THE CHVÁTAL-ERDŐS CONDITION AND 2-FACTORS WITH A SPECIFIED NUMBER OF COMPONENTS

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Abstract

Let G be a 2-connected graph of order n satisfying $\alpha(G) = a \leq \kappa(G)$, where $\alpha(G)$ and $\kappa(G)$ are the independence number and the connectivity of G, respectively, and let $r(m, n)$ denote the Ramsey number. The well-known Chvátal-Erdős Theorem states that G has a hamiltonian cycle. In this paper, we extend this theorem, and prove that G has a 2-factor with a specified number of components if n is sufficiently large. More precisely, we prove that (1) if $n \geq k \cdot r(a + 4, a + 1)$, then G has a 2-factor with k components, and (2) if $n \geq r(2a + 3, a + 1) + 3(k - 1)$, then G has a 2-factor with k components such that all components but one have order three.

Keywords: Chvátal-Erdős condition, 2-factor, hamiltonian cycle, Ramsey number.

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1. Introduction

For a graph G, we denote by $\alpha(G)$ and $\kappa(G)$ the independence number and the connectivity of G, respectively. If the inequality $\alpha(G) \leq \kappa(G)$ holds, we say that G satisfies the Chvátal-Erdős condition, in view of the following well-known theorem.

Theorem A (Chvátal-Erdős Theorem [4]). *Every 2-connected graph G satisfying $\alpha(G) \leq \kappa(G)$ has a hamiltonian cycle.*

If every pair of nonadjacent vertices x and y in a graph G of order n satisfy $\deg_G x + \deg_G y \geq n$, then we say that G satisfies Ore's condition. It is also well-known that every graph of order at least three satisfying Ore's condition has a hamiltonian cycle.

Theorem B (Ore's Theorem [8]). *Let G be a graph of order $n \geq 3$. If $\deg_G x + \deg_G y \geq n$ for every pair of nonadjacent vertices x and y of G, then G has a hamiltonian cycle.*

Though the Chvátal-Erdős condition and Ore's condition look quite different, Bondy [1] proved that they are not independent.

Theorem C (Bondy [1]). *Every graph of order at least three satisfying Ore's condition satisfies the Chvátal-Erdős condition.*

A hamiltonian cycle is a 2-factor with exactly one component. From this point of view, since the Chvátal-Erdős condition and Ore's condition guarantee the existence of a 2-factor with one component, we may suspect that both conditions guarantee the existence of a 2-factor with a specified number of components. Here we remark that we have to take the order of a graph into account. For example, a balanced complete bipartite graph of order at least four satisfies both the Chvátal-Erdős condition and Ore's condition while it has a 2-factor with k components only if its order is at least $4k$. Therefore, when we consider this kind of problem, we always have to consider a graph of a sufficiently large order.

For Ore's condition, Brandt *et al.* [2] proved the following extension of Theorem B.

Theorem D (Brandt *et al.* [2]). *Let k be a positive integer. Then every graph of order at least $4k$ satisfying Ore's condition has a 2-factor with k components.*

Since the balanced complete bipartite graph of order $4k - 2$ does not have a 2-factor with k components, the bound $4k$ of the order in the above theorem is almost best-possible. Later, Enomoto [6] improved the bound to $4k - 1$, which is best-possible.

While the extension of Ore's Theorem to a 2-factor with a specified number of components has been studied in detail, little is known about the extension of the Chvátal-Erdős Theorem in the same direction. Thus, we present the following conjecture.

Conjecture 1. For each positive integer k, there exists a positive integer $f(k)$ such that every 2-connected graph G of order at least $f(k)$ satisfying $\alpha(G) \leq \kappa(G)$ has a 2-factor with exactly k components.

Actually, Kaneko and Yoshimoto [7] tackled this conjecture for $k = 2$, and almost solved it.

Theorem E (Kaneko and Yoshimoto [7]). *Every 4-connected graph of order at least six satisfying $\alpha(G) \leq \kappa(G)$ has a 2-factor with two components.*

Recently, Egawa [5] has pointed out that the proof of Theorem E in [7] misses one case to be considered. But even if this possible flaw is fixed, their proof technique does not work for graphs of connectivity two or three. Therefore, they posed the following conjecture.

Conjecture F. Every 2-connected graph G of sufficiently large order satisfying $\alpha(G) \leq \kappa(G)$ has a 2-factor with two components.

The purpose of this paper is to give a partial solution to Conjecture 1. Let $r(m, n)$ denote the Ramsey number.

Theorem 2. *Let k be a positive integer and let G be a 2-connected graph with $\alpha(G) = a \leq \kappa(G)$.*

(1) *If $|G| \geq k \cdot r(a + 4, a + 1)$, then G has a 2-factor with k components.*

(2) *If $|G| \geq r(2a + 3, a + 1) + 3(k - 1)$, then G has a 2-factor with k components such that $k - 1$ components have order exactly three.*

We remark that the above theorem is not a complete solution of Conjecture 1 since in the conjecture the lower bound of the order in the assumption only depends on k, while in both (1) and (2) of Theorem 2 the lower bound depends not only on k but also on the independence number.

In this paper, we actually prove the following theorem.

Theorem 3. *Let G be a graph of independence number a, and let C be a cycle in G.*

(1) *If $|C| \geq k \cdot r(a + 4, a + 1)$, then there exist k disjoint cycles $C_1, \ldots, C_k$ with $V(C) = \bigcup_{i=1}^{k} V(C_i)$.*

(2) *If $|C| \geq r(2a + 3, a + 1) + 3(k - 1)$, then there exist k disjoint cycles $C_1, \ldots, C_k$ such that $V(C) = \bigcup_{i=1}^{k} V(C_i)$ and $|C_i| = 3$ for $1 \leq i \leq k-1$.*

We obtain Theorem 2 by applying Theorem 3 to a hamiltonian cycle, whose existence is guaranteed by the Chvátal-Erdős Theorem.

In the next section, we give notation and several definitions which we use in the proofs. In Section 3, we prove Theorem 3.

2.   Terminology and Definitions

For graph-theoretic terminology not explained in this paper, we refer the reader to [3]. We define a walk to be a sequence of vertices for which consecutive vertices are adjacent. We express a walk as a sequence of vertices. Let $C = x_0 x_1 \ldots x_{l-1} x_0$ be a cycle with an implied orientation. We define $x_i^+ = x_{i+1}$, $x_i^- = x_{i-1}$ and $x_i^{+n} = x_{i+n}$ (subscripts counted modulo l).

The subpath $x_i x_{i+1} \ldots x_{j-1} x_j$ in C is denoted by $x_i \overrightarrow{C} x_j$. The same path traversed in the reverse order is denoted by $x_j \overleftarrow{C} x_i$. We also adopt the same notation for a path.

When there is no possibility of confusion, we sometimes consider a path and a cycle as graphs. Thus, for example, if x is a vertex in a path P, we write $x \in V(P)$.

Let H be a subgraph of a graph G, and let T be a cycle or a path in G. Then for $u, v \in V(T)$, a subpath $u \overrightarrow{T} v$ is said to be an H-cluster of T if $V(u \overrightarrow{T} v) \subset V(H)$ and the vertices v^- and v^+, if they exist, are not in H. In other words, an H-cluster is a maximal subpath of T contained in H.

3. Proof

In this section, we prove Theorem 3.

Proof of Theorem 3. For (1), we proceed by induction on k. If $k = 1$, the conclusion is trivial. Suppose $k \geq 2$. Let D be a cycle in G with $V(D) = V(C)$, and let P be a subpath of D of order $r(a + 4, a + 1)$.

Let u and v be the first and the last vertices of P, respectively. Also, let H be the subgraph of G induced by $V(P)$. Then either there exists a clique of order $a + 4$ in H, or there exists an independent set of order $a + 1$ in H. However, since $\alpha(H) \leq \alpha(G) = a < a + 1$, the latter case does not occur, and hence there exists a clique K of order $a + 4$ in H. Let $\mathcal{I} = \{I_1, \ldots, I_l\}$ be the set of K-clusters of P, and let x_i and y_i be the first and the last vertices of I_i, respectively $(1 \leq i \leq l)$. Now we choose (D, P, K) so that l, the number of K-clusters, is as small as possible.

Suppose $\mathcal{I}$ has two K-clusters of order at least two. We may assume $|I_1| \geq 2$ and $|I_2| \geq 2$, and I_1 precedes I_2 along P. Let $C_1 = y_1 \overrightarrow{P} x_2 y_1$ and $C' = x_2^+ \overrightarrow{D} y_1^- x_2^+$. Then $|C_1| \leq |P| = r(a + 4, a + 1)$ and $|C'| \geq (k - 1) \cdot r(a + 4, a + 1)$. By the induction hypothesis, C' can be decomposed into $k - 1$ disjoint cycles $C_2, \ldots, C_k$. Then $C_1, C_2, \ldots, C_k$ give a required decomposition. Therefore, we may assume that P has at most one K-cluster of order two or more. We may assume that I_1 is a largest K-cluster. Then $I_2, \ldots, I_l$ all consist of single vertices.

Suppose $|I_1| \geq 5$. Let $C_1 = x_1^+ x_1^{++} x_1^{+3} x_1^+$ and $C' = x_1 x_1^{+4} \overrightarrow{D} x_1$. Then $|C_1| = 3$ and $|C'| \geq k \cdot r(a + 4, a + 1) - 3 \geq (k - 1) \cdot r(a + 4, a + 1)$. By the induction hypothesis, C' can be decomposed into $k - 1$ disjoint cycles $C_2, \ldots, C_k$. Then $C_1, C_2, \ldots, C_k$ give a required decomposition of C.

Therefore, we may assume $|I_1| \leq 4$. It follows that $|K| = \sum_{i=1}^{k} |I_i| \leq l + 3$. Since $|K| = a + 4$, we have $l \geq a + 1$. Let $A = \{y_1^+, \ldots, y_l^+\}$. Then $|A| \geq a + 1 > \alpha(G)$ and hence A is not an independent set. Let $y_i^+ y_j^+ \in E(G)$, $1 \leq i < j \leq l$. We may assume that I_i precedes I_j along P. Let $C' = y_j^+ \overrightarrow{D} y_i y_j \overleftarrow{D} y_i^+ y_j^+$ and $P' = u \overrightarrow{P} y_i y_j \overleftarrow{P} y_i^+ y_j^+ \overrightarrow{P} v$. Then $V(C') = V(D) = V(C)$, P' is a subpath of C' of order $r(a + 4, a + 1)$ and the set of K-clusters of P' is $\mathcal{I} - \{I_i, I_j\} \cup \{x_i \overrightarrow{P} y_i y_j \overleftarrow{P} x_i\}$. This contradicts the minimality of the number of K-clusters, and the conclusion for (1) follows.

For (2), we also proceed by induction on k. The conclusion trivially holds if $k = 1$. Suppose $k \geq 2$. Let H be the subgraph induced by $V(C)$. Since $|H| = |C| \geq r(2a + 3, a + 1) + 3(k - 1) > r(2a + 3, a + 1)$, either there exists a clique of order $2a + 3$ in H, or there exists an independent set of order $a + 1$ in H. However, since $\alpha(H) \leq \alpha(G) < a + 1$, the latter case does not occur. Therefore, H has a clique K of order $2a + 3$. Note that H has a hamiltonian cycle C.

Take a hamiltonian cycle D of H so that

(a) the number of K-clusters of D is as small as possible, and

(b) the order of a largest K-cluster of D is as large as possible, subject to (a).

Let $\mathcal{I} = \{I_1, I_2, \ldots, I_l\}$ be the set of K-clusters of D, and let x_i and y_i be the first and the last vertices of I_i, respectively ($1 \leq i \leq l$).

Suppose some K-cluster I_i has five or more vertices. Let $C_1 = x_i^+ x_i^{++} x_i^{+3} x_i^+$ and $C' = x_i x_i^{+4} \overrightarrow{D} x_i$. Then $|C_1| = 3$ and $|C'| \geq r(2a + 3, a + 1) + 3(k - 2)$. By the induction hypothesis, C' can be decomposed into $k - 1$ cycles $C_2, \ldots, C_k$ such that $|C_2| = |C_3| = \cdots = |C_{k-1}| = 3$. Then $C_1, C_2, \ldots, C_k$ form a required cycle decomposition. Therefore, we may assume that $|I_i| \leq 4$ for each i, $1 \leq i \leq k$. We may assume that I_1 is a largest K-cluster.

Assume $|I_i| \geq 3$ for some i, $2 \leq i \leq l$. Let $D' = y_1 \overrightarrow{D} x_i x_i^{++} \overrightarrow{D} x_1 x_i^+ x_1^+ \overrightarrow{D} y_1$. Then D' is a hamiltonian cycle of H. Furthermore, D' has the same number of K-clusters as D and $x_1 x_i^+ x_1^+ \overrightarrow{D} y_1$ is a K-cluster, which is larger than I_1. This contradicts the choice (b) of D. Therefore, $|I_i| \leq 2$ for each i, $2 \leq i \leq l$. It follows that $2a + 3 = |K| = \sum_{i=1}^{l} |I_i| \leq 2l + 2$, which implies $l \geq a + 1$. Then by the same argument as in the last part of the proof of (1), we have a hamiltonian cycle C' of H such that the number of K-clusters of C' is $l - 1$. This contradicts the choice (a) of D. Therefore, the conclusion follows. ∎

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