On H-Immersions

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Abstract: For a fixed multigraph H, possibly containing loops, with $V(H) = \{h_1, \ldots, h_k\}$, we say a graph G is H-linked if for every choice of k vertices $v_1, \ldots, v_k$ in G, there exists a subdivision of H in it G such that v_i represents h_i (for all i). An H-immersion in G is similar except that the paths in G, playing the role of the edges of H, are only required to be edge disjoint. In this article, we extend the notion of an H-linked graph by determining minimum degree conditions for a graph G to contain an H-immersion with a bounded number of vertex repetitions on any choice of k vertices. In particular, we extend results found in [2,3,5]. © 2007 Wiley Periodicals, Inc. J Graph Theory 57: 245–254, 2008

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1. INTRODUCTION AND TERMINOLOGY

A graph is k-linked if for every sequence of $2k$ vertices, $v_1, \ldots, v_k, w_1, \ldots, w_k$, there are internally disjoint paths $P_1, \ldots, P_k$ such that P_i joins v_i and w_i. The class of k-linked graphs is interesting and important, hence the topic has drawn considerable attention (e.g., see [7]). In this article we wish to generalize this class.

Let H be a multigraph, possibly containing loops, with the order of H being $|H| = k$. For any graph G, let $\mathcal{P}(G)$ denote the set of paths in G.

Definition 1.1. *An H-subdivision in G is a pair of mappings $f : V(H) \to V(G)$ and $g : E(H) \to \mathcal{P}(G)$ such that:*

(i) *f is injective;*
(ii) *for every edge $xy \in E(H)$, $g(xy)$ is an $f(x) - f(y)$ path in G and distinct edges of H map to internally disjoint paths in G.*

A graph G is *H-linked* if every injective map $f : V(H) \to V(G)$ can be extended to an H-subdivision. The notion of an H-linked graph is a generalization of the idea of a k-linked graph, since if H is a matching with k edges, G is k-linked if and only if G is H-linked.

Our result will depend on a parameter $\eta(H)$, defined as follows. If H is connected, we define $\eta(H)$ to be the maximum size of an edge cut in H. If H has several components, say $H_1, \ldots, H_c$, where at least one H_i contains an even cycle, we define $\eta(H) = u(H) + \sum_{i=1}^{c} \eta(H_i)$, where $u(H)$ denotes the number of components of H that contain no even cycles. For the remainder of this article, we will refer to components of this type as *uneven* components. If each component of H is uneven, we will let $\eta(H) \cdot |V(H)|\ 1$.

The following was shown [3].

Theorem 1.2. *Let H be a multigraph and G be a simple graph such that $n = |G| \geq 10(|V(H)| + |E(H)|)$. If $\delta(G) \geq \frac{n+\eta(H)-2}{2}$, then G is H-linked.*

This result (and earlier versions, found in [2,5] which were concerned solely with connected multigraphs) along with extension theorems (see [4,5]) established a new framework for viewing path and cycle problems as particular strong connectivity problems. This framework also supplied a way to generalize a number of results in this area at one time (see [4]).

We now extend the notion of an H-linked graph through the idea of an H-immersion, and thus broaden this new framework even more.

Definition 1.3. *An H-immersion in G is a pair of mappings $f : V(H) \to V(G)$ and $g : E(H) \to \mathcal{P}(G)$ such that:*

(i) *f is injective;*
(ii) *for every edge $xy \in E(H)$, $g(xy)$ is an $f(x) - f(y)$ path in G and distinct edges of H map to edge-disjoint paths in G.*

For an immersion $\mathcal{I}$ of H in a graph G, we let $S = f(V(H))$. For a vertex $v \in G - S$, define the vertex repetition number, $r(v, \mathcal{I})$, to be one less than the number of paths in $g(E(H))$ containing v, if v lies on a path of $\mathcal{I}$, and zero otherwise. We then define the vertex repetition number of an immersion $\mathcal{I}$, denoted $r(\mathcal{I})$, to be the sum of the vertex repetition numbers taken over the vertices of $G - S$. An H-immersion with vertex repetition number 0 is in fact an H-subdivision, establishing the connection between these two structures.

In this article we determine a sharp minimum degree condition for a graph G to contain an H-immersion $\mathcal{I}$ on any $|V(H)|$ vertices such that $r(\mathcal{I})$ is bounded from above.

For a graph G, let $\delta(G)$ denote the minimum degree of a vertex in G. For two subsets A and B of the vertex set of G, let $\delta(A, B) = \min_{v \in A} |N(v) \cap B|$, where $N(v)$ denotes the set of neighbors of v in G. The subgraph induced by a set A of vertices is denoted by $\langle A \rangle$. For terminology not defined here, the reader should consult [1].

2. IMMERSIONS AND MINIMUM DEGREE

As mentioned above, an H-subdivision in a graph G is simply an H-immersion $\mathcal{I}$ with $r(\mathcal{I}) = 0$. Our goal is to prove the following extension of Theorem 1.2 to include immersions $\mathcal{I}$ with $r(\mathcal{I}) \geq 1$. In what follows, we say a graph is *nontrivial* if it contains at least one edge.

Theorem 2.1. *Let H be a loopless nontrivial multigraph of order k and G a simple graph such that $|G| = n \geq \max\{8|E(H)|^2 + 8|E(H)|k + 2k^2 - |E(H)| + 2, 34|E(H)| + 4k + 5\}$. If λ is an integer such that $0 \leq \lambda \leq \eta(H) - k + 1$, and $\delta(G) \geq \frac{n+\eta(H)-\lambda-2}{2}$, then any injective map $f : V(H) \to V(G)$ can be extended to an immersion $\mathcal{I}$ with $r(\mathcal{I}) \leq \lambda$.*

To see that this minimum degree is required, suppose first that H has maximum edge cut size $\eta(H)$, and that this cut determines a partition of $V(H)$ into sets X and Y. Let G be a graph formed from two complete graphs G_1 and G_2 of order m that intersect on $\eta(H) - \lambda - 1$ vertices. If S, the image of $V(H)$ under f, is chosen such that the vertices of $f(X)$ lie in $G_1 - G_2$, and the vertices of $f(Y)$ lie in $G_2 - G_1$, then clearly $G_1 \cap G_2$ is not large enough to allow an immersion of H with repetition, number at most λ.

In the case where $\eta(H)$ is not the size of a maximum edge-cut in H, specifically when H is disconnected and has at least one uneven component, we proceed in a slightly different manner. Let $H_1, \ldots, H_t$ denote the components of H. Furthermore, let G be defined in a manner identical to the previous case. If no component of H contains an even cycle, choose any two adjacent vertices v_1 and v_2 in H and represent v_1 by some vertex in $G_1 - G_2$ and v_2 by some vertex in $G_2 - G_1$. Then assign any $\eta(H) - \lambda - 1 = |V(H)| - \lambda - 2$ other vertices in H to the vertices in $G_1 \cap G_2$ and assign the remaining vertices arbitrarily. This will preclude the possibility of an H-immersion on this choice of vertices.

Assume that there is some component of H, say H_1, that contains an even cycle. We will then embed as many of the vertices of the components that do not contain an even cycle in $G_1 \cap G_2$ and embed the rest of the vertices of these components arbitrarily, if necessary. For each other component H_i, let X_i and Y_i partition $V(H_i)$ such that $e(X_i, Y_i) = \eta(H_i)$, and embed each X_i in G_1 and each Y_i in G_2. As above, this precludes the possibility of an H-immersion on this choice of vertices.

In each case, $\delta(G) = m - 1$ and $n = |V(G)| = 2m - \eta(H) + \lambda + 1$. We can then see that $\delta(G) = \frac{n+\eta(H)-\lambda-3}{2}$, proving the necessity of the stated degree condition.

In the proof of Theorem 2.1, let G be a graph on n vertices satisfying the stated conditions and let $f : V(H) \to V(G)$ be an injection. As above, we will let S denote $f(V(H))$. The following well-known lemma is useful.

Lemma 2.2. *If G is a graph such that $\delta(G) \geq \frac{n+\kappa-2}{2}$, then G is κ-connected.*

If C is a cutset of G of minimum order, Lemma 2.2 implies that $|C| \geq \eta - \lambda \geq k - 1$. Using the following theorem, we can assume that $|C| < 2|E(H)| + |H|$.

Theorem 2.3 (Pfender [6]). *If $|G| = n \geq 2k$, G is $2k$-connected, and $\delta(G) \geq \frac{n}{3} + 10k$, then G is k-linked.*

Suppose $\kappa(G) = |C| \geq 2|E(H)| + |H|$. Then the graph $G - S$ is $2|E(H)|$-connected and

$$\begin{aligned}\delta(G - S) &\geq \frac{n + \eta - \lambda - 2}{2} - k \\ &\geq \frac{(n - k) + \eta - \lambda - 2 - k}{2} \\ &\geq \frac{n - k}{3} + 10|E(H)|\end{aligned}$$

as $n \geq 60|E(H)| + 5k + 5$. By Theorem 2.3, $G - S$ is $|E(H)|$-linked, implying that G is H-linked.

Thus, $\eta - \lambda \leq |C| < 2|E(H)| + |H|$. The minimum degree conditions imply that $G - C$ consists of exactly two components, which we call A and B. We now bound the number of adjacencies that each vertex of A and B must have to C.

Lemma 2.4. $\delta(A, C), \delta(B, C) \geq \eta - \lambda$.

Proof. The minimum degree conditions require that $|B| + |C| \geq \frac{n+\eta-\lambda-2}{2} + 1$, which implies that $|A| \leq \frac{n-\eta+\lambda}{2}$. Since the minimum degree of a vertex in A is $\frac{n+\eta-\lambda-2}{2}$, this means that each vertex in A must be adjacent to at least $\eta - \lambda$ vertices in $|C|$. A similar argument proves the bound on $\delta(B, C)$. ■

Suppose $|C| = \eta - \lambda + t$ for some $t \geq 0$. Let $S_A = S \cap A$, $S_B = S \cap B$, and $S_C = S \cap C$. We call a vertex in C *bad to* A (respectively, *bad to* B) if it has fewer than $2|E(H)| + k$ neighbors in A (resp. B). We call a vertex of C *good* if it is neither bad to A nor bad to B. Let R_A and R_B denote those vertices in C that are bad to A and B, respectively and let J denote the set of good vertices in S_C. Then, if $M_A = S_C \cap R_A$ and $M_B = S_C \cap R_B$, we can write $S_C = J \cup M_A \cup M_B$.

Note that if $x \in R_A \cap R_B$, then $d_A(x) \leq 2|E(H)| + k$ and the same bound holds for $d_B(x)$. Furthermore, x has less than $2|E(H)| + k$ neighbors in C. Thus, $6|E(H)| + 3k > d(x) \geq \delta(G)$. But then from the bound on n we see that R_A and R_B must be disjoint.

Lemma 2.5. $|R_A| + |R_B| \le t$.

Proof. Suppose $|R_A| + |R_B| > t$. Then, as $n \ge 8|E(H)|^2 + 8|E(H)|k + 2k^2 - |E(H)| + 2$ and each A and B have order at least $\frac{n+\eta-\lambda}{2}$, there exist vertices $v_A \in A$ and $v_B \in B$ such that v_A has no neighbors in R_A and v_B has no neighbors in R_B. Then

$$\begin{aligned} d(v_A) + d(v_B) &\le |A| + |B| - 2 + 2|C| - (|R_A| + |R_B|) \\ &= n + |C| - 2 - (|R_A| + |R_B|) \\ &= n + (\eta - \lambda + t - (|R_A| + |R_B|) - 2 \\ &< n + \eta - \lambda - 2 \\ &= 2\delta, \end{aligned}$$

contradicting the minimum degree condition. ■

Before describing how we find the H-immersion in G, we prove a lemma about the parameter $\eta(H)$ that will be useful.

Lemma 2.6. *Let M be a graph and let W, Z be a vertex partition of $V(M)$. Then for any $F \subseteq V(M)$,*

$$e(W - F, Z - F) \le \eta(M) - |F|.$$

Proof. It suffices to show that the result holds for $F = \{v\}$. We assume, without loss of generality, that $v \in W$ and let W' denote $W \setminus \{v\}$.

Assume that the theorem is false, that is, that $e(W', Z) \ge \eta(M)$. Note as well, that $\eta(M)$ is always at least the size of a maximum edge cut in M, so it must be that $e(W', Z) = \eta(M)$. If v is an isolated vertex, then by definition $\eta(M)$ exceeds the size of a maximum edge-cut in M, contradicting the above assumption. Therefore, assume that v has at least one neighbor x in M. If x lies in Z, then $e(W' \cup \{v\}, Z) > \eta(M)$, a contradiction. Similarly, if x lies in W', then $e(W', Z \cup \{v\}) > \eta(M)$, again, a contradiction. ■

One should note that $\langle A \rangle$ and $\langle B \rangle$ are extremely dense and hence satisfy the conditions of Theorem 1.2. We will exploit this fact.

Let $e = xy$ be an edge of H. The construction of the path in G that we use to represent this edge depends heavily on the location of the images of x and y under f. In many cases, we will select neighbors of $f(x)$ and $f(y)$ in A or B to serve as "proxies" for these vertices, as we will be able to use Theorem 1.2 to construct a path between the proxies, and thus between $f(x)$ and $f(y)$. Several cases are required.

Case 1. If $f(x) \in (S_C \setminus R_A)$ ($f(y) \in (S_C \setminus R_B)$, respectively) and $f(y) \in (S_A)$, $f(x) \in (S_B)$ then we choose some vertex $a^1_{e,x}$ in $A \setminus S_A$ adjacent to $f(x)$ ($b^1_{e,y}$ in $B \setminus S_B$ adjacent to $f(y)$).

In this case, we wish to link a vertex in S_A (or S_B) to a vertex in S_C that is good to that particular set. This is not difficult as the vertex in S_C will have a large number of neighbors to choose from in A or B, respectively.

Case 2. If $f(x)$, $f(y) \in (S_C \setminus R_A)$, $(S_C \setminus R_B)$, respectively then we choose vertices a'_e and a''_e (b'_e and b''_e) in $A \setminus S_A$ ($B \setminus S_B$) adjacent to $f(x)$ and $f(y)$, respectively.

Here, we would like to find a path between two vertices in S_C that have a large number of neighbors in the same set (A or B). We cannot assure that x and y are adjacent or have a common neighbor, but the density of A and B will make it relatively simple to link their proxies.

To this point, we have not been forced to make any repetitions among the vertices we have chosen. This is because we have been able to select vertices from A and B. In the remaining cases, in order to represent the remaining edges with paths in an immersion, we must construct paths that cross from A to B through C. In the procedure below, some of the vertices labeled v_e, which lie in C, may be repeated. We will choose these vertices such that the number of repetitions is minimized.

Case 3. If $f(x) \in (M_B)$ and $f(y) \in (M_A)$ then we choose some vertex v_e in $C \setminus (R_A \cup R_B \cup S_C)$ having neighbors $a^1_{e,x}$ in $A \setminus S_A$ and $b^1_{e,y}$ in $B \setminus S_B$, and vertices $a^2_{e,x}$ in $A \setminus S_A$ and $b^2_{e,y}$ in $B \setminus S_B$, adjacent to $f(x)$ and $f(y)$, respectively (Fig. 1).

Here, we would like to construct a path between two vertices in S_C. However, we may not be able to choose proxies for these vertices in the same sets. Therefore, we will choose proxies for x and y in the sets where they have many neighbors. Our goal is to then construct paths from $a^1_{e,x}$ to $a^2_{e,x}$ and $b^1_{e,y}$ to $b^2_{e,y}$, which would complete a path from x to y.

Case 4. If $f(x) \in (S_A)$ and $f(y) \in (S_B)$, then we choose some vertex v_e in $C \setminus (R_A \cup R_B \cup S_C)$ having neighbors $a^1_{e,x}$ in $A \setminus S_A$ and $b^1_{e,y}$ in $B \setminus S_B$.

In this case, we simply wish to link a vertex in S_A to a vertex in S_B using a path through the vertex v_e.

Case 5. If $f(x) \in (M_B)$ and $f(y) \in (S_B)$ ($f(x) \in (S_A)$ and $f(y) \in (M_A)$, respectively) then we choose some vertex v_e in $C \setminus (R_A \cup R_B \cup S_C)$ having neighbors $a^1_{e,x}$ in $A \setminus S_A$ and $b^1_{e,y}$ in $B \setminus S_B$, and a vertex $a^2_{e,x}$ in $A \setminus S_A$ adjacent to $f(x)$ ($b^2_{e,y}$ in $B \setminus S_B$ adjacent to $f(y)$) (Fig. 2).

Here, we would like to link a vertex in S_A or S_B to a vertex in S_C that is bad to A or B, respectively. We must then choose a proxy for the vertex in S_C in the other set, and construct a path through the vertex v_e.

As mentioned above, we have chosen the vertices v_e in such a way to minimize the number of times they are repeated. We will also choose the vertices $a^1_{e,x}$, $a^2_{e,x}$, a', and a'' (resp. $b^1_{e,y}$, $b^2_{e,x}$, b', and b'') to be distinct, which is not difficult as n is sufficiently large. It remains to show that we have no more than λ total repetitions, as desired.

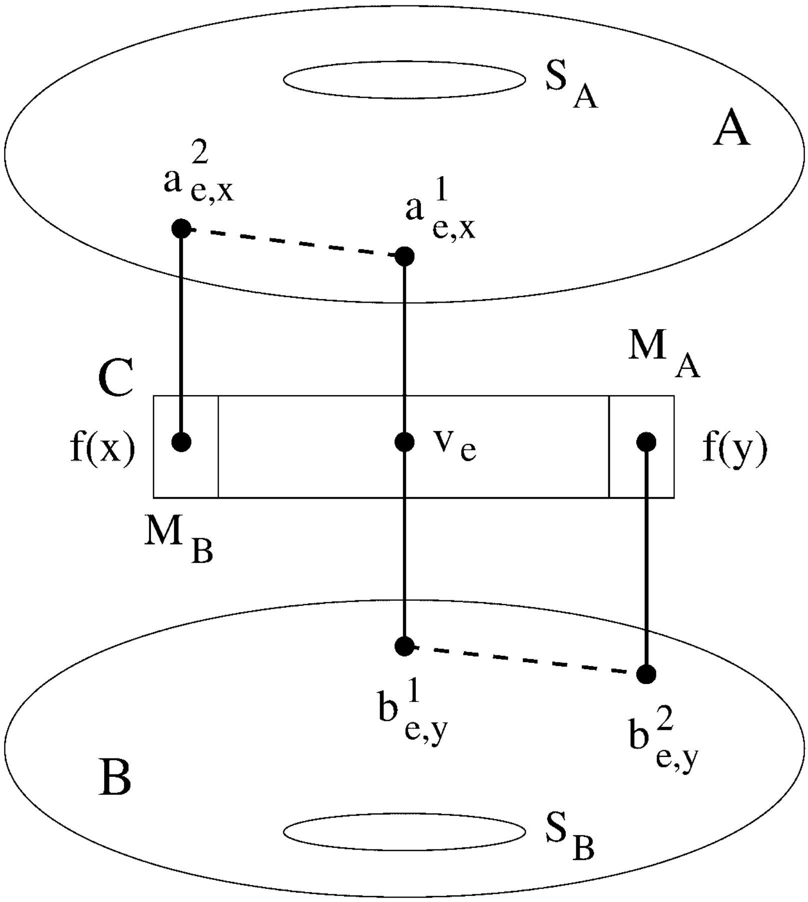

FIGURE 1. Case 3: Dashed lines indicate paths to build.

Lemma 2.7. *In the procedure above, there are at most λ repetitions of the vertices v_e.*

Proof. Let Q be the set of good vertices of C that are not in S_C. We must show that Q is large enough. Note that

$$
\begin{aligned}
|C| &= |S_C| + |Q| + |R_A \setminus M_A| + |R_B \setminus M_B| \\
&= \eta(H) - \lambda + t.
\end{aligned}
$$

By Lemma 2.5, we have that

$$
\begin{aligned}
|C \setminus (R_A \cup R_B)| &\geq \eta(H) - \lambda \\
|J| + |Q| &\geq \eta(H) - \lambda,
\end{aligned}
$$

which implies that $|Q| \geq \eta(H) - \lambda - |J|$.

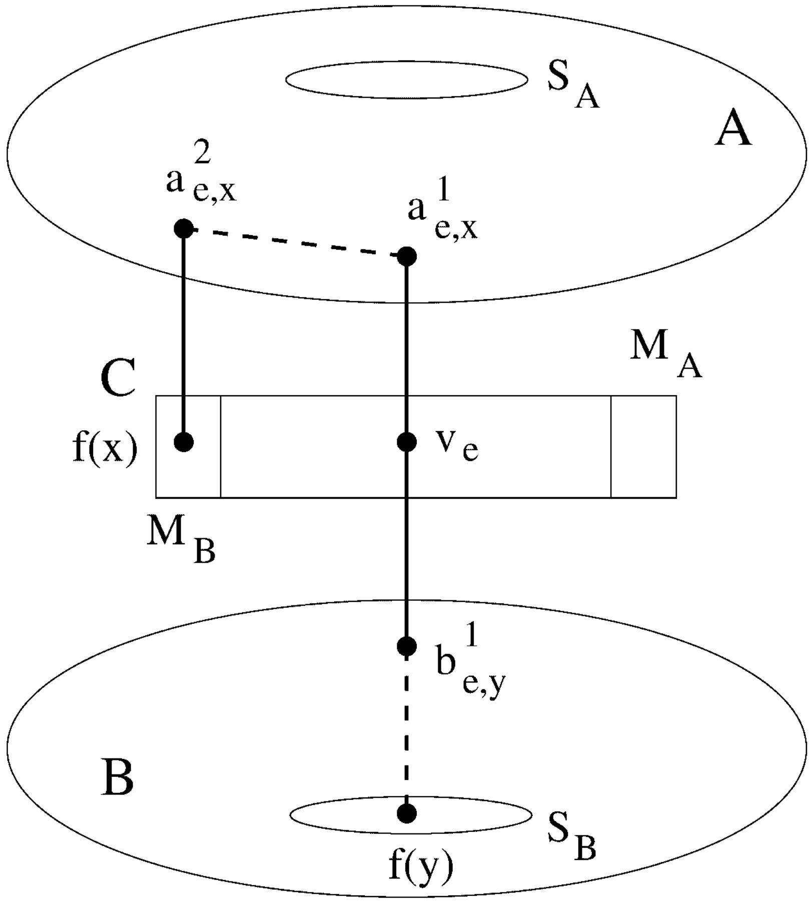

FIGURE 2. Case 5: Dashed lines indicate paths to build.

In constructing this immersion of H, none of the good vertices in S_C are ever involved in paths containing a v_e. Consequently, if we apply Lemma 2.6 to the partition of $H - f^{-1}(S_C)$ defined by $f^{-1}(S_A)$ and $f^{-1}(S_B)$, we can see that there are only $\eta - |J|$ vertices v_e that must be chosen. Thus, the selection of the v_e can be carried out with at most λ repetitions. ■

Let H_A be a graph with vertex set $V(H_A) = \cup_{e \in H}(a^1_{e,x} \cup a^2_{e,x} \cup a'_e \cup a''_e) \cup S_A$. The edges of H_A are determined as follows:

(1) If $w, z \in H_A \cap S_A$, then $wz \in E(H_A)$ if and only if $f^{-1}(w)f^{-1}(z) \in E(H)$.
(2) If $w \in H_A \cap S_A$, and $f^{-1}(w)$ is an endvertex of an edge $e = f^{-1}(w)x \in E(H)$, then w is adjacent to $a^1_{e,x}$ in H_A.
(3) The vertex $a^1_{e,x}$ is adjacent to $a^2_{e,x}$.
(4) The vertex a'_e is adjacent to a''_e.

Similarly, let H_B be a graph with vertex set $V(H_B) = \cup_{e \in H}(b^1_{e,y} \cup b^2_{e,y} \cup b'_e \cup b''_e) \cup S_B$. The edges of H_B are determined as follows:

(1) If $w, z \in H_B \cap S_B$, then $wz \in E(H_B)$ if and only if $f^{-1}(w)f^{-1}(z) \in E(H)$.
(2) If $w \in H_B \cap S_B$, and $f^{-1}(w)$ is an endvertex of an edge $e = f^{-1}(w)y \in E(H)$, then w is adjacent to $b^1_{e,y}$ in H_B.
(3) The vertex $b^1_{e,y}$ is adjacent to $b^2_{e,y}$.
(4) The vertex b'_e is adjacent to b''_e.

Claim 2.8. *If $\langle A \rangle$ is H_A-linked and $\langle B \rangle$ is H_B-linked, then f can be extended to an immersion $\mathcal{I}$ with $r(\mathcal{I}) \leq \lambda$.*

Proof. For every edge $e = xy$ in H, we must show that there is a path from $f(x)$ to $f(y)$, and that there are at most λ repetitions of internal vertices. Let F_A be a subdivision of H_A in $\langle A \rangle$ and let F_B be a subdivision of H_B in $\langle B \rangle$. We consider several cases.

Case 1. Suppose $f(x), f(y) \in S_A$ (S_B, respectively). Then F_A (F_B) contains an $f(x) - f(y)$ path.

Case 2. Suppose $f(x) \in S_A$, $f_1(y) \in S_B$. Then F_A contains an $f(x) - a^1_{e,x}$ path, F_B contains an $f(y) - b^1_{e,y}$ path, and these paths are joined by the edges $a^1_{e,x}v_e$ and $v_e b^1_{e,y}$.

Case 3. Suppose $f(x) \in S_C \setminus R_A$, $f(y) \in S_A$ ($f(x) \in S_B$, $f(y) \in S_C \setminus R_B$, respectively). Then F_A (F_B) contains an $f(y) - a^1_{e,x}$ path (a $f(x) - b^1_{e,y}$ path), which combines with the edge $a^1_{e,x}f(x)$ ($b^1_{e,y}f(y)$) to form the desired path.

Case 4. Suppose $f(x) \in S_A$, $f(y) \in M_A$ ($f(x) \in M_B$, $f(y) \in S_B$, respectively). Then F_A (F_B) contains an $f(x) - a^1_{e,x}$ path ($b^1_{e,y} - f(y)$ path), F_B (F_A) contains a $b^1_{e,y} - b^2_{e,y}$ path (a $a^2_{e,x} - a^1_{e,x}$ path), and these combine with the edges $a^1_{e,x}v_e$, $v_e b^1_{e,y}$, and $b^2_{e,y}f(y)$ ($f(x)a^2_{e,x}$, $a^1_{e,x}v_e$, and $v_e b^1_{e,y}$) to form the required path.

Case 5. Suppose $f(x) \in M_B$, $f(y) \in M_A$. Then F_A contains an $a^2_{e,x} - a^1_{e,x}$ path, F_B contains a $b^1_{e,y} - b^2_{e,y}$ path, and these combine with the edges $f(x)a^2_{e,x}$, $a^1_{e,x}v_e$, $v_e b^1_{e,y}$, and $b^2_{e,y}f(y)$ to form the required path.

Case 6. Suppose $f(x), f(y) \in S_C \setminus R_A$ ($f(x), f(y) \in M_A$, respectively). Then F_A (F_B) contains an $a'_e - a''_e$ path (a $b'_e - b''_e$ path), which combines with the edges $f(x)a'_e$ and $a''_e f(y)$ ($f(x)b'_e$ and $b''_e f(y)$) to form the required path.

Note that these paths form an H-immersion on the required vertices in G. The only vertices that are possibly repeated are the v_e's, but by Lemma 2.7, these vertices could be chosen such that at most λ repetitions are used. ■

To complete the proof, we must now show that $\langle A \rangle$ is H_A-linked and $\langle B \rangle$ is H_B-linked. To do this, we wish to apply Theorem 2.3 and Lemma 2.2 to $\langle A \rangle$ and $\langle B \rangle$.

Note that a vertex in A can be adjacent to every vertex in C, so we want:

$$\delta(\langle A\rangle) - k \geq \frac{|A|}{3} + 10|E(H)|.$$

But, since $|C| \leq 2|E(H)| + k$, this implies that

$$\begin{aligned}\frac{n+\eta(H)+\lambda-2}{2} - (2|E(H)|+k) - k \geq{}& \frac{n}{3} \\ &- \frac{n+\eta(H)+\lambda-2}{6} - \frac{1}{3} + 10|E(H)|\end{aligned}$$

and hence that

$$n \geq 34|E(H)| + 4k + 5.$$

Note that a similar argument applies to $\langle B\rangle$. Thus, $\langle A\rangle$ is H_A-linked and $\langle B\rangle$ is H_B-linked, which completes the proof. ■

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