A NOTE ON THE RAMSEY NUMBER FOR THE UNION OF GRAPHS VERSUS MANY GRAPHS

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1. Introduction.

All graphs in this article are finite simple graphs. In a coloring of the edges of a graph G with t-colors $(t > 0)$, a copy of some graph H, each of whose edges is colored i will be termed an i-colored H. The ramsey number, $r(G_1, G_2, \ldots, G_t)$, is the smallest integer p such that in any coloring of the edges of K_p from a set of t-colors, there exists an i-colored copy of G_i for some i, $1 \leq i \leq t$. Ramsey numbers have received a great deal of attention recently. A result of interest includes:

Theorem A (Burr and Erdos [2]). If $N = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$ and $r(G, K_N) = (|V(G)| - 1)(N - 1) + 1$, then $r(G, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = (|V(G)| - 1)(N - 1) + 1$.

In [6] the following definition was given: $\mathcal{G}_\beta(H) = \{g \mid g$ a connected graph and $r(g, H) = (|V(g)| - 1)(\chi(H) - 1) + \beta + t_1(H) - 1\}$, where $t_1(H)$ is the minimum, over all critical colorings of H, of the order of the smallest color class. The following theorem was then shown:

Theorem B ([6]). If $g_i \in \mathcal{G}_{\beta_i}(H)$, $G = \bigcup_{i=1}^{R} g_i$ and

$$p = \max_{i \leq j \leq c(G)} \{(j - 1)(\chi(H) - 2) + \sum_{i=j}^{c(G)} i k_i\} + t_1(H) - 1,$$

where $c(G)$ is the order of the largest component of G and k_i is the number of components of G of order i, then

$$p \leq r(G, H) \leq p + \max_i \{\beta_i\}.$$

The purpose of this paper is to establish bounds for the ramsey number $r(G, H_1, H_2, \ldots, H_t)$, where G is the disjoint union of

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graphs. We further obtain the multicolor ramsey numbers for various cases.

2. Upper and Lower Bounds

Let $H_1, H_2, \ldots, H_t$ $(t \geq 2)$ be graphs and let

$r = r(K_{\omega_1}, K_{\omega_2}, \ldots, K_{\omega_t})$, where ω_i = clique number of H_i. Define $\mathcal{G}_\beta(H_1, H_2, \ldots, H_t) = \{g \mid g$ is connected and $r(g, H_1, H_2, \ldots, H_t) = (|V(g)| - 1)(r-1) + \beta + 1\}$.

(1) For i such that $1 \leq i \leq \ell$, let $g_i \in \mathcal{G}_\beta(H_1, H_2, \ldots, H_t)$ and let $G = \bigcup_{i=1}^{\ell} g_i$.

Theorem 1. If G is as defined in (1) then

$$r(G, H_1, H_2, \ldots, H_t) \geq \max_{1 \leq j \leq c(G)} \left[(j-1)(r-2) + \sum_{i=1}^{c(G)} ik_i\right]$$

where k_i is the number of components of G with order i, and $c(G)$ is the size of the largest component of G.

Proof. For convenience let $p_j = \sum_{i=j}^{c(G)} ik_i$ and choose j_0 such that

$$p = (j_0 - 1)(r-2) + p_{j_0} = \max_{1 \leq j \leq c(G)} \{(j-1)(r-2) + \sum_{i=j}^{c(G)} ik_i\}.$$

Consider the following factorization

$K_{p-1} = R \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_t$ where $R = K_{p_{j_0}-1} \cup (r-2)K_{j_0-1}$ and the B_i's are formed as follows: Consider a factorization of $K_{r-1} = D_1 \oplus D_2 \oplus \ldots \oplus D_t$ such that $K_{\omega_i} \nsubseteq D_i$ for $i = 1, 2, \ldots, t$. There are $(r-1)$ components in R; associate with each component a vertex in the above factorization of K_{r-1}. The edges of B_i are precisely those edges between components of R for which there is a corresponding edge in D_i.

It is clear that $K_{\omega_i} \nsubseteq B_i$ hence $H_i \nsubseteq B_i$, for each i, for this would require choosing two vertices from one component of R and the edge between these two vertices would not be in B_i.

To see that $G \nsubseteq R$, we concentrate on the subgraph G_{j_0} of G, which consists of all components of G with j_0 or more vertices. Clearly $G_{j_0} \not\subseteq K_{p_{j_0}-1}$ since $|V(G_{j_0})| > p_{j_0} - 1$. Further, K_{j_0-1} is too small to contain any component of G_{j_0}. Thus $G_{j_0} \not\subseteq R$,

hence $G \not\subseteq R$ and the result follows. □

Theorem 2. If G is as defined in (1) then

$$r(G, H_1, H_2, \ldots, H_t) \leq \max_{1\leq j\leq c(G)} \left[(j-1)(r-2) + \sum_{i=j}^{c(G)} ik_i\right] + \beta$$

where k_i is the number of components of G with order i, and $c(G)$ is the order of the largest component of G.

Proof. We assume the notation developed in Theorem 1. Let G_j denote the subgraph of G consisting of all components of order at least j. Then the order of G_j is p_j. Consider an arbitrary factorization of $K_p = R \oplus B_1 \oplus \ldots \oplus B_t$ in which $H_i \not\subseteq B_i$ for $i = 1, 2, \ldots, t$, and where $p = (j_0 - 1)(r-2) + \sum_{i=j_0}^{c(G)} ik_i + \beta$. We show that $G \subseteq R$ by descending induction on j.

First suppose $G = G_\ell$ where $\ell = c(G)$. By an easy induction on k_ℓ, the total number of components of G, we show $G = G_\ell \subseteq R$. This is clear for $k_\ell = 1$ by definition. If $k_\ell > 1$ and g is an arbitrary component of G, then the factorization $K_p = R \oplus B_1 \oplus \ldots, \oplus B_t$ induces a factorization of $K_p - V(g)$ with $|V(K_p) - V(g)| = (\ell-1)(r-2) + \ell(k_\ell - 1) + \beta$. Hence, if $G = G_\ell$ then $G \subseteq R$.

To complete the induction assume $G_{j+1} \subseteq R$, for some j, $1 \leq j < c(G)$. Clearly $G_j \subseteq R$ when $G_j = G_{j+1}$, so we may assume that $G_j - V(G_{j+1})$ consists of $k_j (> 0)$ components, each of order j. The graph $G_{j+1} \subseteq R$, so again the factorization $K_p = R \oplus B_1 \oplus \ldots, \oplus B_t$ induces a factorization on $K_p - V(G_{j+1})$ with $|V(K_p) - V(G_{j+1})| = p - \sum_{i=j+1}^{c(G)} ik_i \geq (j-1)(r-2) + jk_j + \beta$, because of our choice of p. As above $G_j - V(G_{j+1}) \subseteq (K_p - V(G_{j+1})) \cap R$. Therefore, $G_j \subseteq R$ and the induction is complete. □

Corollary 3. If $g_1, g_2, \ldots, g_k \in \mathcal{G}_0(H_1, H_2, \ldots, H_t)$ and $G = \bigcup_{i=1}^{k} g_i$ then

$$r(G, H_1, H_2, \ldots, H_t) = \max_{1 \le j \le c(G)} \left[(j-1)(r-2) + \sum_{i=j}^{c(G)} ik_i \right]$$

where $c(G)$, r and k_i are as defined before.

3. Applications and Conclusions.

Theorem A allows us to conclude that $T_m \in \mathcal{G}_0(K_{n_1}, \ldots, K_{n_t})$ where T_m is any tree on m vertices. Hence we may determine $r(F, K_{n_1}, K_{n_2}, \ldots, K_{n_t})$ where F is any forest and $n_1, n_2, \ldots, n_t$ are positive integers.

Corollary 4. If F is any forest and $n_1, n_2, \ldots, n_t$ are positive integers then

$$r(F, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = \max_{1 \le j \le c(G)} \left[(j-1)(r-2) + \sum_{i=j}^{c(F)} ik_i \right]$$

where $r = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$ and k_i is the number of components of F of order i.

In [1] it was shown that $C_m \in \mathcal{G}_0(K_n)$ when $m > n^2 - 2$. Now applying Corollary 3, we may obtain the ramsey number for $G = (\bigcup_{i=1}^{k_1} T_{s_i}) \cup (\bigcup_{j=1}^{k_2} C_{m_j})$, $(m_j > n^2 - 2)$, versus $K_{n_1}, K_{n_2}, \ldots, K_{n_t}$, with $r = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$.

Burr and Erdos [2] have shown that any sufficiently large graph homeomorphic to a connected graph is in $\mathcal{G}_0(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$. Hence we may determine the ramsey number for unions of these graphs versus many complete graphs.

Finally, we state a theorem bounding the ramsey number and allowing one to vary the $\mathcal{G}_\beta$ classes.

Theorem 5. If $g_i \in \mathcal{G}_{\beta_i}(H_1, H_2, \ldots, H_t)$ where $1 \le i \le k$, $G = \bigcup_{i=1}^{k} g_i$ and $p = \max_{1 \le j \le c(G)} \left[(j-1)(r-2) + \sum_{i=j}^{c(G)} ik_i \right]$ then

$$p \le r(G, H_1, H_2, \ldots, H_t) \le p + \max_i \{\beta_i\}$$

where $c(G)$, r and k_i are as before.

The proof of Theorem 5 is analogous to that of Theorems 1 and 2 with $\max_i \{\beta_i\}$ substituted for β.

In conclusion, we feel an interesting direction for future work would be to vary the definition of $\mathcal{G}_\beta(H_1, H_2, \ldots, H_t)$, in some fashion, to increase the number of known graphs in $\mathcal{G}_0(H_1, H_2, \ldots, H_t)$. Perhaps increased knowledge on the orders of the color classes would be of help.

Another possibility might be to determine classes of graphs, other than complete graphs, for which T_m (or other graphs) is in $\mathcal{G}_0(H_1, H_2, \ldots, H_t)$. This might be attempted in the manner of [1], [3], [5], [7] and [8].

References

1. A. Bondy and P. Erdös, Ramsey Numbers for Cycles in Graphs, J.C.T. 14B, 1973, 46-54.

2. S. A. Burr and P. Erdös, Generalizations of a Ramsey Theoretic Result of Chvátal, (preprint).

3. G. Chartrand, R. J. Gould, and A. D. Polimeni, On Ramsey Numbers of Forest Versus Nearly Complete Graphs, J. Graph Theory 4(1980), no. 2, 233-239.

4. V. Chvátal, Tree-Complete Graph Ramsey Numbers, J. Graph Theory 1, (1977), 93.

5. R. J. Faudree, C. C. Rousseau, and R. H. Schelp, Generalizations of a Ramsey Result of Chvátal, to appear.

6. R. J. Gould and M. S. Jacobson, Bounds for the Ramsey Number of a Disconnected Graph Versus Any Graph, J. Graph Theory (to appear).

7. R. J. Gould and M. S. Jacobson, On the Ramsey Number of Trees Versus Various Graphs with Clique Number n, (preprint).

8. M. S. Jacobson, On Various Extensions of Ramsey Theory, Doctoral Dissertation, Emory University, 1980.

9. S. Stahl, On the Ramsey Number $r(F, K_m)$ Where F is a Forest, Can. J. Math, vol. 27, no. 3, 1975, 585-589.