On the Extremal Number of Edges in 2-Factor Hamiltonian Graphs

Ralph J. Faudree, Ronald J. Gould and Michael S. Jacobson

Abstract. In this paper we consider the question of determining the maximum number of edges in a hamiltonian graph of order n that contains no 2-factor with more than one cycle, that is, 2-factor hamiltonian graphs. We obtain exact results for both bipartite graphs, and general graphs, and construct extremal graphs in each case.

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1. Introduction

In this paper, we determine the maximum number of edges in a hamiltonian graph of order n containing no 2-factor with more than one component. The question of the structure of hamiltonian graphs with no 2-factors with more than one component has been receiving attention lately, for example see [2], [3], [4] and [5].

A hamiltonian cycle is interpreted as a 2-factor with one component. In [4], the question of the minimum degree in a hamiltonian graph sufficient to ensure the existence of a 2-factor with two cycles is considered. A 4-regular hamiltonian graph with no other 2-factor with less than $n/5$ cycles is shown. However, the exact minimum degree condition remains an open question. Hendry [6] provided sharp results for the maximum number of edges in a graph with a unique 2-factor.

We say a graph is *2-factor isomorphic* if it contains a 2-factor X, but contains no 2-factor that is not isomorphic to X. If X is a hamiltonian cycle, then of course, there are no 2-factors with more than one cycle. In this instance we will refer to such graphs as 2-factor hamiltonian graphs.

The following is a special case of a result in [1].

Theorem 1.1. *If G is a hamiltonian graph with $\delta(G) \geq 8$, then G is not 2-factor hamiltonian.*

While in [3] the following was shown.

Theorem 1.2. *Let G be a 2-factor hamiltonian k-regular graph. Then $k \leq 3$.*

We consider the nonregular case for 2-factor hamiltonian graphs and determine the maximum number of edges in such graphs. In addition, we present examples of the extremal graphs and show that when $n \equiv 2 \ mod\ 4$ and bipartite, the extremal graph is unique. The extremal graphs are shown not to be unique in all other cases studied here.

Let G be a graph. We denote the minimum degree of G by $\delta(G)$. For a vertex x of G, we denote by $N(x)$ and $\deg x$ the neighborhood of x and the degree of x in G, respectively. Given a vertex x on a cycle C with an orientation, $\overrightarrow{C}$, then the successor of x on C will be denoted by x^+ and the predecessor by x^-.

For convenience we establish the following notation. Let C be a cycle with a given orientation and $v \in V(C)$. A *t-chord associated with v* will be an edge $e = v^{+(t-1)/2}v^{-(t-1)/2}$ such that e forms a t-cycle containing v and the cycle uses only the edge e and edges of the cycle C. Note, this is only defined for odd t. Similarly, a *t-chord associated with an edge $f = xy \in E(C)$* is an edge $e = x^{-(t-2)/2}y^{+(t-2)/2}$ such that e forms a t-cycle containing f and the cycle uses only the edge e and edges of the cycle C.

2. Extremal Graph Constructions

In this section we present several different constructions of graphs which will be shown to be 2-factor hamiltonian and attain the maximum size in Sections 3 (the bipartite case) and 4 (the nonbipartite case).

Let B_n be bipartite of even order n with partite sets $\{u_1, u_2, \ldots, u_{2m}\}$ and $\{v_1, v_2, \ldots, v_{2m}\}$ if $n \equiv 0\ mod\ 4$ and $\{u_1, u_2, \ldots, u_{2m+1}\}$ and $\{v_1, v_2, \ldots, v_{2m+1}\}$ if $n \equiv 2\ mod\ 4$. Define the adjacencies in B_n as follows:

$N(u_1) = \{v_1, v_2\}$,
$N(u_2) = \{v_1, v_2, v_3\}$, $N(u_3) = \{v_1, v_2, v_4\}$,
$N(u_4) = \{v_1, v_2, v_3, v_4, v_5\}$, $N(u_5) = \{v_1, v_2, v_3, v_4, v_6\}$, ...
$N(u_{2j}) = \{v_1, v_2, \ldots, v_{2j}, v_{2j+1}\}$, $N(u_{2j+1}) = \{v_1, v_2, \ldots, v_{2j}, v_{2j+2}\}$, ... ,
$N(u_{2m-2}) = \{v_1, v_2, \ldots, v_{2m-2}, v_{2m-1}\}$, $N(u_{2m-1}) = \{v_1, v_2, \ldots, v_{2m-2}, v_{2m}\}$
and
$N(u_{2m}) = \{v_1, v_2, \ldots, v_{2m}\}$ (if $n \equiv 0\ mod\ 4$)
while
$N(u_{2m}) = N(u_{2m+1}) = \{v_1, v_2, \ldots, v_{2m+1}\}$ (if $n \equiv 2\ mod\ 4$).

Extremal graphs for the nonbipartite case when $n \equiv 0\ mod\ 4$ can be obtained by inserting all possible edges into either one of the partite sets of a copy of B_n of the appropriate order. We designate these two graphs as $S_{n,u}$ ($S_{n,v}$) when the set

$\{u_1, \ldots, u_{2m}\}$ ($\{v_1, \ldots, v_{2m}\}$) is complete. When $n \equiv 2 \ mod\ 4$ the graphs $S_{n,u}$ and $S_{n,v}$ are isomorphic.

Next we consider the case for odd n. When $n \equiv 1 \mod 4$ we form the graph O_n as follows: take a copy of $S_{n-1,v}$ along with a new vertex x and we join x to each vertex of the complete set $\{v_1, \ldots, v_{2m}\}$ and we join x to u_{2m}. When $n \equiv 3 \ mod\ 4$ we form O_n by taking a copy of $S_{n-1,v}$ along with a new vertex x where x is joined to each of $v_1, \ldots, v_{2m}$ and u_{2m+1}.

3. Bipartite 2-Factor Hamiltonian Graphs

We now turn to the question of establishing the upper bounds on the size of a bipartite 2-factor hamiltonian graph.

Theorem 3.1. *If G is a bipartite 2-factor hamiltonian graph of order $n \equiv 0 \ mod\ 4$, then*

$$|E(G)| \leq n^2/8 + n/2$$

and the bound is sharp.

Proof. Assume G is a bipartite 2-factor hamiltonian graph of order $n = 2k$ (we use a more general condition to establish a setting useful in the subsequent theorem as well) with partite sets $\{u_1, u_2, \ldots, u_k\}$ and $\{v_1, v_2, \ldots, v_k\}$. Let $C^*:\ v_1, u_k, v_2, u_{k-1}, \ldots, v_k, u_1, v_1$ be a hamiltonian cycle in G.

Define parallel classes of pairs of vertices as follows:

$$P_1 = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$$

and let P_i be the parallel class containing the pair u_1v_i, obtained from the rotation of P_1. Note that every edge of the complete graph on these vertices is in exactly one parallel class.

As G is 2-factor hamiltonian, it follows that at most $\lceil \frac{(n/2-2)}{2} + 2 \rceil = \lceil \frac{n}{4} + 1 \rceil$ of the pairs in any parallel class can then be edges of G, for otherwise a 2-factor with two cycles formed using consecutive parallel edges would clearly result. It now follows that $|E(G)| \leq (n/4 + 1)n/2 = n^2/8 + n/2$, when $n \equiv 0 \ mod\ 4$.

To see that this is optimal consider B_n, $n \equiv 0 \ mod\ 4$, as defined in the previous section. The cycle $C : u_1, v_2, u_3, v_4, \ldots, u_{2m-1}, v_{2m}, u_{2m}, v_{2m-1}, u_{2m-2}, \ldots, v_3, u_2, v_1, u_1$ shows that this graph is hamiltonian. To see that there is no nonisomorphic 2-factor observe that the edges u_1v_1 and u_1v_2 must be in any 2-factor. If the edges u_2v_3 and u_3v_4 are not in the 2-factor, then one of u_2 or u_3, say u_2, would be adjacent to v_1 and v_2 in the 2-factor. However, this would imply that u_3 would have degree one in the remaining graph and a 2-factor could not be formed. Now a similar argument applies to u_4 and u_5 forcing the edges u_4v_5 and u_5v_6 to be used. Subsequently, $u_{2t}v_{2t+1}$ and $u_{2t+1}v_{2t+2}$ would also be used. Therefore, any 2-factor must contain the path $v_{2m-1}, u_{2m-2}, v_{2m-3}, \ldots, u_2, v_1, u_1, v_2, u_3, \ldots, v_{2m}$. Hence, it follows that the only possible 2-factor is a hamiltonian cycle. Furthermore, this graph has $n^2/8 + n/2$ edges, demonstrating the extremal number. □

Theorem 3.2. *If G is a bipartite 2-factor hamiltonian graph of order $n \equiv 2 \mod 4$, then*

$$|E(G)| \leq n^2/8 + n/2 + 1/2.$$

Further, the graph B_n is the unique extremal graph in this case.

Proof. Consider $B_n,\ n \equiv 2 \mod 4$, as defined in the previous section. The cycle $C^* : u_1, v_2, u_3, v_4, \ldots,\ u_{2m-1}, v_{2m}, u_{2m+1}, v_{2m+1}, u_{2m},\ v_{2m-1},\ u_{2m-2},\ \ldots, v_3,\ u_2,$ $v_1,\ u_1$ shows that this graph is hamiltonian.

To see that there is no nonisomorphic 2-factor, observe that the edges u_1v_1 and u_1v_2 must be in any 2-factor. If the edges u_2v_3 and u_3v_4 are not in the 2-factor, then one of u_2 or u_3, say u_2, would be adjacent to v_1 and v_2 in the 2-factor. However, this would imply that u_3 would have degree one in the remaining graph and a 2-factor could not be formed. Now a similar argument applies to u_4 and u_5 forcing the edges u_4v_5 and u_5v_6 to be used. Subsequently, $u_{2t}v_{2t+1}$ and $u_{2t+1}v_{2t+2}$ would also be used. Therefore, any 2-factor must contain the path

$$u_{2m}, v_{2m-1}, u_{2m-2}, v_{2m-3}, \ldots, u_2, v_1, u_1, v_2, u_3, \ldots, v_{2m}.$$

Hence, it follows that the only possible 2-factor is a hamiltonian cycle. Furthermore, this graph has $n^2/8 + n/2 + 1/2$ edges, demonstrating the extremal number of edges is at least this number.

Now let G be a bipartite 2-factor hamiltonian graph of order $n = 4m + 2$ containing the extremal number of edges. Let C be a hamiltonian cycle in G with the given ordering $v_1, u_k, v_2, \ldots, v_k, u_1, v_1$, and note, to avoid a 2-factor with two cycles, each parallel class as defined in the previous theorem admits at most $m+2$ edges into the graph G.

For each edge e of the hamiltonian cycle, let F_e be the family of edges of the parallel class containing e which are included in the graph G. In this case, we call the edge e *strong* if $|F_e| = m + 2$, and *weak* otherwise. When e is weak, $|F_e| \leq m + 1$. Also note that when e is strong, F_e contains the 4-chord associated with e and alternate edges of the parallel class must also be in F_e, in particular, the 8-chord associated with e must also be in F_e. Observe that if e' is the antipodal edge of e on C, e is strong if and only if e' is strong. Note by the edge count, at least $2m + 2$ of the edges must be strong.

There cannot be three consecutive strong edges or a 2-factor with two cycles results (see Figure 1).

If consecutive edges e_1 and e_2 are strong, then the families

$$F_{e_3}, F_{e_5}, F_{e_7}, \ldots, F_{e_{2m+1}}, F_{e_{2m+4}}, F_{e_{2m+6}}, \ldots, F_{e_{4m+2}}$$

cannot contain the associated 4-chords. See Figure 2 for the e_5 and e_7 cases. All other cases work similarly, using one chord from the strong parallel class of e_1, one chord from the strong parallel class of e_2 and the 4-chord in question from e_{2j+1} for an appropriate j, to produce a 2-factor with two cycles. Thus, each of these edges must be weak. This implies there are at most $2m + 2$ strong edges but as we discussed above, we can conclude that there are precisely $2m + 2$ strong edges.

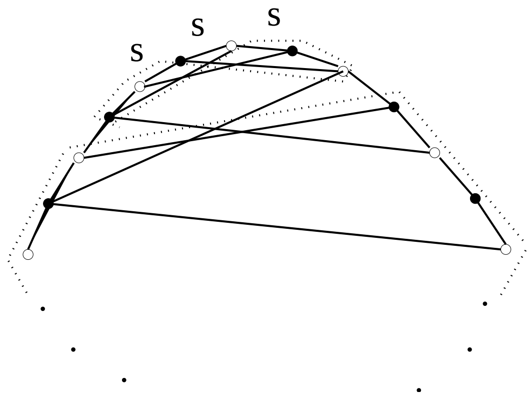

FIGURE 1. For $n \geq 10$, 3 consecutive strong edges produce a contradiction.

Each of the families associated with the remaining weak edges must have precisely $m+1$ edges and no 4-chords. Thus, G has size at most $n^2/8 + n/4 + 1/2$.

The extremal case arises when strong and weak edges alternate, with the exception of two consecutive strong edges (and their corresponding two consecutive strong antipodal edges). Furthermore, the weak classes are completely determined as alternating edges in the family. Thus, this graph is unique and hence must be isomorphic to B_n. □

See Figure 3 for the $n = 14$ case. This figure shows the graph for this case, displays a hamiltonian cycle C, as well as the strong and weak edges. Consequently, the graph of Figure 3 and that displayed in Figure **??** restricted to the case when $n = 14$, are seen to be isomorphic.

We conclude this section with a summary of the results.

Theorem 3.3. *If G is a bipartite 2-factor hamiltonian graph of order n, then*

$$|E(G)| \leq \begin{cases} n^2/8 + n/2 & \text{if} \quad n \equiv 0 \bmod 4, \\ n^2/8 + n/2 + 1/2 & \text{if} \quad n \equiv 2 \bmod 4, \end{cases}$$

and the bounds are sharp in each case.

4. The General Case

We now consider the question of establishing the upper bound on the size of a 2-factor hamiltonian graph of order n.

Theorem 4.1. *If G is a 2-factor hamiltonian graph of order n, then*

$$|E(G)| \leq \lceil n^2/4 + n/4 \rceil$$

and the bound is sharp for all $n \geq 6$.

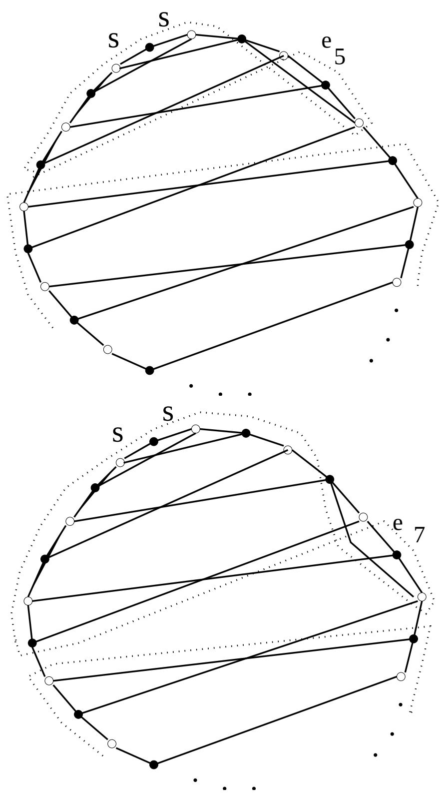

FIGURE 2. A 4-chord in F_{e_5} and F_{e_7} cases.

Proof. First suppose that n is even. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let C : $v_1, v_2, \ldots, v_n, v_1$ be a hamiltonian cycle in G. We next partition $V(G)$ into two sets, $V_1 = \{2, 4, \ldots, n-2, n\}$ and $V_2 = \{1, 3, 5, \ldots, n-1\}$.

Besides the parallel classes for edges as in the bipartite case, we now must also consider parallel classes of edges within the sets V_1 and V_2. That is, parallel classes defined relative to a vertex rather than an edge.

Hence, we recognize two types of parallel classes. Define the classes relative to edges of C (hence relative to the bipartite structure of V_1 and V_2) just as we did in the bipartite case. From Theorem 3.3 we have a bound on the maximum number of these edges that may be included in G.

Next, define the classes relative to a vertex of C as: $P_n = \{(1, n-1), (2, n-2), (3, n-3), \ldots\}$ and let P_i be obtained by a translation of P_n to contain the pair $(i-1, i+1)$. These edges partition the edges whose ends both are within the set V_1 or both within the set V_2. Let Q_i be the edges in G that are in P_i. Note that in this case a parallel class Q_i has at most $\lfloor n/4 \rfloor$ edges.

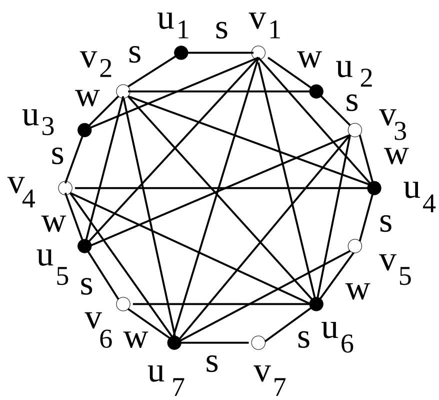

FIGURE 3. The graph when $n = 14$.

First suppose that $n \equiv 2 \ mod \ 4$. Now the bipartite parallel classes may contribute a total of at most $n^2/8 + n/2 + 1/2$ edges. The $n/2$ distinct classes Q_i may contribute at most $(n-2)/4$ edges each. Hence, we may have at most:

$$\begin{aligned} |E(G)| &\leq n^2/8 + n/2 + 1/2 + ((n-2)/4)(n/2) && (4.1) \\ &= n^2/8 + n/2 + 1/2 + n^2/8 - n/4 && (4.2) \\ &= n^2/4 + n/4 + 1/2. && (4.3) \end{aligned}$$

Note that the split graphs S_n (and similarly O_n) are 2-factor hamiltonian because the corresponding B_n are 2-factor hamiltonian. Since the split graph S_n, $n \equiv 2 \ mod \ 4$ achieves this size, this bound is sharp.

If $n = 4m$, then the $n/2$ classes Q_i cannot all contain $n/4$ edges, as a 2-factor with two cycles is then easily produced. We say a parallel class Q_i is *full* (F) if it contains m edges. Clearly, in this case, Q_i cannot contain more than m edges, for otherwise a 2-factor with two cycles results. Note that if Q_i is full, then the antipodal class $Q_{i+n/2}$ is also full. A class Q_i is called *near-full* (N) if it contains exactly $m-1$ edges and is called *partially full* (P) if it contains at most $m-2$ edges. Observe that for any i, families Q_i, Q_{i+1} and Q_{i+2} cannot all be full (see Figure 4). In fact, if Q_{i+1} is full, then we know that at least one of Q_i and Q_{i+2} must not contain the associated 3-chord.

Suppose we have two consecutive full classes, without loss of generality say Q_n and Q_1. Now consider the class Q_2. Since Q_2 does not contain the 3-chord associated with vertex 2, we know Q_2 is not full. We now show that Q_2 is a partially full class. Assume to the contrary that it contains more than $m-2$ edges. By the remarks above, $Q_{n/2+2}$ cannot contain its associated 3-chord. Hence, for Q_2 to be near-full, it must contain precisely the $5, 9, 13, \ldots$-chords associated with vertex 2. Thus, we see that $C_1 : 1, 2, 3, , n-3, n-2, 4, n, n-1, 1$ and $C_2 :$

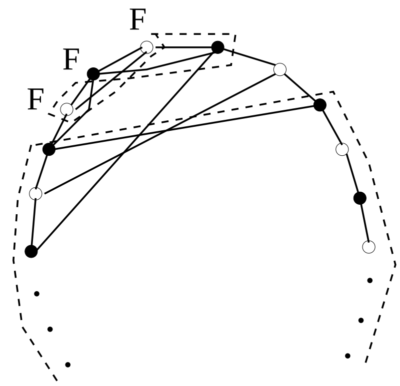

FIGURE 4. Three consecutive full families

$5, 6, 7, n-7, n-6, 10, 11, n-11, n-10, \ldots, n/2-2, n/2-1, n-(n/2-1), n-(n/2-2), n/2, n-(n/2-4), n-(n/2-3), n/2-3, n/2-4, \ldots, n-8, n-9, 9, 8, n-4, n-5, 5$ forms a 2-factor with two cycles, a contradiction. (see Figure 5 for the $n = 24$ case.) Hence, Q_2 must be a partially full class.

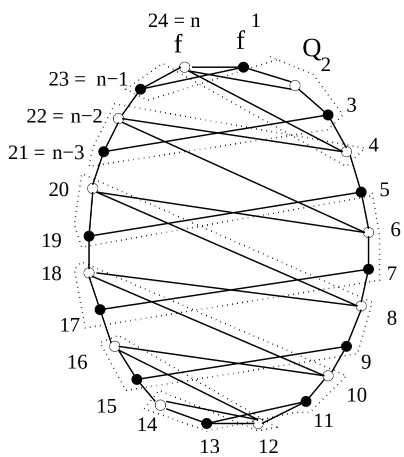

FIGURE 5. Q_2 cannot be near full.

Therefore, the vertices can be partitioned into intervals around C containing one, two or three consecutive vertices of C into patterns of the form FN, FFP, P or N. Each of the intervals must average $m - 1/2$ edges in order to achieve the size of S_n.

Thus, we see that the edge average for each of these patterns is: $m - 2/3$ for FFP , $m - 1/2$ for FN, $m - 1$ for N, and finally at most $m - 2$ for P.

Hence, the upper bound on the size can only be obtained by having the pattern FN repeated around the cycle. Thus, we must have $n/4$ full classes and $n/4$ near-full classes with $(n/4 - 1)$ edges each, implying

$$\begin{aligned} |E(G)| &\leq n^2/8 + n/2 + (n/4)(n/4) + (n/4)(n/4 - 1) && (4.4)\\ &= n^2/8 + n/2 + n^2/16 + n^2/16 - n/4 && (4.5)\\ &= n^2/4 + n/4. && (4.6) \end{aligned}$$

Again the split graph on $n = 4m$ vertices with the bipartite structure from the previous section achieves this bound.

Now suppose that $n \equiv 3 \ mod\ 4$. Let C be a hamiltonian cycle. In this case the parallel classes defined relative to a vertex may contain at most $(n+1)/4$ edges. Since there are n such classes, we see that $|E(G)| \leq n(n+1)/4 = n^2/4 + n/4$ as desired. The graph O_n shows that the bound is sharp in this case.

Finally, suppose that $n \equiv 1 \ mod\ 4$, say $n = 4m + 1$. First, we call a vertex strong provided its parallel class contains exactly $(n + 3)/4$ edges. It is weak otherwise. Note that this includes the antipodal edge on C.

As before, there cannot be three consecutive strong vertices, or a 2-factor with two cycles is immediate. Next we show that there are at most $2m + 1 = (n + 1)/2$ strong vertices. Otherwise, if there were more than $(n+1)/2$ strong vertices, then either there are three consecutive strong vertices, or there are at least two places around the cycle where there are two consecutive strong vertices. Therefore, there exists a strong vertex, separated from two consecutive strong vertices by an even number of vertices. For convenience let the two consecutive vertices be labeled v_n and v_1 and suppose there is another strong at vertex v_{2t}. Observe that the chord $c = v_{2t+(n-3)/2}v_{2t-(n-3)/2}$ is an edge of G since v_{2t} is strong. Note that there is a chord c_1 from the parallel class of vertex v_1 between the vertices $v_{(n-5)/2-2t+1}$ and $v_{2t+(n+5)/2}$. Further, there is a chord c_2 from the parallel class of v_n between the vertices $v_{2t+(n-1)/2}$ and $v_{(n-5)/2-2t+2}$. We now form a 2-factor with two cycles as follows. For one cycle we take the chord c_1 and the path on the hamiltonian cycle between the ends of c_1 and containing v_1. The second cycle is formed by taking the chord c_2, following the hamiltonian cycle back to $v_{2t+(n+3)/2}$, then taking the chord c, and now following the hamiltonian cycle back to $v_{(n-5)/2-2t+2}$.

Therefore, if there are two consecutive vertices, say v_n and v_1, then by our previous observation, vertices $v_2, v_4, \ldots, v_{n-1}$ must all be weak. Hence, there can be at most $(n + 1)/2$ strong vertices. (If there are not two consecutive strong vertices, then we can have at most $(n - 1)/2$ strong vertices.) Consequently,

$$|E(G) \leq (\frac{n+1}{2})(\frac{n+3}{4}) + (\frac{n-1}{2})(\frac{n-1}{4}).$$

Hence, $|E(G)| \leq \lceil n^2/4 + n/4 \rceil$ as desired. The graph O_n for this case shows that this bound is sharp. $\square$

5. Remarks

We conclude with a few remarks. First, it is clear from our proofs that any hamiltonian graph with more than the extremal number of edges, must contain a 2-factor with exactly two cycles.

Further, note that B_n for $n \equiv 2 \bmod 4$ was the only unique extremal graph. This is easily seen since in all other cases there were parallel classes which allowed some flexibility in exactly where the edges were placed. This flexibility allows the existence of nonisomorphic extremal graphs in these cases. For example, in the graph B_{4m}, the edge $u_{2m-2}v_{2m-2}$ can be removed and then the edge $u_{2m-1}v_{2m-1}$ can be inserted, forming B'. The graph B' can be further altered by removing $u_{2m-4}v_{2m-4}$ and inserting $u_{2m-3}v_{2m-3}$. Each of these graphs can easily be seen to be 2-factor hamiltonian. Further, we can continue this edge exchange process until the edge u_2v_2 is removed and u_3v_3 inserted. At this point a graph isomorphic to B_{4m} has been constructed, with the role of the partite sets interchanged.

Finally, consider S_n. Here we note that the edge $v_{n/2-1}v_{n/2}$ can be removed and the edge $u_{n/2-1}u_{n/2}$ can be inserted. These two graphs are clearly nonisomorphic and it is again easy to see the new graph is 2-factor hamiltonian.

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Ralph J. Faudree
Office of the Provost
University of Memphis, Memphis, TN 38152
e-mail: `rfaudree@memphis.edu`

Ronald J. Gould
Department of Mathematics and Computer Science
Emory University, Atlanta GA 30322
e-mail: `rg@mathcs.emory.edu`

Michael S. Jacobson
Department of Mathematics
University of Colorado at Denver, Denver, CO 80217
e-mail: msj@math.cudenver.edu