Discrete Mathematics 309 (2009) 3013–3022

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Minimum degree and pan-k-linked graphs

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ARTICLE INFO

Article history:
Received 17 January 2007
Received in revised form 30 July 2008
Accepted 31 July 2008
Available online 5 September 2008

Keywords:
Path system
Extendability
Minimum degree
k-linked
Panconnected

ABSTRACT

For a k-linked graph G and a vector $\vec{S}$ of $2k$ distinct vertices of G, an $\vec{S}$-linkage is a set of k vertex-disjoint paths joining particular vertices of $\vec{S}$. Let T denote the minimum order of an $\vec{S}$-linkage in G. A graph G is said to be pan-k-linked if it is k-linked and for all vectors $\vec{S}$ of $2k$ distinct vertices of G, there exists an $\vec{S}$-linkage of order t for all t such that $T \leq t \leq |V(G)|$. We first show that if $k \geq 1$ and G is a graph on n vertices with $n \geq 5k-1$ and $\delta(G) \geq \frac{n+k}{2}$, then any nonspanning path system consisting of k paths, one of which has order four or greater, is extendable by one vertex. We then use this to show that for $k \geq 2$ and $n \geq 5k-1$, a graph on n vertices satisfying $\delta(G) \geq \frac{n+2k-1}{2}$ is pan-k-linked. In both cases, the minimum degree result is shown to be best possible.

1. Introduction

Many well-known concepts in graph theory deal with the existence of certain structures (i.e. cycles and paths) of all possible orders in a particular graph. As an example, a graph G of order n is *panconnected* if for each pair of distinct vertices u and v of G, there exists a $[u, v]$-path of order l for each l satisfying $\mathrm{dist}(u, v) \leq l \leq n-1$. Hence, a panconnected graph contains paths of all possible orders between pairs of distinct vertices. Along the same lines, a graph G on n vertices is *pancyclic* if it contains a cycle of length l for each $3 \leq l \leq n$. That is, it contains cycles of all possible orders. In [1], Brandt defines a generalization of pancyclic graphs, namely a *weakly pancyclic* graph, which contains cycles of every length from the girth to the circumference.

In studying these and similar properties, much attention has been placed on finding minimum degree conditions which imply that a graph has these properties. Williamson provided a minimum degree condition for panconnectedness:

Theorem 1.1 ([2]). *If G is a graph on $n \geq 4$ vertices with $\delta(G) \geq \frac{n+2}{2}$, then G is panconnected.*

For weakly pancyclic graphs, Brandt, Faudree, and Goddard showed the following:

Theorem 1.2 ([3]). *Every nonbipartite graph with $\delta(G) \geq \frac{n+2}{3}$ is weakly pancyclic and has girth 3 or 4.*

In this paper, we provide a minimum degree condition for a property similar to panconnected and pancyclic. However, instead of looking for paths or cycles of all orders, we are looking for particular path systems of all possible orders. Our goal

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.07.038

will be to show that when a graph has a particular minimum degree, we can control the number of vertices in these path systems.

Before continuing further, though, we now list some necessary assumptions, definitions, and notation. Unless noted otherwise, G will denote a simple, loopless graph with $|V(G)| = n$. The *order* of a graph is the number of vertices in the graph. The order of a graph G will be denoted by $|V(G)|$ or just $|G|$. The complete graph on n vertices will be denoted K_n, and the complement of a graph G will be denoted $\overline{G}$. A *path* is an alternating sequence of vertices and edges, beginning with a vertex and ending with a vertex, such that each edge joins the vertices immediately before and after it in the sequence and no edges or vertices are repeated in the sequence. Note in particular that a path contains only the edges in the alternating sequence and no other edges. Consequently, as a subgraph, a path is not necessarily an induced subgraph. The *length* of a path is the number of edges in the path. A path between x and y in G which includes x and y will be denoted $[x, y]_G$. If we exclude x and y, we will use $(x, y)_G$. We will use $\sigma_2(G)$ to denote the minimum sum of the degrees of any two nonadjacent vertices of G. Recall that an *independent set* is a set of pairwise nonadjacent vertices. The independence number of G, which is the maximum size of an independent set, will be denoted $\alpha(G)$. Two edges are *independent* if they share no endpoints.

For two (not necessarily disjoint) subgraphs A, B of G, let $d(A, B) = |\{uv \in E(G) : u \in V(A), v \in V(B)\}|$. If $A = \{u\}$, we will use $d(u, B)$. We will use $\delta(G)$ to indicate the minimum degree of a vertex in G and for two subgraphs A, B of G, let $\delta(A, B) = \min_{u\in A} d(u, B)$. Let $E(A, B) = \{uv \in E(G)|u \in V(A), v \in V(B)\}$. The neighborhood of a vertex x in the set of vertices S will be given by $N(x, S)$. For two graphs G_1 and G_2, the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph formed by taking $G_1 \cup G_2$ and joining each vertex of G_1 to each vertex of G_2. For a connected subgraph H of G and distinct vertices $x, y \in V(H)$, the distance between x and y in H, which is the length of the shortest path between x and y using only the edges in H, will be denoted $\mathrm{dist}_H(x, y)$. We will use $\mathrm{dist}(x, y)$ to indicate this distance when $H = G$. The expression $G - H$ will be used to denote the subgraph induced by the vertex set $V(G) - V(H)$. Additionally, a set containing k elements will be referred to as a *k-set*. The integers from one to k will be denoted $[k]$. See [4] for other terms or notation not defined here.

Now, we will formally define what we mean by a *path system*.

Definition 1.3. A family of vertex-disjoint paths $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ is a *path system* of G. We will denote the set of vertices of $\mathcal{P}$ by $V(\mathcal{P})$.

It is important to note that the paths in a path system are paths as defined above where all internal path vertices have exactly degree two and end-vertices of paths have exactly degree one.

Path systems and k-connectedness are related concepts, and Menger's Theorem [5] provides a specific relationship between the two:

Theorem 1.4. *A graph G is k-connected if and only if for any pair (A, B) of disjoint k-sets of $V(G)$, there exists a path system $\mathcal{P} = \{P_1, P_2, \ldots P_k\}$ such that for all $i \in [k]$, P_i is a path joining some vertex of A to some vertex of B.*

Note that the paths of the path system may join *any* vertex of A to *any* vertex of B, and note that the definition of path system assures that paths are vertex disjoint. As a stronger connectivity property, we may require that each path of a path system joins *specific* vertices of A and B. This idea gives us the following two definitions:

Definition 1.5. Let $\vec{S} = \langle a_1, a_2, \ldots, a_k, b_1, b_2, \ldots b_k\rangle$ be a vector of $2k$ distinct vertices of G. We say that a path system $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ is an $\vec{S}$*-linkage* if for all i, P_i is an $[a_i, b_i]$-path.

Definition 1.6. A graph G is said to be *k-linked* if there is an $\vec{S}$-linkage for every vector $\vec{S}$ of $2k$ distinct vertices of G.

There are numerous papers on k-linked graphs. Many of these deal with the minimum connectivity required to imply that a graph is k-linked. See [6–8] for some of these results. Recently, Kawarabayashi et al. [9] and Gould and Whalen [10] independently proved useful Ore-type degree conditions for a graph to be k-linked. The theorem below is by Gould and Whalen:

Theorem 1.7 ([10]). *If G is a graph on $n \geq 4k$ vertices with $\sigma_2(G) \geq n + 2k - 3$, then G is k-linked. Further, this bound on $\sigma_2(G)$ is best possible.*

Note that this result implies a minimum degree condition ($\delta(G) \geq \frac{n+2k-3}{2}$) for a graph to be k-linked. In this paper, we will concentrate on the following property:

Definition 1.8. A graph G is said to be *pan-k-linked* if it is k-linked and for all vectors $\vec{S}$ of $2k$ distinct vertices of G, there exists an $\vec{S}$-linkage of order t for all t such that $T \leq t \leq |V(G)|$, where T denotes the minimum order of an $\vec{S}$-linkage in G.

We prove a sharp minimum degree bound for a graph to be pan-k-linked. Our approach is to prove that the minimum degree condition allows us to extend any $\vec{S}$-linkage by one vertex at a time. This leads to the concept of 1-*extendable*.

Definition 1.9. A path system $\mathcal{P}$ is 1-*extendable* if there exists a path system $\mathcal{P}'$ which has the same endpoints as $\mathcal{P}$ and $|\mathcal{P}'| = |\mathcal{P}| + 1$.

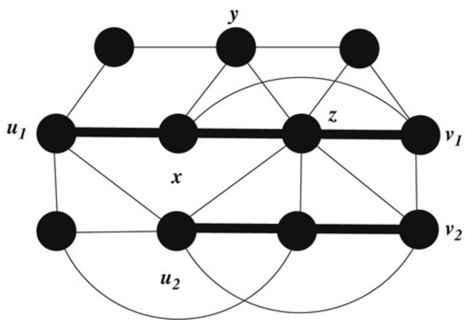

Fig. 1. Graph with a 1-extendable $\vec{S}$-linkage.

An example of a 1-extendable path system is given in Fig. 1. The actual edges of the path system are in bold. Note that this path system is an $\vec{S}$-linkage for $S = \langle u_1, u_2, v_1, v_2\rangle$. The order of this $\vec{S}$-linkage is seven, and the path system contains five edges. As will often be the case, the path system is not an induced subgraph of the larger graph. We can 1-extend this $\vec{S}$-linkage by replacing the edge xz of the $[u_1, v_1]$-path with the edges xy and yz.

In [11], Hendry proves results which focus on a single path and a stronger variation of 1-extendable. One example is the following minimum degree result:

Theorem 1.10 *([11]). If G is a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+2}{2}$, then any nonspanning path in G is 1-extendable in such a way that all of the vertices of the original path are in the extended path.*

In [10], Gould and Whalen examine degree conditions required to extend a path system so that it spans G. In [12], they examine the more general problem of finding a spanning subgraph in G which is isomorphic to a subdivision of a multigraph H. In the case where H is k independent edges, this is the same as finding a spanning path system containing k paths. We now present a useful proposition from [12]:

Proposition 1.11 *([12]). Let $\mathcal{P}$ be an $\vec{S}$-linkage. Let $R \subseteq V(G - \mathcal{P})$ and $A = V(\mathcal{P}) \cup R$. If $\delta(R, A) > \alpha(\mathcal{P}) + |R| - 1$, then there exists an $\vec{S}$-linkage $\mathcal{P}'$ where $V(\mathcal{P}') = V(\mathcal{P}) \cup R$.*

Proposition 1.11 is an important result as it will be used to place vertices into an existing path system without having to explicitly find the edges needed to incorporate them into the path system. When we use Proposition 1.11 to find a new path system which incorporates a set of vertices R, we will say that we have *inserted* R into the path system.

An important and useful fact about the independence number of a path system is the following, the proof of which is straightforward:

Fact 1.12. *If $\mathcal{P} = \{P_1, P_2, \ldots P_k\}$ is a path system with $|V(\mathcal{P})| = p$, then*

$$\alpha(\mathcal{P}) = \sum_{i=1}^{k} \left\lceil \frac{|P_i|}{2} \right\rceil \leq \sum_{i=1}^{k} \frac{|P_i| + 1}{2} = \frac{p+k}{2}.$$

Further, there is equality if, and only if, every P_i has odd order. In this case, there is a unique maximum independent set which contains the end-vertices of the paths.

Note that the independence number in Fact 1.12 is calculated only with respect to the edges and vertices which make up the paths in the path system. This is consistent with the definition of path system presented earlier. Due to this, Fact 1.12 follows by a simple parity argument. In addition to Fact 1.12, we will also use the following lemma:

Lemma 1.13. *Let F be a graph and let A and B be sets of vertices such that $V(F) = A \cup B$, A is a set of independent vertices of F, and $|B| \geq |A|$. If*

$$\delta(B, F) \geq |B| + 1, \tag{1.1}$$

then there exist three mutually adjacent vertices u, u', v such that $u, u' \in B$ and $v \in A$.

Proof. We will proceed by induction on the size of A. Assume that $|B| \geq |A|$. Note that any vertex b in B can be adjacent to at most $|B| - 1$ other vertices in B, and so Inequality (1.1) implies that b must have at least two neighbors in A. Thus, $|A| \geq 2$. Further, it is straightforward to verify that the lemma is true if $|A| = 2$. Hence, suppose that the lemma is true for all sets A of size bigger than or equal to two and smaller than k. Now, we will examine what happens when $|A| = k$.

Let $u \in V(B)$, $A_u = N(u, A)$, and $B_u = N(u, B)$. Also, let $A' = A - A_u$, $B' = B_u$, and F' be the subgraph of F on $A' \cup B'$. Observe that Inequality (1.1) ensures $A_u \geq 2$, so $|A'| < k$. If $E(A_u, B_u) \neq \emptyset$, then we are done. So, suppose this set is empty. Then for any $u' \in B'$,

$$\begin{aligned} d(u', F') &= d(u', F) - d(u', B - B') \\ &\geq |B| + 1 - (|B| - |B'|) \\ &= |B'| + 1. \end{aligned}$$

Thus, if we can show that $|B'| \geq |A'|$, then we can apply the induction hypothesis to $F' = A' \cup B'$. Note that

$$d(u, F) = |A_u| + |B_u| \geq |B| + 1 > |A|.$$

Using this, we have

$$|B'| = |B_u| > |A| - |A_u| = |A'|.$$

Therefore, the result follows by induction. $\square$

Lemma 1.13 will be used to find a way to 1-extend path systems which contain relatively few vertices.

2. Results

We will first show that for any positive integer k and any graph containing $n \geq 5k - 1$ vertices, every nonspanning path system $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ in G which contains a path of order four or more is 1-extendable if $\delta(G) \geq \frac{n+k}{2}$. To show that the minimum degree condition is best possible, we use an example shown in [12]. Consider the split graph $G = K_b + \overline{K_a}$ where $a = \frac{n-k+1}{2}$ and $b = \frac{n+k-1}{2}$. So, G has n vertices with $\delta(G) = \frac{n+k-1}{2}$. Suppose that the endpoints of each path of $\mathcal{P}$ are in K_b. Then for every $P \in \mathcal{P}$, $|P \cap \overline{K_a}| \leq |P \cap K_b| - 1$. Thus, $|\mathcal{P} \cap \overline{K_a}| \leq |\mathcal{P} \cap K_b| - k$. However, if $\mathcal{P}$ covered all of the vertices of K_b, then

$$|\mathcal{P} \cap \overline{K_a}| \leq \frac{n+k-1}{2} - k = \frac{n-k-1}{2} < |\overline{K_a}|.$$

So, $\mathcal{P}$ cannot cover all of the vertices of G. Thus, at some point, the path system is not 1-extendable.

Theorem 2.1. *Let $k \geq 1$ and $n \geq 5k - 1$. Suppose $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ is a nonspanning path system in G which contains at least one path of order four or greater. If $\delta(G) \geq \frac{n+k}{2}$, then $\mathcal{P}$ is 1-extendable. Furthermore, the minimum degree condition is best possible.*

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a nonspanning path system in G which contains at least one path of order four or greater. Let $Q = G - \mathcal{P}$, $p = |\mathcal{P}|$, and $q = |Q|$. We prove the result by contradiction. Thus, assume that $\mathcal{P}$ is not 1-extendable.

Claim 2.2. *For all $w \in V(Q)$, $N(w, \mathcal{P})$ is an independent set of $\mathcal{P}$.*

Proof. Suppose that $w \in Q$ is adjacent to two adjacent vertices x and y on some P_i. Then, we can extend our path system by using the path xwy which contradicts our choice of $\mathcal{P}$. $\square$

By Fact 1.12 and Claim 2.2,

$$\Delta(Q, P) \leq \alpha(\mathcal{P}) \leq \frac{p+k}{2}, \tag{2.1}$$

so for all $w \in V(Q)$,

$$d(w, Q) \geq \frac{n+k}{2} - \alpha(\mathcal{P}) \geq \frac{n+k}{2} - \left(\frac{p+k}{2}\right) \geq \frac{q}{2}. \tag{2.2}$$

Claim 2.3. *If $w \in Q$ is adjacent to $x, y \in V(P_i)$ where $(x, y)_{P_i}$ has no neighbors in Q, then $\mathrm{dist}_{P_i}(x, y) = 2$.*

Proof. Let R be the vertices of $(x, y)_{P_i}$, and let $r = |R|$. Note that $r \neq 0$ because otherwise $\mathcal{P}$ is 1-extendable. Assume that $r \neq 1$. Suppose first that $2 \leq r \leq q$. Remove R from P_i and use the edges xw and wy to complete the path P_i. This gives a new path system $\mathcal{P}'$ with $|\mathcal{P}'| = |\mathcal{P}| - r + 1 = p - r + 1$. Now, we need to show that we can insert the vertices of R into $\mathcal{P}'$. In order to use Proposition 1.11 to do this, we must show that

$$\delta(R, R \cup V(P')) > \alpha(P') + |R| - 1.$$

Recall that by assumption R has no edges to the original set $Q = G - \mathcal{P}$ and in particular R has no edges to the vertex w. Hence,

$$\delta(R, R \cup V(P')) = \delta(R, R \cup V(P)) \geq \frac{n+k}{2} = \frac{p+q+k}{2}.$$

Using this inequality and the fact that $r \leq q$, we have

$$\begin{aligned}\delta(R, R \cup V(P')) &\geq \frac{p+q+k}{2} \\ &\geq \frac{p+r+k}{2} \\ &= \frac{p-r+k+1}{2} + r - \frac{1}{2} \\ &> \frac{(p-r+1)+k}{2} + r - 1 \\ &\geq \alpha(P') + |R| - 1.\end{aligned}$$

By Proposition 1.11, we may insert R into the path system $\mathcal{P}'$ to get a path system $\mathcal{P}''$ where $|\mathcal{P}''| = |\mathcal{P}| + 1$. Thus, $\mathcal{P}$ is 1-extendable if $2 \leq r \leq q$. Therefore, we must have $r \geq q + 1$. Then, for $w \in Q$,

$$\begin{aligned}d(w, \mathcal{P}) &\geq \frac{n+k}{2} - d(w, Q) \\ &\geq \frac{n+k}{2} - (q-1) \\ &= \frac{p-q+k+2}{2} \\ &> \frac{p-r+k+2}{2} \\ &> \alpha(\mathcal{P} - R).\end{aligned}$$

Since w has no neighbors in R, w must be adjacent to two consecutive vertices in $\mathcal{P} - R$. However, this cannot occur or else we could 1-extend $\mathcal{P}$. Therefore, $r = 1$ and $\mathrm{dist}_{P_i}(x, y) = 2$. □

Claim 2.4. *Let $w \in V(Q)$ and $\{x, y, z\} \subseteq N(w, P_i)$ for some path P_i of $\mathcal{P}$, appearing in that order in P_i. Then* $\mathrm{dist}_{P_i}(x, z) \geq 5$.

Proof. Claim 2.2 forces $\mathrm{dist}_{P_i}(x, z) \geq 4$, so assume we have equality. That is, assume that $\mathcal{P}$ has a path P_i containing a sub-path $xuyvz$ where $wx, wy, wz \in E(G)$. Claim 2.2 shows that $uw \notin E(G)$. If $d(u, Q) = 0$, then

$$d(u, \mathcal{P} - u) \geq \frac{n+k}{2} > \frac{p-1+k+1}{2} \geq \alpha(\mathcal{P} - u).$$

Thus, u could be inserted somewhere else in $\mathcal{P} - u$, and we could use the edges xw and wy to get a path system of order $|\mathcal{P}| + 1$.

Therefore, $d(u, Q) \geq 1$ and so let u' be a vertex in $N(u, Q)$. If $N(w, Q) \cap N(u', Q) = \emptyset$, then Inequality (2.2) implies that $wu' \in E(G)$ and $d(w, Q) = d(u', Q) = \frac{q}{2}$. This is because there are only $q - 2$ vertices in $Q - \{u', w\}$, so if u' and w have no neighbors in common, then in order to exceed the lower bound in Inequality (2.2), u' and w must be adjacent and both must be adjacent to exactly $\frac{q-2}{2}$ other vertices in Q. Yet, if this is the case, then $d(w, \mathcal{P}) = d(u', \mathcal{P}) \geq \frac{p+k}{2} \geq \alpha(\mathcal{P})$. However, Claim 2.2 implies that $d(u', \mathcal{P}) = d(w, \mathcal{P}) = \alpha(\mathcal{P})$ and $N(w, \mathcal{P}) = N(u', \mathcal{P})$. However, this cannot occur since $uu' \in E(G)$ but $wu \notin E(G)$.

Hence, $N(w, Q) \cap N(u', Q) \neq \emptyset$, so there must be a path $ww'u'$ in Q, and then replacing the sub-path $xuyvz$ of P_i with $xuu'w'wz$ yields a path system of order $|\mathcal{P}| + 1$. □

Claim 2.5. *Let $w \in V(Q)$ and $\{x, y\} \subseteq N(w, P_i)$ for some path P_i of $\mathcal{P}$. Then* $\mathrm{dist}_{P_i}(x, y) \neq 3$.

Proof. Suppose $\mathcal{P}$ has a path P_i containing a sub-path $xuvy$ such that, for some $w \in V(Q)$, $wx, wy \in E(G)$. Note that $N(\{u, v\}, w \cup N(w, Q)) = \emptyset$, or a path system of order $|\mathcal{P}| + 1$ would exist. Now, by Inequality (2.2), $d(w, Q) \geq \frac{q}{2}$. So, since u and v are not adjacent to w and have no neighbors in $N(w, Q)$, $d(u, Q), d(v, Q) \leq \frac{q}{2} - 1$ which, with the minimum degree condition, implies that $d(u, \mathcal{P}), d(v, \mathcal{P}) \geq \frac{p+k+2}{2}$. Now, remove u and v from P_i and use the edges wx and wy to complete the path. This gives a new path system $\mathcal{P}'$ with $|\mathcal{P}'| = |\mathcal{P}| - 2 + 1 = p - 1$. We need to insert u and v into $\mathcal{P}'$. In order to use Proposition 1.11 to do this, we need to show that

$$\delta(\{u, v\}, \{u, v\} \cup V(\mathcal{P}')) \geq \alpha(\mathcal{P}') + 2 - 1 = \alpha(\mathcal{P}') + 1.$$

Since u and v are adjacent to w, we get the following chain of inequalities:

$$\begin{aligned}\delta(\{u,v\},\{u,v\}\cup V(P')) &= \delta(\{u,v\},\{u,v\}\cup V(P))\\ &\geq \frac{p+k+2}{2}\\ &= \frac{(p-1)+k+3}{2}\\ &> \frac{(p-1)+k}{2}+1\\ &\geq \alpha(\mathcal{P}')+1.\end{aligned}$$

Consequently, Proposition 1.11 shows that u and v can be inserted in $\mathcal{P}'$ to get a path system $\mathcal{P}''$ where $|\mathcal{P}''| = |\mathcal{P}|+1$. This contradicts our assumption that $\mathcal{P}$ is not 1-extendable. □

Claim 2.6.

$$\Delta(Q,\mathcal{P}) \leq \frac{p+k-2}{2}.$$

Proof. Assume to the contrary that $\Delta(Q,\mathcal{P}) > \frac{p+k-2}{2}$. By Inequality (2.1),

$$\frac{p+k-1}{2} \leq \Delta(Q,\mathcal{P}) \leq \frac{p+k}{2}. \tag{2.3}$$

Let w be a vertex of Q with $d(w,\mathcal{P}) = \Delta(Q,\mathcal{P})$. By Fact 1.12 and Inequality (2.3), w is adjacent to a maximum independent set of $\mathcal{P}$ and $\mathcal{P}$ has at most one path of even order.

Assume without loss of generality that P_1 is a path of largest order in $\mathcal{P}$. Note that $|P_1| \geq 4$. If $|P_1| \geq 5$, Claims 2.4 and 2.5 imply that w cannot be adjacent to a maximum independent set of P_1. Thus, $|P_1| = 4$, the $k-1$ other paths of $\mathcal{P}$ must have order three, and $|V(\mathcal{P})| = 3k+1$.

Let $P_1 = a_1xyb_1$. Since $d(w,\mathcal{P}) = \alpha(\mathcal{P})$, w is adjacent to one of the vertices a_1 and b_1. Without loss of generality, assume $a_1w \in E(G)$. By Claim 2.5, $b_1w \notin E(G)$, so $wy \in E(G)$ in order to ensure that w is adjacent to a maximum independent set of $\mathcal{P}$. Note that $b_1x \notin E(G)$ since otherwise the path b_1xywa_1 would 1-extend the path system. Thus, $d(b_1,\mathcal{P}) \leq |\mathcal{P}|-2 = 3k-1$. Since $n \geq 5k-1$,

$$d(b_1,Q) \geq \frac{n+k}{2} - (3k-1) \geq \frac{1}{2}.$$

Thus, b_1 has a neighbor in Q. However, $b_1w \notin E(G)$ implies that there exists a vertex w' in Q, with $w' \neq w$ such that $b_1w' \in E(G)$.

Since $d(w,\mathcal{P}) = \alpha(\mathcal{P}) = 2k$, then

$$d(w,Q) \geq \frac{n+k}{2} - 2k = \frac{p+q+k}{2} - 2k = \frac{q+1}{2}.$$

Note that $d(w',Q) \geq \frac{q+1}{2}$ as well since $d(w',\mathcal{P}) \leq d(w,\mathcal{P})$. Thus, w and w' have a common neighbor w'' in Q, and the path $b_1w'w''wa_1$ can be used to 1-extend the path system. However, this contradicts the assumption that $\mathcal{P}$ is not 1-extendable. Hence, $\Delta(Q,\mathcal{P}) \leq \frac{p+k-2}{2}$. □

With these previous claims, we can now gain greater freedom in 1-extending path systems by showing that Q is panconnected.

Claim 2.7. *The subgraph Q has at least four vertices, $\delta(Q,Q) \geq \frac{q+2}{2}$, and Q is panconnected.*

Proof. Since $\delta(G) \geq \frac{n+k}{2}$, Claim 2.6 implies that

$$\delta(Q,Q) \geq \frac{n+k}{2} - \Delta(Q,\mathcal{P}) \geq \frac{q+2}{2}. \tag{2.4}$$

Now, $q \neq 0$ since $\mathcal{P}$ is nonspanning. Also, since $\delta(Q,Q) \leq q-1$, from Inequality (2.4) we have

$$q-1 \geq \frac{q+2}{2}.$$

This inequality implies that $q \geq 4$. Thus, by Theorem 1.1, Q is panconnected. □

This claim implies the following:

Claim 2.8. *If there exist two independent edges uu' and vv' such that $u', v' \in V(Q)$ and for some path P_i of $\mathcal{P}$, $u, v \in V(P_i)$, then* $\mathrm{dist}_{P_i}(u, v) \le 2$.

Proof. Let $R = (u, v)_{P_i}$ and $r = |R|$. Assume that $r \ge 2$ and that r is minimal under these conditions. That is, assume that there does not exist any other edge independent of uu' and vv' with one end-vertex in Q and one end-vertex in R since otherwise r would not be minimal.

If $r \le q - 1$, then by Claim 2.7, there is a $[u', v']$-path P' in Q of order $r + 1$. Hence, replacing P_i by

$$(P_i - R) \cup uu' \cup P' \cup v'v,$$

we get a path system of order $|\mathcal{P}| + 1$.

Hence, assume $r \ge q$. Since $q \ge 4$ by Claim 2.7, we may find a vertex $w \in V(Q) - u' - v'$. By the minimality of r, we see that $d(w, R) = 0$. Thus, since $r \ge q$,

$$\begin{aligned} d(w, \mathcal{P} - R) &= d(w, \mathcal{P}) \\ &= d(w) - d(w, Q) \\ &\ge \frac{n+k}{2} - q + 1 \\ &= \frac{p - q + k + 2}{2} \\ &\ge \frac{p - r + k + 2}{2} \\ &> \alpha(\mathcal{P} - R), \end{aligned}$$

which cannot occur by Claim 2.2. □

Claim 2.9. *For all* $1 \le i \le k$, $|N(Q, P_i)| \le 3$.

Proof. Suppose $|N(Q, P_i)| \ge 4$ for some $1 \le i \le k$. Let $v_1, v_2, v_3, v_4 \in N(Q, P_i)$. Assume that the v_i's are chosen to minimize $\mathrm{dist}_{P_i}(v_1, v_4)$. Let

$$P_i = S_1 v_1 S_2 v_2 S_3 v_3 S_4 v_4 S_5$$

where each S_i is a possibly empty sub-path of P_i. By the minimality of $\mathrm{dist}_{P_i}(v_1, v_4)$, Q has no neighbors on S_2, S_3, or S_4. In addition, note that $N(v_1, Q) = N(v_4, Q) = \{w\}$ for some $w \in V(Q)$ by Claim 2.8.

Suppose $v_2 w \in E(G)$, then $|S_2| = 1$ by Claim 2.3. If $v_3 w \in E(G)$, then Claim 2.3 implies that $|S_3| = 1$. However, then w and v_1, v_2, v_3 violate Claim 2.4. Thus, $v_3 w' \in E(G)$ for some $w' \in Q - \{w\}$. However, this gives two independent edges wv_1, $w'v_3$ with $\mathrm{dist}_{P_i}(v_1, v_3) \ge 3$. This cannot occur by Claim 2.8.

Therefore, $v_2 w \notin E(G)$. So, for some $w_1 \in Q - \{w\}$, $v_2 w_1 \in E(G)$. By Claim 2.8, $|S_2| \in \{0, 1\}$. Suppose first that $|S_2| = 0$. Then, by Claim 2.8, $|S_3| = |S_4| = 0$. Note that if there exists a vertex $\overline{w} \in N(v_2, Q) \cap N(w, Q)$, then we can 1-extend by using $v_1 v_2 \overline{w} w v_4$. So, $N(v_2, Q) \cap N(w, Q) = \emptyset$. However, since $v_2 w \notin E(G)$ and, by Claim 2.7, $d(w, Q) \ge \frac{q+2}{2}$, we have

$$d(v_2, Q) \le q - \left(\frac{q+2}{2} + 1\right) = \frac{q-4}{2}.$$

Similarly, we have that $N(v_3, Q) \cap N(w, Q) = \emptyset$, $v_3 w \notin E(G)$, and $d(v_3, Q) \le \frac{q-4}{2}$. So,

$$\delta(\{v_2, v_3\}, \mathcal{P}) \ge \frac{n+k}{2} - \frac{q-4}{2} = \frac{p+k}{2} + 2 > \alpha(\mathcal{P}) + |\{v_2, v_3\}| - 1.$$

However, by Proposition 1.11, we can insert v_2 and v_3 into our path system and use $v_1 w$, $w v_4$ to 1-extend.

Thus, we must have $|S_2| = 1$. As before, $|S_3| = |S_4| = 0$ by Claim 2.8. Consequently, $v_3 w \notin E(G)$ since otherwise w would be adjacent to two consecutive vertices on P_i. Similarly, $v_3 w_1 \notin E(G)$. So, $v_3 w_2 \in E(G)$ for some $w_2 \in Q - \{w, w_1\}$. However, we then have two independent edges $v_1 w$, $v_3 w_2$ with $\mathrm{dist}_{P_i}(v_1, v_3) = 3$, which cannot occur by Claim 2.8.

Therefore, we must have $|N(Q, P_i)| \le 3$ for all $1 \le i \le k$. □

Claim 2.10. *Every vertex in $\mathcal{P}$ has a neighbor in Q.*

Proof. Since $\mathcal{P}$ has a path of order at least four, Claim 2.9 implies that there exists at least one vertex $v \in V(\mathcal{P})$ such that $d(v, Q) = 0$. Now, $\frac{n+k}{2} \le d(v, \mathcal{P}) \le p - 1$, and so $p \ge \frac{n+k+2}{2}$. Since $n = p + q$, we have

$$n \ge 2q + k + 2. \tag{2.5}$$

By Claim 2.9, $d(w, P_i) \leq 3$ for any $w \in Q$ and any P_i in $\mathcal{P}$. Suppose that $d(w, P_i) = 3$ for some $w \in Q$ and P_i. Then, Claim 2.9 also implies that no other vertex on P_i outside of $N(w, P_i)$ has a neighbor in Q. Now, Claim 2.3 implies that the distance between consecutive neighbors of w on P_i must be exactly two. So, if x and y are the two neighbors of w on P_i which are furthest apart on P_i, then $\mathrm{dist}_{P_i}(x, y) = 4$. However, this violates Claim 2.9 which states that the distance between these two furthest neighbors must be greater than or equal to five. Thus, $d(w, P_i) \leq 2$ for all $w \in Q$ and all paths P_i.

So, we have

$$\delta(Q, Q) \geq \frac{n+k}{2} - 2k = \frac{n-3k}{2} = \frac{p+q-3k}{2}.$$

This inequality and $\delta(Q, Q) \leq q - 1$ give us

$$p - q \leq 3k - 2. \tag{2.6}$$

Now, let F be an auxiliary graph with vertex set $B \cup A$, where each vertex in B corresponds to a specific vertex of Q and $A = \{v_1, \dots, v_k\}$ is a set of vertices distinct from B, each corresponding to a path of $\mathcal{P}$. Let the subgraph of F induced by $V(B)$ be an isomorphic copy of Q and place an edge between a vertex $u \in B$ and $v_i \in A$ if, and only if, the vertex $w \in Q$ corresponding to u satisfies $d(w, P_i) = 2$.

Let w be any vertex of Q and u the corresponding vertex of B. We have $d(u, A) \geq d(w, \mathcal{P}) - k$ and $d(u, B) = d(w, Q)$. Hence, $\delta(B, F) \geq \delta(Q, G) - k \geq \frac{n-k}{2}$, so using Inequality (2.5), we get $\delta(B, F) \geq q + 1 = |B| + 1$. Also, $n \geq 5k - 1$ and Inequality (2.6) imply that

$$q = \frac{n - (p-q)}{2} \geq \frac{n - (3k-2)}{2} \geq \frac{5k - 1 - 3k + 2}{2} > k.$$

Thus, we get

$$|B| = |Q| = q > k = |A|,$$

and the conditions needed to apply Lemma 1.13 are met.

So, by Lemma 1.13, there are three mutually adjacent vertices $u, u' \in B$ and $v_i \in A$ in F. Let $w, w' \in Q$ be the vertices corresponding to u and u'. By the definition of F, $ww' \in E(G)$ and $d(w, P_i) = d(w', P_i) = 2$. Now, Claim 2.9 implies that w and w' must have at least one neighbor in common on P_i. Let x be the common neighbor of w and w' on P_i, and let z be the other neighbor of w on P_i. Note that the edges $w'x$ and wz are independent edges. By Claim 2.8, $\mathrm{dist}_{P_i}(x, z) \leq 2$. Note that $\mathrm{dist}_{P_i}(x, z) \neq 1$ since otherwise w would be adjacent to two consecutive vertices on P_i, which would violate Claim 2.2. Thus, $\mathrm{dist}_{P_i}(x, z) = 2$ and there is exactly one vertex, say y, between x and z on P_i. However, this means that we can find a path system of order $|\mathcal{P}| + 1$ by replacing P_i with $(P_i - y) \cup xw'wz$. □

Claims 2.9 and 2.10 imply that for all i, $|P_i| \leq 3$. However, this contradicts the fact that we have a path of order four and concludes the proof of Theorem 2.1. □

We now proceed to show that for all $k \geq 2$, $n \geq 5k - 1$, and any vector $\vec{S}$ of $2k$ distinct vertices, an $\vec{S}$-linkage exists and is 1-extendable if $\delta(G) \geq \frac{n+2k-1}{2}$. To see that the minimum degree condition is best possible, we consider the join of several graphs. Let A be a complete graph on the vertices $\{a_1, a_2, \dots, a_k\}$, B be a complete graph on the vertices $\{b_1, b_2, \dots, b_k\}$, C be a complete graph on $\frac{n-2k}{2}$ vertices, and D be a complete graph on $\frac{n-2k}{2}$ vertices. Consider the graph $G = (A + B) \cup (B + C) \cup (C + D) \cup (A + D)$, and notice that $\delta(G) = \frac{n+2k-2}{2}$. Let $\vec{S} = \langle a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k \rangle$. Note that the edges $a_1b_1, a_2b_2, \dots, a_kb_k$ are an $\vec{S}$-linkage $\mathcal{P}$ of order $2k$. In order to extend this $\mathcal{P}$ by one more vertex, we need to replace an edge a_ib_i with the path a_ixb_i for some $x \in V(C) \cup V(D)$. However, the adjacencies of G ensure that any $[a_i, b_i]$-path that contains a vertex of C or D must have at least one vertex from both C and D. Thus, no $[a_i, b_i]$-path of order three exists and consequently, $\mathcal{P}$ is not 1-extendable.

Theorem 2.11. *If $k \geq 2$ and G is a graph on $n \geq 5k - 1$ vertices with $\delta(G) \geq \frac{n+2k-1}{2}$, then G is k-linked and for every vector $\vec{S}$ of $2k$ distinct vertices, every nonspanning $\vec{S}$-linkage is 1-extendable. Further, this minimum degree result is best possible.*

Proof. Note that since $\delta(G) \geq \frac{n+2k-1}{2} > \frac{n+2k-3}{2}$ and $n \geq 5k - 1$, G is k-linked by Theorem 1.7. If for every vector $\vec{S}$ of $2k$ distinct vertices of G, we have that every nonspanning $\vec{S}$-linkage is 1-extendable, then we are done. So, assume there exists a vector $\vec{S}$ which has a nonspanning $\vec{S}$-linkage that is not 1-extendable. Among all $\vec{S}$-linkages with this property, choose the $\vec{S}$-linkage $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ such that $|V(\mathcal{P})|$ is minimized.

Let $Q = G - \mathcal{P}$. Also, let $p = |V(\mathcal{P})|$ and $q = |V(Q)|$. We now wish to show by contradiction that we can extend P by one vertex. If $\mathcal{P}$ consists only of edges (that is, $p = 2k$), then every $x \in \mathcal{P}$ satisfies $d(x, \mathcal{P}) \leq 2k - 1$. So, $d(x, Q) \geq \frac{n+2k-1}{2} - 2k + 1 = \frac{n-2k+1}{2}$. However, this means that any two vertices of $\mathcal{P}$ must have a common neighbor in Q, and consequently, $\mathcal{P}$ is 1-extendable.

Thus, assume that $p \geq 2k + 1$. If $\mathcal{P}$ contains a path of order four or more, then since $\delta(G) \geq \frac{n+2k-1}{2} > \frac{n+k}{2}$, then $\mathcal{P}$ is 1-extendable by Theorem 2.1. Therefore, all paths of $\mathcal{P}$ must have order three or less, and so $p \leq 3k$.

Since we have previously taken care of the case where the path system consists only of edges, we assume there is a path xyz of order three. Let λ_{xz} equal 1 if $N(x, Q) \neq N(z, Q)$ and 0 otherwise. We will also consider $d(x, z)$ which equals one if $xz \in E(G)$ and zero if $xz \notin E(G)$. We choose the path xyz so that $d(x, z)$ is minimized.

We have

$$d(x, Q), d(z, Q) \geq \frac{n + 2k - 1}{2} - (p - 2) - d(x, z) \tag{2.7}$$

since x and z can be adjacent to at most $p - 2$ vertices in $\mathcal{P} - \{x, z\}$ and $d(x, z)$ subtracts an additional edge from the above inequality if $xz \in E(G)$. Assuming, without loss of generality, that $|N(x, Q)| \geq |N(z, Q)|$, we let $w \in N(x, Q)$ such that $w \notin N(z, Q)$ if $N(x, Q) \neq N(z, Q)$. If there is an edge $uv \in E(N(x, Q), N(z, Q))$ then replacing xyz with $xuvz$ will extend our path system by one vertex. Hence, $E(N(x, Q), N(z, Q)) = \emptyset$, so $d(w, Q) \leq q - d(z, Q) - \lambda_{xz}$. Note that λ_{xz} is used here to account for the possibility that $N(x, Q) = N(z, Q)$. Using Fact 1.12, Inequality (2.7), and the minimum degree condition, we get

$$\begin{aligned} \frac{n + 2k - 1}{2} &\leq d(w) \\ &= d(w, \mathcal{P}) + d(w, Q) \\ &\leq \frac{p + k}{2} + q - \left(\frac{n + 2k - 1}{2} - p + 2 - d(x, z)\right) - \lambda_{xz}, \end{aligned}$$

which is equivalent to

$$n + 2k - 1 \leq n - 2 + d(x, z) - \lambda_{xz} + \frac{p + k}{2}.$$

Thus, we have that

$$3k + 2(\lambda_{xz} - d(x, z)) + 2 \leq p.$$

Since we know that $p \leq 3k$, we must have $\lambda_{xz} - d(x, z) \leq -1$. However, by the definitions of $d(x, z)$ and λ_{xz}, we have $\lambda_{xz} - d(x, z) \geq -1$. Thus, $p = 3k$, $d(x, z) = 1$, and $\lambda_{xz} = 0$. That is, $\mathcal{P}$ must be composed of k paths of order three, $xz \in E(G)$, and $N(x, Q) = N(z, Q)$. Since $d(x, z)$ was chosen to be minimal and $d(x, z) = 1$, each path $v_1v_2v_3 \in \mathcal{P}$ has v_1 adjacent to v_3. Note that Fact 1.12 implies that

$$d(w, Q) \geq \frac{n + 2k - 1}{2} - d(w, \mathcal{P}) \geq \frac{n + 2k - 1}{2} - \frac{p + k}{2} \geq \frac{q + k - 1}{2}$$

for all $w \in Q$. Thus, in particular, any two vertices $w_1, w_2 \in N(x, Q) = N(z, Q)$ have k common neighbors in Q. Therefore, we may find a path P' of length two in Q between w_1, w_2. We then replace xyz with $xP'z$. This gives us a net gain of two vertices. Take another path $v_1v_2v_3 \in \mathcal{P}$. Since $v_1v_3 \in E(G)$, replace $v_1v_2v_3$ with v_1v_3. Since this removes another vertex from $\mathcal{P}$, this gives us a path system with exactly one more vertex, a contradiction. □

Corollary 2.12. *If $k \geq 2$ and G is a graph on $n \geq 5k - 1$ vertices with $\delta(G) \geq \frac{n+2k-1}{2}$, then G is pan-k-linked. Further, this minimum degree result is best possible.*

In closing, note that the condition $k \geq 2$ is needed in the last two results. In [13], Faudree and Schelp showed that there exist graphs with minimum degree $\delta(G) = \frac{n+1}{2}$ which are not panconnected. Hence, the minimum degree condition $\delta(G) \geq \frac{n+2k-1}{2}$ cannot work for $k = 1$.

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments and suggestions for improving the manuscript.

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