SIAM J. DISCRETE MATH.
Vol. 20, No. 3, pp. 682–689

CYCLE EXTENDABILITY OF HAMILTONIAN INTERVAL GRAPHS*

GUANTAO CHEN†, RALPH J. FAUDREE‡, RONALD J. GOULD§, AND MICHAEL S. JACOBSON¶

Abstract. A graph G of order n is pancyclic if it contains cycles of all lengths from 3 to n. A graph is called cycle extendable if for every cycle C of less than n vertices there is another cycle C^* containing all vertices of C plus a single new vertex. Clearly, every cycle extendable graph is pancyclic if it contains a triangle. Cycle extendability has been intensively studied for dense graphs while little is known for sparse graphs, even very special graphs. We show that all Hamiltonian interval graphs are cycle extendable. This supports a conjecture of Hendry that all Hamiltonian chordal graphs are cycle extendable.

Key words. interval graph, Hamiltonian, cycle extendable

AMS subject classification. 05C38

DOI. 10.1137/S0895480104441450

1. Introduction. All graphs considered in this paper are finite and simple. We will generally follow the notation and definitions of West [14]. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any vertex v of G, $N(v)$ (or $N_G(v)$) denotes the neighborhood of v (neighborhood of v in G) and $d(v)$ (or $d_G(v)$) denotes the degree of v (degree of v in G). For any $X \subseteq V(G)$, let $G[X]$ denote the subgraph induced by X. If H is a subgraph of G, we define $G[H] := G[V(H)]$.

A graph is chordal if every cycle of length at least 4 contains a chord. An *interval graph* is a graph whose vertices correspond to a family of intervals so that vertices are adjacent if and only if the corresponding intervals intersect. It is well known that all interval graphs are chordal graphs.

In a graph G, a *Hamiltonian cycle* is a cycle containing all vertices of G. A graph is *Hamiltonian* if it has a Hamiltonian cycle. Determining when graphs are Hamiltonian is one of the fundamental problems in graph theory. Although it is NP-hard to decide whether a graph is Hamiltonian, finding conditions sufficient to imply a graph is Hamiltonian has been intensively studied in the last thirty years. While studying Hamiltonicity, many related properties have also been heavily explored. For example, a graph G of order n is *pancyclic* if it contains cycles of all lengths from 3 to n. Clearly, every pancyclic graph is Hamiltonian, but the converse is not true. Being pancyclic provides a lot more cycle structure to graphs. Although there are many

*Received by the editors February 21, 2004; accepted for publication (in revised form) October 17, 2005; published electronically September 15, 2006.
http://www.siam.org/journals/sidma/20-3/44145.html

†Department of Computer Science and Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, and Faculty of Mathematics and Statistics, Huazhong Normal University, Wuhan, China (gchen@gsu.edu). The research of this author was partially supported by NSA grant H98230-04-1-0300 and NSF grant DMS-0500951.

‡University of Memphis, Memphis, TN 38152 (rfaudree@memphis.edu).

§Department of Math and Computer Science, Emory University, Atlanta, GA 30322 (rg@mathcs.emory.edu).

¶Department of Mathematics, University of Colorado at Denver, Denver, CO 80217-3364 (msj@math.cudenver.edu).

Hamiltonian graphs which are not pancyclic, the known sufficient degree conditions implying each of the properties are often similar. For example, the classic result of Ore [10] says that a graph G of order $n \geq 3$ is Hamiltonian if $d(u) + d(v) \geq n$ for every nonadjacent pair $u, v \in V(G)$. Bondy [2] showed the same condition implies that G is either pancyclic or a complete bipartite graph $K_{n/2,n/2}$. A common method of showing that a graph G is pancyclic is described below:

- Show that G has a triangle.
- Suppose that G has a cycle of length $k < n$, and find a special cycle of length k $(< n)$ and a special vertex $v \notin V(C)$ such that $G[V(C) \cup \{v\}]$ is Hamiltonian.

Motivated by the above observations, Hendry [7] gave the following definitions. In a graph G, a non-Hamiltonian cycle C is *extendable* if there exists a vertex $v \notin V(C)$ such that $G[V(C) \cup \{v\}]$ is Hamiltonian. A graph G is *cycle extendable* if all non-Hamiltonian cycles are extendable. In the same paper, Hendry showed that a graph G of order $n \geq 3$ is cycle extendable if $d(u) + d(v) \geq n + 1$ for every pair of nonadjacent vertices u and v. Graphs satisfying the above degree conditions must be very dense (in edges). To study the cycle structure of graphs less dense, usually some other structural properties are imposed, for example, planarity.

In 1931, Whitney [15] proved that every 4-connected plane triangulation contains a Hamiltonian cycle. In 1956, Tutte [13] extended that result to 4-connected planar graphs. Malkevitch [9] conjectured that every 4-connected graph containing a C_4 is pancyclic. Combining results from [12, 11, 3], we know that every 4-connected planar graph of order $n \geq 9$ contains cycles of length $n - i$ for $i = 1, \ldots, 6$. These results use the approach of finding shorter cycles from long cycles. However, this approach cannot demonstrate why C_4s should play an important role in 4-connected planar graphs being pancyclic. Thus, constructing larger cycles from smaller cycles might be a better approach. Hence, cycle extendable graphs take on added importance.

For any graph H, let $c(H)$ denote the number of connected components of H. Let $t > 0$ be a positive number. We say a graph is *t-tough* if $|A| \geq t \cdot c(G - A)$ for all cuts $A \subseteq V(G)$. Clearly, every Hamiltonian graph is 1-tough. On the other hand, a longstanding conjecture of Chvátal [5] states that there exists a constant t such that every t-tough graph is Hamiltonian. Although this conjecture remains open, Chen et al. [4] showed that all 18-tough chordal graphs are Hamiltonian. Note that a chordal graph containing a cycle C_k also contains a cycle C_{k-1} if $k \geq 4$. Repeating this argument, we see that all chordal Hamiltonian graphs are pancyclic. Hendry [7] gave the following conjecture.

CONJECTURE 1.1. *All Hamiltonian chordal graphs are cycle extendable.*

The purpose of this paper is to prove that Conjecture 1.1 is true for a special class of chordal graphs, namely interval graphs.

THEOREM 1.2. *All Hamiltonian interval graphs are cycle extendable.*

The proof of Theorem 1.2 will be given in section 3. In section 2 we will develop necessary properties of interval graphs.

Keil [8] designed a linear algorithm to find a Hamiltonian cycle in an interval graph. One consequence of his algorithm is that an interval graph is Hamiltonian if and only if it is 1-tough. We will heavily use this fact in our proof. For 1-tough Hamiltonian graphs, a cut A of G is called *critical* if $c(G - A) = |A|$. Let C be a Hamiltonian cycle of G and A be a critical cut of G; then the vertex sets of the components of $G - A$ are exactly those of the components of $C - A$. The following lemma regarding critical cuts on Hamiltonian graphs will be needed in our proof, and its proof is straightforward.

LEMMA 1.3. *Let G be a Hamiltonian graph with a Hamiltonian cycle C. If A is a cut of G such that all segments of $C - A$ induce components of $G - A$ and A does not contain two consecutive vertices of C, then A is a critical cut of G.*

For any two disjoint intervals A and B on the real number line, we let $d(A, B)$ denote the distance between A and B. Let G be an interval graph. For each vertex $v \in V(G)$, let $I(v)$ denote the corresponding interval called the *representation* of v. For each $W \subseteq V(G)$, let $I(W) = \bigcup_{v \in W} I(v)$. For each subgraph H of G, we define $I(H) = I(V(H))$. Clearly, $I(H)$ is also an interval of the real line if H is connected. Since only finite simple graphs will be considered in this paper, we assume that $I(v)$ is a closed interval for each $v \in V(G)$. For each interval $I = [a, b]$, we call a the left-end of I and b the right-end of I. We say a vertex v is on the *left side* of w (or equivalently w is on the *right side* of v) if $a \leq b$ for all $a \in I(v)$ and $b \in I(w)$. For any two vertex subsets U and W, we say that U is on the *left side* of W if u is on the left side of w for any $u \in U$ and $w \in W$.

2. Paths and cycles in interval graphs. In this section we will review some properties of interval graphs. Most of these properties are given in [8]. A clique D is a subgraph of G such that all vertices in D are mutually adjacent. This is equivalent to the property that the intersection of the corresponding intervals is not empty. Thus, a clique D can be represented by a point p which is contained in each of the intervals corresponding to the vertices of D. Note, however, that different cliques may have the same representative. A clique is *maximal* if there is no other clique containing this clique as a proper subgraph. It is not difficult to see that different maximal cliques must have different representatives. By selecting a representative p for each maximal clique D and ordering all maximal cliques from left to right on the real number line by their representative points, Gilmore and Hoffman [6] obtained the following property.

LEMMA 2.1. *The maximal cliques of an interval graph G can be linearly ordered, such that, for every vertex x of G, the maximal cliques containing x occur consecutively.*

We name such an ordering $D_1, D_2, \ldots, D_m$ the *linear order* of cliques, where a maximal clique is named D_i if its representative point p_i is the ith smallest representative of the maximal cliques of G.

A vertex v that appears in a maximal clique D_i is called a *conductor* for D_i if v also appears in the maximal clique D_{i+1}. Clearly, the interval corresponding to v contains the interval $[p_i, p_{i+1}]$. Let

$$L(D_i) := \{D_1, D_2, \ldots, D_i\} \quad \text{and } \tilde{L}(D_i) := \{D_{i+1}, \ldots, D_m\}.$$

A path P in G is *spanning* for $L(D_i)$ if P contains all vertices of G not appearing in $\tilde{L}(D_i)$ and P has two conductors of D_i as endvertices. Let R_i be the set of representatives of the maximal cliques containing vertex v_i. A point *embedding* Q of a path P : $v_1 v_2 \ldots v_n$ is an assignment of a real number $q(v_i) \in R_i$ to v_i such that $q(v_i) \in R_{i+1}$ for $1 \leq i \leq n - 1$. A path is *straight* if it has a point embedding Q with the property that $q(v_r) \leq q(v_{r+1})$ for $1 \leq r \leq n - 1$. The following lemma is due to Keil [8].

LEMMA 2.2. *Given a path P with point embedding Q, in an interval graph G, with an endpoint v_1 that appears only in D_1, there exists a straight path P', with v_1 as an endpoint, that has the same vertex set as P and has a point embedding Q' that has the same point set as Q.*

A path P, with endvertices u and v, that spans $L(D_i)$ is said to be *U-shaped* if there exists a vertex x in P that appears only in D_1 such that the two subpaths of

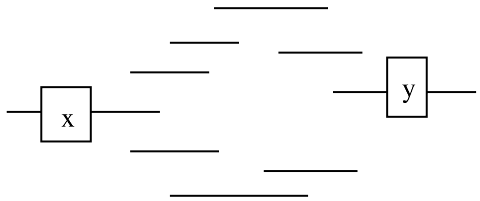

FIG. 1. *A standard cycle.*

P from x to u and from x to v are straight. Such a vertex x is called the *base* of the U-shaped path P. The point embedding of w in the U-shaped path P is the point embedding of v in the path from x to u if w lies on this path; otherwise it is the point embedding of v in the path from x to v. We denote the embedding by q_P. The following result is also due to Keil [8].

LEMMA 2.3. *If G is an interval graph with m maximal cliques, then G has a Hamiltonian cycle if and only if there exists a U-shaped spanning path for $L(D_i)$, $1 \leq i \leq m-1$.*

Based on Lemma 2.3, for every Hamiltonian interval graph there is a Hamiltonian cycle C and two vertices $x \in D_1$ and $y \in D_m$ such that both x-y paths induced by C are straight, x appears only in D_1, and y appears only in D_m. We name such a Hamiltonian cycle a *standard Hamiltonian cycle* (see Figure 1) and denote it by $(C : x, y)$ with distinguished vertices x and y. We also denote the embedding by q_C. Keil [8] also showed the following lemma.

LEMMA 2.4. *An interval graph with at least* 3 *vertices is Hamiltonian if and only if it is* 1*-tough.*

LEMMA 2.5. *Let G be a* 2*-connected chordal graph and e an edge of G. Then e is on a triangle of G.*

Proof. Let T be a smallest cycle containing e. Since every cycle of length at least 4 must contain a chord, T is a triangle. $\square$

3. Proof of Theorem 1.2. Suppose, to the contrary, there is a Hamiltonian interval graph G and a non-Hamiltonian cycle C of G such that C is not extendable. Furthermore, we assume that $|G|$, the order of G, is minimum with respect to this assumption.

The strategy of the proof is to find a critical cut A of $H = G[V(C) \cup \{v\}]$ such that $H - A$ has $|A|$ components, there is a component of $G - V(H)$ adjacent only to vertices in A, and every other component of $G - V(H)$ is adjacent only to A and vertices in at most one component of $H - A$. Thus, $G - A$ has more components than $|A|$, a contradiction to the fact that G is 1-tough (violating Lemma 2.4).

Since C is a Hamiltonian cycle in $G[V(C)]$, we can assume that there exist two vertices x and y such that $(C : x, y)$ is a standard Hamiltonian cycle of $G[V(C)]$. Further, x appears only in D_1 and y appears only in D_m, where the ordering of $D_1, D_2, \ldots, D_m$ is the linear ordering of maximal cliques of $G[V(C)]$. Let P_1 and P_2 be the two x-y paths induced by C. Let q_i be an embedding of P_i for each $i = 1$, 2, respectively. Since x appears only in D_1, all neighbors of x are adjacent. So, without loss of generality, we assume that $q_1(x) = q_2(x)$. Similarly, we assume that $q_1(y) = q_2(y)$. For convenience, we define $q_C(v) = q_i(v)$ if $v \in P_i$.

Let B be a Hamiltonian cycle of G and assume that B has a given orientation. Since B is a cycle, $B - V(C)$ is a union of disjoint segments. Let $B(a_i, b_i)$, $i = 1, 2, \ldots$, denote those nonempty segments, where a_i and b_i are in $V(C)$. A segment $B(a_i, b_i)$ is a *type*-1 segment if a_i and b_i are adjacent in G. Otherwise, we call $B(a_i, b_i)$ a *type*-2 *segment*.

CLAIM 3.1. *If $B(a_i, b_i)$ is a type-1 segment, then there is a vertex $c_i \in B(a_i, b_i)$ such that c_i is adjacent to both a_i and b_i.*

Proof. Since $a_i B(a_i, b_i] a_i$ is a cycle and G is a chordal graph, by Lemma 2.5, $a_i b_i$ is on a triangle in the subgraph induced by this cycle. Let c_i be the other vertex of this triangle. Clearly, c_i is adjacent to both a_i and b_i. ∎

CLAIM 3.2. *All $B(a_i, b_i)$ are type-2 segments.*

Proof. Suppose, to the contrary, that $B(a_1, b_1)$ is a type-1 segment. Let $G^* = G - V(B(a_1, b_1))$ and $B^* = B \cup \{a_1 b_1\} - V(B(a_1, b_1))$. Clearly, B^* is a Hamiltonian cycle of G^* and $V(G^*) \supset V(C)$. If $|G^*| > |C|$, G^* is cycle extendable by the induction hypothesis. Thus, C is extendable in G^*, so it is extendable in G, a contradiction. If $|G^*| = |C|$, then by Claim 3.1, there exists $c_1 \in B(a_1, b_1)$ such that $a_1 c_1$, $b_1 c_1 \in E$. Then, $C^* = B[b_1, a_1] c_1 b_1$ is an extension of C, a contradiction. ∎

Let $H := G[V(C)]$ and for any $v \in V(G) - V(C)$ let $H_v := G[V(C) \cup \{v\}]$. The following claim is a direct consequence of the fact that cycles are 1-tough.

CLAIM 3.3. *If A is a critical cut of H, then A does not contain two consecutive vertices of C and all segments of $C - A$ induce components of $H - A$. Thus, all segments of $C - A$ induce disjoint intervals on the real line.*

CLAIM 3.4. *If $v \notin V(H)$ has at least two neighbors in H, there exists a nontrivial critical cut A of H such that $N(v) \subseteq A$.*

Proof. Since H_v is a non-Hamiltonian interval graph, it is not 1-tough. Hence, there is a cut A of H_v such that $c(H_v - A) \geq |A| + 1$. Since H is a Hamiltonian interval graph, it is 1-tough. Thus, v itself is a component of $H_v - A$, A is a critical cut of H, and $N(v) \subseteq A$. ∎

CLAIM 3.5. *For each segment $B(a_i, b_i)$ there exists $c_i \in B(a_i, b_i)$ such that c_i has two neighbors on C. Thus, H has a nontrivial critical cut.*

Proof. Since $I(a_i) \cap I(b_i) = \emptyset$, let I denote the interval between $I(a_i)$ and $I(b_i)$. Then, $I \subseteq I(B(a_i, b_i))$. Let $c_i \in B(a_i, b_i)$ such that $I(c_i) \cap I(a_i) \neq \emptyset$ and $I(c_i) \cap I \neq \emptyset$. Since C is connected, $I \subseteq I(C)$, so that there exists $d_i \in V(C) - \{a_i, b_i\}$ such that $c_i d_i \in E$. Thus, $|N_C(c_i)| \geq 2$, and we are done by Claim 3.4. ∎

CLAIM 3.6. *Let A be a nontrivial critical cut of H. Then, $I(S)$ are disjoint intervals for all components $S \subseteq H - A$. If there exists a path P in $G - V(C)$ connecting two components S and T of $H - A$, then $I(S)$ and $I(T)$ must be two consecutive intervals in $I(H - A)$.*

Proof. The first part of Claim 3.6 is trivial. To prove the second part of the claim, suppose, to the contrary, there is a component R of $H - A$ such that $I(R)$ is between $I(S)$ and $I(T)$. So $I(R) \subset I(P)$. Let $r \in R$. Then, $q_C(r) \in I(r) \subset I(P)$, so that there is a vertex $w \in P$ such that $q_C(r) \in I(w)$. Since $q_C(r)$ is contained in two consecutive vertices of C, w can be inserted into cycle C to make a larger cycle, which is a contradiction. ∎

Recall that $(C : x, y)$ is a standard Hamiltonian cycle in H. If A is a critical cut of H, A does not contain two consecutive vertices of C and each component of $C - A$ induces a component of $H - A$.

CLAIM 3.7. *For any nontrivial critical cut A of H, $x \notin A$ and $y \notin A$.*

Proof. Since $x \in D_1$, all neighbors of x in H are adjacent. Thus, $x \notin A$. Similarly, $y \notin A$. ∎

CLAIM 3.8. *For every component D of $G - V(C)$, there exists a nontrivial critical cut A of H such that $N(D) \subseteq A$; i.e., all neighbors of D are in A.*

Proof. Let D be a component of $G - V(C)$ and $v \in D$. We assume, without loss of generality, $v \in B(a_1, b_1)$. By Claim 3.4, let $A := A_v$ be a critical cut of H such that $N_C(v) \subseteq A$.

Note that $I(H - A)$ is a union of disjoint intervals and each such interval corresponds to a component of $H - A$. Let L be the component of $H - A$ such that $I(L)$ is the closest interval of $I(H - A)$ on the left side of v and let R be the component of $H - A$ such that $I(R)$ is the closest interval of $I(H - A)$ on the right side of v. Since A is critical and $N_C(v) \subseteq A$, such components L and R exist. We assume that $|V(L)| + |V(R)|$ is at its minimum over all nontrivial critical cuts $A := A_v$.

We claim that $N_H(D) \subseteq A$. Suppose, to the contrary, that $N_H(D) \not\subseteq A$; then we have $N_H(D) \cap V(L \cup R) \neq \emptyset$. Assume, without loss of generality, that for $w \in D$, we have $N_C(w) \cap V(R) \neq \emptyset$ and $\mathrm{dist}_D(v, w)$ is minimum with this property. Let $P[v, w]$ be a shortest path in D connecting v and w. Then, $N(P[v, w)) \cap R = \emptyset$.

Since $N(w) \cap R \neq \emptyset$ and $N(P[v, w)) \cap R = \emptyset$, $I(w)$ must contain the left-end of $I(R)$. Since there are two paths from x to R along C, then $|N_C(w)| \geq 2$. By Claim 3.4, let $A^* := A_w$, be a nontrivial critical cut of H such that $N_C(w) \subseteq A^*$. Let

$$A_L = \{a \in A \ : \ a \text{ is on the left side of } w\},$$

$$A^*_R = \{a^* \in A^* : \ a^* \text{ is not on the left side of } a\},$$

$$X = A_L \cup A^*_R.$$

We will show that X is a critical cut of H. Note that

- each component S of $H - X$ such that $I(S)$ is on the left side of $I(w)$ is a component of $H - A$,
- each component S of $H - X$ such that $I(S)$ is on the right side of $I(w)$ is a component of $C - A^*$, and
- there is no component S of $H - X$ such that $I(S)$ is between $I(v)$ and $I(w)$.

Thus, X is a cut of H, and, by Claim 3.3, in order to show that X is a critical cut, we need only show that X does not contain two consecutive vertices of C. Suppose, to the contrary, there are two consecutive vertices a and b on C and $a, b \in X$. Without loss of generality, we assume that $a \in A \setminus A^*$ and $b \in A^* \setminus A$. By the definition of X, a is on the left side of w and b is not on the left side of w. Thus, $b \in R$. Since $q_C(a)$ is on the left side of w and $q_C(a) \in I(b)$ (because a and b are consecutive on C), $I(P[v, w)] \cap I(b) \neq \emptyset$, which contradicts the minimality of $P[v, w]$.

Let R^* be the component of $H - X$ such that $I(R^*)$ is the closest interval of $I(H - X)$ on the right side of w and let $I := I(P[v, w])$. Note that if $x \in V(C)$ such that $I(x) \cap I \neq \emptyset$, then either $x \in A^*$ or $x \in A$. In any case, we have that $x \in X$. Note that R is induced by a segment of C. Let y_0 be the first vertex along the segment of R from left to right such that $y_0 w \in E(G)$. Without loss of generality, we assume that $y_0 \in P_1$. Let x_0 be the predecessor of the segment R along P_1 from x to y and let x_0^- be the predecessor of x_0. Since X does not contain two consecutive vertices of C and $x_0 \in X$, $q_C(x_0^-)$ must lie on the left side of $I(R)$. Since C is a standard cycle of H, $q_C(x_0^-) \notin I$. Thus, $q_C(x_0^-)$ is on the left side of the interval of c_1. Thus, $x_0 \in A \cap A^*$. Let S be the segment of R from the first vertex of R to the predecessor of y_0. We first note that $S \neq \emptyset$ (since X does not contain two consecutive vertices). Thus, S is a component of $H - X$.

We claim that $|V(R^*)| < |V(R)|$, which leads to a contradiction of the minimality of $|V(L)| + |V(R)|$. This is certainly true if $R^* = S \subset R$. Suppose $R^* \neq S$. Then,

$I(R^*)$ is between $I(w)$ and $I(S)$. From the definition of R and S, we have $R^* \subseteq A$. Since A does not contain two consecutive vertices of C, $|V(R^*)| = 1$. Since $|V(R)| \geq |V(S)| + 1 \geq 2$, we have $|V(R^*)| < |V(R)|$, as desired. □

Let $D_1, D_2, \ldots, D_m$ be the components of $G - V(C)$. Assume, without loss of generality, that $I(D_i)$ is on the left side of $I(D_j)$ whenever $i < j$. By Claim 3.8, for each D_i, $I(D_i) \subseteq I(C)$ and there exists a nontrivial critical cut A_i of H such that $N(D_i) \subseteq A_i$. Let $LA_i = \{a \in A_i \mid a \text{ is on the left side of } A_i\}$ and $RA_i = A - LA_i$. We now inductively define B_i for each $i = 1, 2, \ldots, m$ as follows: $B_1 = A_1$ and, for each $i > 1$, if D_i is adjacent to at most one component of $H - B_{i-1}$, let $B_i = B_{i-1}$. Otherwise, let

$$B_i = \{b \in B_{i-1} \mid b \text{ is on the left side of } D_i\} \cup RA_i.$$

CLAIM 3.9. *B_i is a nontrivial critical cut for each $i = 1, 2, \ldots, m$.*

Proof. Claim 3.9 is true for $i = 1$. Suppose it is true for $i - 1 \geq 1$. If $B_i = B_{i-1}$, then it is also true for i. So, we assume that $B_i \neq B_{i-1}$. In this case, let L and R be two components of $H - B_{i-1}$ such that $N(D_i) \cap L \neq \emptyset$ and $N(D_i) \cap R \neq \emptyset$. By Claim 3.6, $I(L)$ and $I(R)$ are two consecutive intervals of $I(H - B_{i-1})$. Furthermore, $I(D_i)$ contains the interval between L and R as a subinterval. Note that components of $H - B_i$ on the left side of D_i are those of $H - B_{i-1}$ and components of $H - B_i$ on the right side of D_i are those of $H - A_i$. In order to show that B_i is a critical cut, we only need show that B_i does not contain two consecutive vertices of C. Suppose, to the contrary, a and b are two consecutive vertices on C such that $a \in B_{i-1} \setminus A_i$ and $b \in A_i \setminus B_{i-1}$. Since $b \notin B_{i-1}$, $q_C(a) \in I(a) \cap I(b)$ must be on the right side of L. Similarly, $q_C(a)$ must be on the left side of R. Thus, $q_C(a) \in I(D_i)$, so that there exists $w \in I(D_i)$ adjacent to both a and b. Then, C is extendable, which is a contradiction. □

By the definition, we have $N(D_1) \subseteq B_m$ and, for each $i > 1$, either $N(D_i) \subseteq B_m$ or D_i is adjacent to at most one component of $H - B_m$. Since $H - B_m$ has exactly $|B_m|$ components, $G - B_m$ has at least $|B_m| + 1$ components, which contradicts the fact that G is 1-tough. This contradiction completes the proof. □

Note: Just at the time of submission we were informed of another proof of this result in [1].

Acknowledgment. We thank the referees for their useful suggestions which led to a simpler proof.

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