j \le \frac{t+\ell}{2} + 1.$$

This implies that

$$|N_{Y'}(x_i) - w_i - Z_i| \ge \delta(X', Y') - 1 - |Z_i| > \frac{t+\ell}{2} - 1 - |Z_i| > 0. \quad \square$$

The following corollary is obtained from Lemma 3.3 by induction on k:

Corollary 3.4. *Let $G = (X \cup Y, E)$ be a bipartite graph, let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be k edge-disjoint path systems, and let $Y' \subset Y - \bigcup_{i=1}^{k} int(\mathcal{P}_i)_Y$. For all $i \in [k]$ let $X_i \subset (\partial\mathcal{P}_i)_X$ and $|X_i| = s_i + t_i$, where s_i is the number of vertices of X_i arising from paths of $\mathcal{P}_i$ consisting of a single vertex. Furthermore let ℓ_i denote the number of vertices of Y_i' that are endpoints of some non-trivial path in $\mathcal{P}_i$. If for all $i \in [k]$,*

$$\delta(X_i, Y') > \frac{t_i + \ell_i}{2} + s_i + 2(k-1)$$

then there exist k edge-disjoint path systems $\mathcal{P}_1', \ldots, \mathcal{P}_k'$ such that for all $i \in [k]$, $\mathcal{P}_i \le \mathcal{P}_i'$ and $(\partial\mathcal{P}_i')_{X_i} = \emptyset$.

3.2. The degree-product lemma

The remainder of the proof of Theorem 1.4 relies on a result pertaining to degree products as opposed to degree sums. We feel it would be interesting to investigate similar results.

Lemma 3.5. *If G has no proper pair (u, v) such that $d_L(u)d_L(v) \ge 12k(n - 12k)$ then G has k edge-disjoint hamiltonian cycles.*

Proof. Suppose G has no such vertices. Let A be the subgraph of G generated by the vertices of degree less than $16k$, and B the subgraph generated by the vertices of degree greater or equal to $16k$. By (3) and (1) no bipartite pairs (u, v) of A are proper.

Next we show that no bipartite pairs (u, v) of B can be proper. Suppose that (u, v) was a proper bipartite pair and without loss of generality, assume that $d_L(u) \geq d_L(v)$. Since v has degree at least $16k$ in G, we have that $d_L(v) \geq 14k + 2$ and by Fact 1 we know that $d_L(u) \geq \frac{n-2k+3}{2}$. If $d_L(u) \geq \frac{6n}{7} - 2k + 3$ then since n is at least $128k^2$, $d_L(u) > \frac{12k(n-12k)}{14k+2}$ then $d_L(u)dL(v) > 12k(n - 12k)$, a contradiction. If, otherwise, $d_L(u) < \frac{6n}{7} - 2n + 3$ then Fact 1 implies that $d_L(v) > \frac{n}{7}$, so that

$$d_L(u)d_L(v) > \frac{n}{7}\frac{n-2k+3}{2}$$

which exceeds $12k(n - 12k)$ since n is at least $128k^2$.

Thus A and B induce complete bipartite graphs. Assume without loss of generality, that $|A_X| \geq |A_Y|$, and set $\lambda = |A_X| - |A_Y| = |B_Y| - |B_X|$. We can assume $\lambda < 4k - 3$ since otherwise we could find a proper non-adjacent pair $(x, y) \in V(B_X) \times V(A_Y)$ with $d_G(x) + d_G(y) \geq |B_Y| + \lambda + |A_X| + \lambda = n + \lambda \geq n + 4k - 3$, and Fact 2 would imply a hamiltonian cycle in L, hence k edge-disjoint hamiltonian cycles in G. □

Claim 5. *We have* $\delta(A_X, B_Y) \geq \lambda + 2k - 1$ *and* $\delta(A_Y, B_X) \geq 2k - 1 - \lambda$.

Let $x \in A_X$ such that $d(x, B_Y) = \delta(A_X, B_Y)$. By (3), every vertex $y \in B_Y - N(x, B_Y)$ must verify

$$\begin{aligned} d_G(y) &\geq n + 2k - 1 - d_G(x) \\ &= n + 2k - 1 - |A_Y| - d(x, B_Y) \\ &= |B_Y| + 2k - 1 - \delta(A_X, B_Y), \end{aligned}$$

so

$$\begin{aligned} d_G(y, A_X) &\geq |B_Y| + 2k - 1 - \delta(A_X, B_Y) - |B_X| \\ &= \lambda + 2k - 1 - \delta(A_X, B_Y). \end{aligned}$$

This implies that

$$d_G(A_X - x, B_Y - N(x, B_Y)) \geq (|B_Y| - \delta(A_X, B_Y))(\lambda + 2k - 1 - \delta(A_X, B_Y)) \tag{5}$$

yet, since the vertices of A_X can be adjacent to no more than $\lambda + 4k - 1$ vertices of B_Y (by Fact 2), we see that

$$d_G(A_X - x, B_Y - N(x, B_Y)) \leq (|A_X| - 1)(\lambda + 4k - 1). \tag{6}$$

Thus if $\lambda + 2k - 1 - \delta(A_X, B_Y) > 0$, (5) and (6) imply

$$\begin{aligned} |B_Y| &\leq \frac{(|A_X| - 1)(\lambda + 4k - 1)}{\lambda + 2k - 1 - \delta(A_X, B_Y)} + \delta(A_X, B_Y) \\ &\leq (16k)(8k - 4) + 2k - 2 \end{aligned}$$

which contradicts the fact that $n \geq 128k^2$, hence $\delta(A_X, B_Y) \geq \lambda + 2k - 1$.

The proof of $\delta(A_Y, B_X) \geq 2k - 1 - \lambda$ is similar. $\square_{\text{Claim 5}}$

We distinguish two cases, according to the size of A_X:

Case 1: Suppose $1 \leq |A_Y| \leq 2k - 1$. Then Claim 5 and the completeness of A imply

$$\begin{aligned} \delta(A_Y) &\geq |A_X| + 2k - 1 - \lambda \\ &= |A_Y| + 2k - 1 \\ &> |A_Y| + 2(k - 1). \end{aligned}$$

Now, we apply Corollary 3.4 with $\mathcal{P}_i = X_i = A_Y$ for all i, and let $Y' = X$. This implies, in the language of the corollary, that $\delta(X_i, Y') = \delta(A_Y)$. Thus, we find that there are k edge-disjoint systems $\mathcal{P}_1, \ldots, \mathcal{P}_k$ whose paths have all order three and whose endvertices are all in X.

Further, since A is a complete bipartite graph, we may choose these path systems so that they cover a subset A'_X of $\min(|A_X|, 2|A_Y|)$ vertices of A_X. That is to say, if $|A_X| \leq 2|A_Y|$, $A'_X = A_X$, so these systems each cover A entirely, and if $|A_X| > 2|A_Y|$, we require that they each cover the same proper subset A'_X of A_X having order $2|A_Y|$.

For all $i \in [k]$ we let $\mathcal{P}'_i = \mathcal{P}_i$ when $|A_X| \leq 2|A_Y|$, and $\mathcal{P}'_i = \mathcal{P}_i \cup (A_X - A'_X)$ when $|A_X| > 2|A_Y|$. In either case, we now have k edge-disjoint path systems which cover A.

Again we wish to apply Corollary 3.4 to the $\mathcal{P}'_i$ with $X_i = (\partial\mathcal{P}'_i)_{A_X}$, to extend to a family of k edge-disjoint systems $\mathcal{P}''_1, \ldots, \mathcal{P}''_k$ such that every path in each of these systems has both endvertices in B.

We may do so since if $|A_X| \leq 2|A_Y|$ then all $t_i = |A_X|$ vertices of X_i come from non-trivial paths, and if $|A_X| > 2|A_Y|$ then $t_i = 2|A_Y|$ vertices of X_i also come from non-trivial paths, and $s_i = |A_X| - 2|A_Y|$ of them come from paths consisting of exactly one vertex, so by Claim 5,

$$\begin{aligned} d(A_X, B_Y) &\geq \lambda + 2k - 1 \\ &> \frac{t_i}{2} + s_i + 2(k-1). \end{aligned}$$

Consider some matching $\mathcal{M}_1$ that contains exactly one edge from each non-empty path in $\mathcal{P}'_1$. Clearly, $\vartheta_Y^X(\mathcal{M}_1) = 0$, and therefore by Proposition 2.1 we have that

$$\vartheta(\partial(\mathcal{P}'_1)) = 0 \tag{7}$$

in $G - \mathring{\mathcal{P}}'_1$. Thus, as $\partial(\mathcal{P}'_1) \subset B$, and B induces a complete bipartite graph, we can link the endpoints of the paths in $\mathcal{P}'_1$ to form a Hamiltonian cycle in G.

Suppose then that we have extended $\mathcal{P}'_1, \ldots, \mathcal{P}'_{t-1}$ $(t \leq k)$ to the disjoint Hamiltonian cycles $H_1, \ldots, H_{t-1}$. As above, Proposition 2.1 implies that

$$\vartheta(\partial(\mathcal{P}'_t)) = 0 \tag{8}$$

in $G - \mathring{\mathcal{P}}'_t$. Assume that $\mathcal{P}'_t$ has exactly j paths, and let $\{x_1, y_1\}, \ldots, \{x_j, y_j\}$ denote the pairs of endpoints of these paths. Additionally, let the set $W = \{\{y_1, x_2\}, \{y_2, x_3\}, \ldots, \{y_j, x_1\}\}$. As B induces a complete bipartite graph with each partite set having size at least $n - |A_Y| - \lambda \geq n - 6k$, it is simple to see that there is a W-linkage in $G_t := G - \mathring{\mathcal{P}}'_t - \bigcup_{i=1}^{t-1} E(H_i)$. Note that there are at most $j \leq |A_Y| < 2k$ paths in $\mathcal{P}'_t$, so if we are able to show that G_t is $2k$-extendible we will be done.

By Corollary 2.2, it suffices to show that

$$\sigma_2^2(G_t) > \frac{|V(G_t)| + 6k}{2} \geq \frac{2n - 2k}{2} \geq n - k. \tag{9}$$

In G, the minimum degree of a vertex in the subgraph induced by B is $n - (|A_Y| + \lambda) \geq n - 6k$. In removing the edges from the $t-1$ other hamiltonian cycles, each vertex loses $2t - 2 < 2k - 2$ adjacencies. Thus, it is clear that $\sigma_2^2(G_t)$ certainly exceeds $n - k$, completing this case.

Case 2: Suppose $|A_Y| \geq 2k$. Let A'_X be a subset of $|A_Y|$ vertices of A_X. As A is a complete bipartite graph, there are k edge-disjoint hamiltonian cycles in $(A'_X \times A_Y)_G$, and we let $x_1y_1, \ldots, x_ky_k$ be independent edges of $(A'_X \times A_Y)_G$ such that x_iy_i is an edge of the ith hamiltonian cycle.

Using Claim 5 we get that $\delta(A'_X, B_Y) \geq 2k - 1$ and $\delta(A_Y, B_X) \geq 2k - 1 - \lambda$ so

$$\begin{aligned} \delta(A_Y, B'_X) &\geq |A_X - A'_X| + \delta(A_Y, B_X) \\ &\geq 2k - 1. \end{aligned}$$

Let $B' = G - A'_X - A_Y$. We have

$$\begin{aligned} \sigma_2(B') &\geq \delta(A_X - A'_X, B_Y) + |B_X| \\ &\geq |B_X| + \lambda + 2k - 1 \\ &= |B'| + 2k - 1. \end{aligned}$$

One may then use the edges of $E(A'_X, B_Y)$ and $E(A_Y, X - A'_X)$ along with Theorem 2.3 to find k edge-disjoint hamiltonian cycles in G. □

Before we proceed to prove the main theorem, we give one final technical lemma.

Lemma 3.6. *Let G be a graph containing a Hamiltonian cycle C and let S and R be non-empty disjoint subsets of $V(G)$. If $|S| \leq |E(R, S)| - |R|$ then there are four distinct vertices c_1, c_2, c_3, c_4, encountered in that order on C, such that one of the following holds:*

(a) $c_1, c_3 \in R$, $c_2, c_4 \in S$, $c_1c_2 \in E(G)$, *and* $c_3c_4 \in E(G)$, *or*
(b) $c_1, c_4 \in R$, $c_2, c_3 \in S$, $c_1c_3 \in E(G)$, *and* $c_2c_4 \in E(G)$, *or*
(c) $c_1, c_4 \in S$, $c_2, c_3 \in R$, $c_1c_3 \in E(G)$, *and* $c_2c_4 \in E(G)$.

Proof. First, note that if $R' = \{r \in R : d(r, S) > 0\}$ and $S' = \{s \in S : d(s, R) > 0\}$, then

$$|R'| + |S'| \leq |R| + |S| \leq |E(R, S)| = |E(R', S')|$$

so we may assume that every vertex of R is adjacent to at least one vertex of S, and vice versa. Further, observe that the inequality in the statement of the lemma cannot hold if $|R| = 1$ or $|S| = 1$. Thus, both R and S have at least two vertices.

If $|R| = |S| = 2$, then $|E(R, S)| = 4$, and one of (a), (b), or (c) must occur. So assume without loss of generality that $|R| \geq 3$, and let $R = \{u_1, \ldots, u_r\}$, where the labels on the vertices of R are determined by a chosen orientation of C. Suppose the theorem is not true. Then we claim that C can be traversed such that all of the vertices of R are encountered before all of the vertices of S. Let P and P' be the two $[u_1, u_r]$ paths on C, with P being the path containing all of the u_i for $1 \le i \le r$.

To avoid (a), all of u_1's neighbors in S and all of u_r's neighbors in S must lie either entirely in P or entirely in P'. If $(N(u_1) \cup N(u_r)) \cap S \subset P'$ no vertex of S can lie in P, for then the edge between this vertex and any of its neighbors in R would cause (a), (b), or (c) to occur. But this means that the claim is proven for this case.

So suppose that $(N(u_1) \cup N(u_r)) \cap S \subset P$. Also, define v_i to be the vertex with highest index i such that $v_i \in N(u_1) \cap S$, and let v_j be the vertex with lowest index j such that $v_j \in N(u_r) \cap S$. Then $i \le j$, or else (b) occurs. No vertex of R lies between v_i and v_j, or else (a), (b), or (c) would occur. Then $u_1, \dots, u_k$ lie along the path $[u_1, v_i^-]$, and $u_{k+1}, \dots, u_r$ lie along the path $[v_j^+, u_r]$ for some k between 1 and $r - 1$. All vertices of S on the path $[u_1, v_i]$, must lie on the path $[u_k^+, v_i]$, or else (a), (b) or (c) will occur. Similarly, all vertices of S on the path $[v_j, u_r]$, must lie on the path $[v_j, u_{k+1}^-]$. But this implies that the claim holds. If necessary, relabel the vertices of R such that $P = [u_1, u_r]$ contains no elements of S. Since (b) or (c) will be violated if two chords from R to S cross, a simple count reveals that $|S| \ge |E(R, S)| - (|R| - 1)$, a contradiction. □

3.3. *Proof of* Theorem 1.4

Proof. Let C be a cycle of L of maximal order which minimizes $d_L(T, C)$, where $T = L - C$. By Lemma 3.2

$$t = \frac{|T|}{2} \le 2k - 2. \tag{10}$$

Let $u \in T_X$ and $v \in T_Y$ such that $d_L(u, C) + d_L(v, C)$ is maximal. Let $\alpha = d_L(u, C)$ and $\beta = d_L(v, C)$. We assume, without loss of generality, that $\alpha \le \beta$.

We may assume that

$$\alpha \ge 2k + 4. \tag{11}$$

Indeed, by Fact 1, every vertex of $Y - N_G(u)$ has degree greater or equal to $n - 2k + 3 - t - \alpha$ in L. If $\alpha \le 2k + 3$, this would yield that there are at least $n - t - (2k+3) - 2(k-1) \ge n - 6k$ vertices that have degree at least $n - 2k + 3 - t - (2k+3) \ge n - 6k$ in L. Let $S \subseteq Y$ denote this set of vertices.

Let the vertices x and y, in X and Y respectively, be such that (x, y) is a proper pair in G. Assume first that there is some vertex s in S such that (x, s) is a proper pair in G. Then since $d_L(s) \ge n - 6k$, Fact 2 implies that $d_L(x) < 6k + 1$. Therefore $d_L(x)d_L(y) < (6k + 1)n$.

Suppose then that x is adjacent to every vertex in S. Then $d_G(x) \ge |S| \ge n - 6k$ and hence $d_L(x) \ge n - 8k - 1$. By Fact 2, it follows that $d_L(y) < 8k + 1$ and hence $d_L(x)d_L(y) < (8k + 1)n$. Since $n \ge 128k^2$, both $(6k + 1)n$ and $(8k + 1)n$ are strictly less than $12k(n - 12k)$. Therefore, if $\alpha \le 2k + 3$, G contains k disjoint hamiltonian cycles by Lemma 3.5 and hence we may assume that $\alpha \ge 2k + 4$.

Note that

$$\alpha + \beta \le n - t + 1 \le n - 2k + 3$$

or else C could be extended.

We must have $|N_L(u, C)^+ \cap N_L(v, C)| \le 1$ and $|N_L(u, C) \cap N_L(v, C)^+| \le 1$. Let $R = N_L(v, C)^+ - N_G(u, C)$. Then

$$\begin{aligned} |R| &\ge d_L(v, C) - d_H(u, C) - |N_L(u, C) \cap N_L(v, C)^+| \\ &\ge \beta - 2(k - 1) - 1 \\ &= \beta - 2k + 1. \end{aligned} \tag{12}$$

For every $r \in R$ $ru \notin E(G)$, so by Fact 1,

$$d_L(r) + d_L(u) = d_L(r, T) + d_L(r, C) + d_L(u) \ge n - 2k + 3,$$

hence

$$\begin{aligned} d_L(r, C) &\ge n - 2k + 3 - d_L(u, C) - d_L(u, T) - d_L(r, T) \\ &\ge n - 2k + 3 - \alpha - t - t. \end{aligned} \tag{13}$$

Together with the fact that $\sum_{r \in R} d_L(r, T) \le t - 1$ (since otherwise, we could extend C), we get

$$d_L(R, C) = \sum_{r \in R} d_L(r, C)$$

$$\ge \sum_{r \in R} (n - 2k + 3 - d_L(u, C) - d_L(u, T) - d_L(r, T)) \tag{14}$$

$$= |R|(n - 2k + 3) - |R|(\alpha + t) - \sum_{r \in R} d_L(r, T) \tag{15}$$

$$\ge |R|(n - 2k + 3 - \alpha - t) - t + 1. \tag{16}$$

Let $S = N_L(u, C)$. We have

$$\begin{aligned} d_L(R, S) &\geq d_L(R, C) - |C_X - S| \\ &\geq |R|(n - 2k + 3 - \alpha - t) - t + 1 - (n - t) + |S| \\ &= |R|(n - 2k + 3 - \alpha - t) + |S| + 1 - n. \end{aligned}$$

If Lemma 3.6 with $G = C$, $R = R$, and $S = S^+$ were to hold, then we could extend C. Therefore, the assumption of Lemma 3.6 fails, and we have

$$|S| - (d_L(R, S) - |R| + 1) \geq 0$$
$$|S| - ((|R|(n - 2k + 3 - \alpha - t) + |S| + 1 - n) - |R| + 1) \geq 0$$
$$n - 2 - |R|(n - 2k + 2 - \alpha - t) \geq 0. \tag{17}$$

By (12) and (11), we have $|R| \geq \alpha - 2k + 1 \geq 3$, so (17) yields

$$n - 2 - 3(n - 2k + 2 - \alpha - t) \geq 0 \tag{18}$$
$$3\alpha \geq 2n - 2k + 9 \tag{19}$$
$$\alpha \geq \frac{2}{3}n - \frac{2}{3}k - 3t + 3. \tag{20}$$

Yet, as $\alpha \leq \beta$, $t \leq 2k - 1$, and $n \geq 128k^2 \geq 46k$), this would imply

$$\alpha + \beta \geq \frac{4}{3}n - \frac{4}{3}k - 6(2k - 1) + 6 > n + 2k$$

contradicting (3.3). $\square_{\text{Theorem 1.4}}$

Acknowledgement

The authors would like to extend their thanks to the anonymous referee whose careful reading and thoughtful suggestions have undoubtedly improved the clarity and overall quality of this paper.

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