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GRAPHS WITH PRESCRIBED DEGREE SETS AND GIRTH

by

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Abstract

For a finite nonempty set $\mathfrak{D}$ of integers, each of which is at least two, and an integer $n \geq 3$, the number $f(\mathfrak{D}; n)$ is defined as the least order of a graph having degree set $\mathfrak{D}$ and girth n. The number $f(\mathfrak{D}; n)$ is evaluated for several sets $\mathfrak{D}$ and integers n. In particular, it is shown that $f(\{3, 4\}; 5) = 13$ and $f(\{3, 4\}; 6) = 18$.

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For integers $r \geq 2$ and $n \geq 3$, the integer $f(r, n)$ is defined as the smallest order of an r-regular graph having girth n (the girth being the length of a smallest cycle in the graph). Erdős and Sachs [1] have shown that $f(r, n)$ exists for all integers $r \geq 2$ and $n \geq 3$. The problem of evaluating $f(r, n)$ for various values of r and n has received considerable attention. The r-regular graphs having girth n and order $f(r, n)$ are known as *(r, n)-cages*. The object of this paper is to extend the function $f(r, n)$ and the (r, n)-cages.

The *degree set* $\mathfrak{D}_G = \{a_1, a_2, \ldots, a_k\}$ of a graph G is the set of degrees of the vertices of G. We henceforth assume for $\mathfrak{D}_G = \{a_1, a_2, \ldots, a_k\}$ that $a_1 < a_2 \ldots < a_k$.

For a set $\mathfrak{D} = \{a_1, a_2, \ldots, a_k\}$ of integers with $2 \leq a_1 < a_2 < \ldots < a_k$ and for an integer $n \geq 3$, we define

$$f(\mathfrak{D}; n) = f(a_1, a_2, \ldots, a_k; n)$$

to be the smallest order of a graph having girth n and degree set $\mathfrak{D}$. The existence of $f(\mathfrak{D}; n)$ is guaranteed by the above result of Erdős and Sachs. In particular, if H_i is an a_i-regular graph of girth n, where $V(H_i) \cap V(H_j) = \emptyset$ $(i \neq j)$, then the graph G defined by

$$V(G) = \bigcup_{i=1}^{k} V(H_i) \quad \text{and} \quad E(G) = \bigcup_{i=1}^{k} E(H_i)$$

has degree set $\mathfrak{D}$ and girth n. We shall refer to a graph G of order $f(\mathfrak{D}; n)$ having degree set $\mathfrak{D} = \{a_1, a_2, \ldots, a_k\}$ and girth n as a *$(\mathfrak{D}; n)$-cage* or an *$(a_1, a_2, \ldots, a_k; n)$-cage.*

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In [2] KAPOOR, POLIMENI and WALL showed that for a given set $\mathfrak{D} = \{a_1, a_2, \ldots, a_k\}$ of positive integers (with $a_1 < a_2 < \ldots < a_k$), the minimum order of a graph G with degree set $\mathfrak{D}$ is $1 + a_k$. If v is a vertex of degree a_k in a graph G with degree set $\mathfrak{D}_G = \mathfrak{D}$ containing no vertices of degree 1, then there must be two adjacent vertices which are themselves adjacent to v, producing a 3-cycle. This gives the following observation.

THEOREM 1. *If* $\mathfrak{D} = \{a_1, a_2, \ldots, a_k\}$ *is a set of positive integers with* $2 \leq a_1 < a_2 < \ldots < a_k$, *then* $f(\mathfrak{D}; 3) = 1 + a_k$.

The difficulty of evaluating $f(\mathfrak{D}; n)$ appears to increase sharply when $n > 3$. By placing restrictions on $\mathfrak{D}$, however, we are able to determine $f(\mathfrak{D}; n)$ in certain cases. In particular, if $\mathfrak{D}$ has cardinality two and $a_1 = 2$, the follow-ing result can be obtained.

THEOREM 2. *For* $m \geq 3$, $n \geq 3$,

$$f(2, m; n) = \begin{cases} \dfrac{m(n-2)+4}{2} & \text{if } n \text{ is even,} \\[2mm] \dfrac{m(n-1)+2}{2} & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. We observe that $f(2, m; n) \leq 2 + m(n-2)/2$ for n even and $(2, m; n) \leq 1 + m(n-1)/2$ if n is odd, since the graphs G_1 and G_2 of Fig. 1 have degree set $\{2, m\}$, girth n and the appropriate orders.

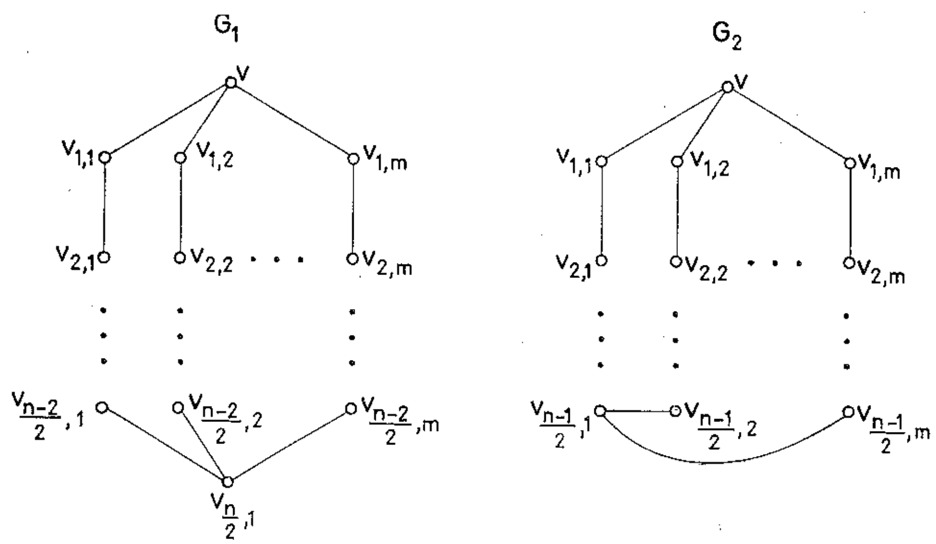

Fig. 1. The $(2, m; n)$-cages for n even and for n odd

Now suppose $n(\geq 4)$ is an even integer and let v be a vertex of degree m in a graph G having degree set $\{2, m\}$ and girth n. Since $n \geq 4$, the vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,m}$ adjacent to v are distinct and pairwise non-adjacent; therefore, G contains more than $m + 1$ vertices, which gives the desired result for $n = 4$. Thus, we assume $n \geq 6$. Since $\mathfrak{D}_G = \{2, m\}$, each vertex $v_{1,i}$ $(i = 1, 2, \ldots, m)$ is adjacent to at least one new vertex $v_{2,i}$. Since $n \geq 6$, the vertices $v_{2,1} v_{2,2}, \ldots, v_{2,m}$ are distinct and pairwise non-adjacent, so that G has order at least $2m + 2$, which gives the required result for $n = 6$.

If $n \geq 8$ we repeat the above process until the vertices

$$v_{\frac{n-2}{2},1},\ v_{\frac{n-2}{2},2}, \ldots, v_{\frac{n-2}{2},m}$$

have been added (see Fig. 1/a). These vertices are distinct and pairwise non-adjacent, for otherwise, an $(n-1)$-cycle is produced. Thus, G has order at east $2 + m(n-2)/2$, i.e.,

$$f(2, m; n) \geq 2 + m(n-2)/2,$$

which completes the proof of the theorem if n is even.

The argument if n is odd is similar and is omitted. ∎

Another case in which $f(\mathfrak{D}, n)$ can be evaluated rather casily occurs when $|\mathfrak{D}| = 2$ and $n = 4$.

THEOREM 3. *For* $2 \leq r < s$,

$$f(r, s; 4) = r + s.$$

PROOF. The complete bipartite graph $K(r, s)$ has degree set $\{r, s\}$ and girth four; hence $f(r, s; 4) \leq r + s$.

In order to show that $f(r, s; 4) \geq r + s$, let G be a graph with degree set $\{r, s\}$ and girth four. Let $u_1 \in V(G)$ such that $\deg u_1 = s$. Let $v_1, v_2, \ldots, v_s$ be the s vertices adjacent to u_1. Since G has no 3-cycles, $\langle\{v_1, v_2, \ldots, v_s\}\rangle$ contains no edges. Since the degree of v_1 is at least r and v_1 is not adjacent to any of $v_2, v_3, \ldots, v_s$, at least r other vertices must be present in G, i.e., $|V(G)| \geq$ $\geq r + s$. Hence $f(r, s; 4) \geq r + s$, giving the desired result. ∎

Since it is well known that $f(r; 4) = 2r$, the above result could be extended to include the case $r = s$.

Due to the difficulty of determining $f(r, n)$ when $n \geq 5$, it is probably not surprising that the problem of evaluating $f(\mathfrak{D}; n)$ when $|\mathfrak{D}| = 2$ and $n \geq 5$ seems to be extremely difficult. We now consider this problem when $\mathfrak{D} = \{3, 4\}$ and $n = 5$ or $n = 6$.

THEOREM 4. $f(3, 4; 5) = 13$.

PROOF. Let G be a graph with degree set $\{3, 4\}$ and girth 5. Let v be a vertex of degree 4 in G, and let v_0, v_1, v_2, v_3 be the vertices adjacent to v. Since G contains no 3-cycles, no two of the vertices v_0, v_1, v_2, v_3 are adjacent to each other. Since every vertex of G has degree 3 or 4, the vertex v_i $(i = 0, 1, 2, 3)$ is adjacent to at least two vertices different from v, say $v_{i,1}$ and $v_{i,2}$. Further, since G contains no 4-cycles, for $i \neq j$, we have $v_{i,k} \neq v_{j,l}$ when $i, j \in \{0, 1, 2, 3\}$ and $k, l \in \{1, 2\}$. Thus G contains at least 13 vertices so that $f(3, 4; 5) \geq 13$.

To show that $f(3, 4; 5) = 13$, it now suffices to verify the existence of a graph of order 13 having girth 5 and degree set $\{3, 4\}$. To the graph partially constructed above, add the edges

$$v_{i,1}\, v_{i+2,2},\ v_{i,1}\, v_{i+3,2},\ v_{i,2}\, v_{i+1,1} \quad \text{and} \quad v_{i,2}\, v_{i+2,1}$$

for $i = 0, 1, 2, 3$, where $i + 1$, $i + 2$ and $i + 3$ are expressed as 0, 1, 2 or 3 modulo 4. The graph H so described is shown in Figure 2. The graph H has order 13 and $\mathfrak{D}_H = \{3, 4\}$. Also $v, v_0, v_{0,1}, v_{3,2}, v_3, v$ is a 5-cycle of H. It remains only to show that H contains no 3-cycles or 4-cycles. It is straightforward to see that H has no 3-cycle or 4-cycle containing any vertex in the set $U = \{v, v_0, v_1, v_2, v_3\}$. If H contains a 3-cycle or 4-cycle, all vertices of such a cycle must belong to the set $V(H) - U$. Such a cycle C must contain a vertex $v_{i,1}$ for $i = 0, 1, 2$ or 3. Thus, C must contain the path

$$v_{i,1},\ v_{i+2,2},\ v_{i+3,1},\ v_{i+1,2}$$

or the path

$$v_{i,1},\ v_{i+3,2},\ v_{i+1,1},\ v_{i,2}$$

which cannot occur if C has length 3 or 4. Thus G has girth 5.

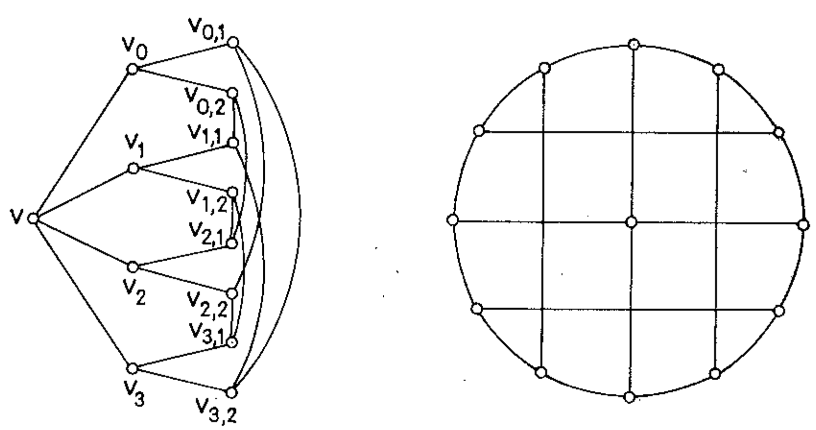

Fig. 2. Two drawings of a (3, 4; 5)-cage

THEOREM 5. $f(3, 4; 6) = 18$.

PROOF. Let G be a graph with degree set $\{3, 4\}$ and girth six. Let v be a vertex of degree 4 in G and let v_0, v_1, v_2, v_3 be the vertices adjacent to v. Since G contains no 3-cycles, no two of the vertices v_0, v_1, v_2, v_3 are adjacent. Since every vertex of G has degree 3 or 4, the vertex v_i $(i = 0, 1, 2, 3)$ is adjacent to at least two vertices different from v, say $v_{i,1}$ and $v_{i,2}$. Further, since G contains no 4-cycles, for $i \neq j$, we have $v_{i,k} \neq v_{j,l}$, where $i, j \in \{0, 1, 2, 3\}$ and $k, l \in \{1, 2\}$.

Again, each $v_{i,j}$ $(i = 0, 1, 2, 3;\ j = 1, 2)$ has degree at least three. Thus at least four additional vertices, must be present in G, say u_0, u_1, u_2, u_3. If G has only these 17 vertices, then each u_k $(k = 0, 1, 2, 3)$ must have degree 4 and be adjacent to exactly four of the vertices $v_{i,j}$ $(i = 0, 1, 2, 3;\ j = 1, 2)$. If any u_k is adjacent to both $v_{i,1}$ and $v_{i,2}$, a 4-cycle is produced. Thus each u_k must be adjacent to exactly one of $v_{i,1}$ and $v_{i,2}$ $(i = 0, 1, 2, 3)$. But then two of the vertices u_k $(k = 0, 1, 2, 3)$ must be adjacent to two of the vertices $v_{i,j}$ $(i = 0, 1, 2, 3;\ j = 1, 2)$, thereby producing a 4-cycle. Thus G must contain at least one additional vertex w, i.e. $|V(G)| \geq 18$.

To show that $f(\{3, 4\}; 6) = 18$, it now suffices to verify the existence of a graph of order 18 having girth 6 and degree set $\{3, 4\}$. To the graph partially constructed above, add the edges

$$u_i v_{i,1},\ u_i v_{i+1,2},\ u_i v_{i+2,2},\ wv_{i,1} \qquad (i = 0, 1, 2, 3)$$

where the subscripts $i + 1$ and $i + 2$ are expressed as 0, 1, 2 or 3 modulo 4.

Observe that $v, v_0, v_{0,1}, u_0, v_{1,2}, v_1, v$ is a 6-cycle. It remains only to show that G contains no r-cycle for $3 \leq r \leq 5$. It is straightforward to see that G

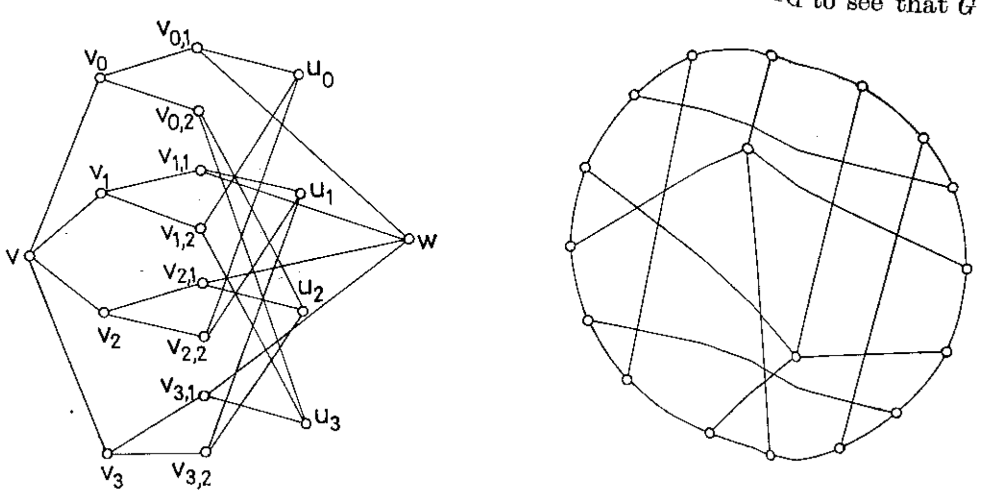

Fig. 3. Two drawings of a (3, 4; 6)-cage

contains no such cycle containing any vertex in the set $M = \{v, v_0, v_1, v_2, v_3\}$. If G contains a cycle of length five or less, all vertices of such a cycle must belong to the set $V(G) - M$.

Such a cycle C must contain a vertex u_i $(i = 0, 1, 2, 3)$. Thus C must contain one of the following paths:

(1) $u_i, v_{i,1}, w, \quad v_{i+k,1}, u_{i+k}$

(2) $u_i, v_{i+1,2}, \quad u_{i+3}, \quad v_{i,2}, u_{i+2}$

(3) $u_i, v_{i+1,2}, \quad u_{i+3}, \quad v_{i+3,1}, w$

(4) $u_i, v_{i+2,2}, \quad u_{i+1}, \quad v_{i+3,2}, u_{i+2}$

(5) $u_i, v_{i+2,2}, \quad u_{i+1}, \quad v_{i+1,1}, w$

where $i = 0, 1, 2, 3$ and $k = 1, 2, 3$ and all subscripts are expressed modulo 4. Since these paths do not extend to a cycle of length less than six, the graph G has girth six. Also $\mathfrak{D}_G = \{3, 4\}$. Thus $f(3, 4; 6) = 18$. ∎

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