Graphs and Combinatorics (2006) 22:217–224
Digital Object Identifier (DOI) 10.1007/s00373-006-0651-6

Graphs and Combinatorics

On H-Linked Graphs

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Abstract. For a fixed multigraph H, possibly containing loops, with $V(H) = \{h_1, \dots, h_k\}$, we say a graph G is H-linked if for every choice of k vertices $v_1, \dots, v_k$ in G, there exists a subdivision of H in G such that v_i represents h_i (for all i). This notion clearly generalizes the concept of k-linked graphs (as well as other properties). In this paper we determine, for a connected multigraph H and for any sufficiently large graph G, a sharp lower bound on $\delta(G)$ (depending upon H) such that G is H-linked.

1. Introduction

For terms not defined here, see [1]. Let H be a multigraph, possibly containing loops. For any graph G, let $\mathcal{P}(G)$ denote the set of paths in G. An H-*subdivision* in G is a pair of mappings $f_1 : V(H) \to V(G)$ and $f_2 : E(H) \to \mathcal{P}(G)$ such that:

(i) f_1 is injective;
(ii) for every edge $xy \in E(H)$, $f_2(xy)$ is an $f_1(x) - f_1(y)$ path in G and distinct edges of H map to internally disjoint paths in G.

A graph G is H-*linked* if every injective map $f_1 : V(H) \to V(G)$ can be extended to an H-subdivision. The vertices in $f_1(V(H))$ are called the *ground* vertices. Thus, we can say that G has a subdivision of H whose ground vertices are prescribed. This idea originated with Jung [2], but had not been considered for arbitrary H until recently, when the concept was considered in [7] and [3].

A graph is k-*linked* if for every sequence of $2k$ vertices, $v_1, \dots, v_k, w_1, \dots, w_k$, there are internally disjoint paths $P_1, \dots, P_k$ such that P_i joins v_i and w_i. Clearly, the concept of graphs being H-linked generalizes that of being k-linked. In fact, if $H = kK_2$, then G is k-linked if, and only if, G is H-linked. Further, a graph G is said to be k-*ordered* if for every sequence of k vertices $v_1, \dots, v_k$ there is a cycle in G that encounters the vertices $v_1, \dots, v_k$ in the specified order. If H is a k-cycle, then G is H-linked if and only if G is k-ordered. Thus, the property of being H-linked also generalizes the property of being k-ordered. Other such connections are explored in [3].

A common question dealing with k-linked graphs is to find the minimum connectivity $f(k)$ such that every $f(k)$-connected graph is k-linked. At this time the best know result is that every $10k$-connected graph is k-linked [6]. The literature also contains numerous results pertaining to the minimum number of edges needed to assure a graph G contains a subdivision of some graph H. For a survey of results of this type, see [5].

The purpose of this paper is to provide a sharp minimum degree condition such that any such sufficiently large graph G will be H-linked. This bound will depend on the multigraph H. In order to present our result, we first need some notation.

All multigraphs in this paper will be assumed to have labeled vertices. An edge is *proper* if it is not a loop. We denote by $d_H(w)$, the *degree* of w in H, which is the number of proper edges incident to w, plus twice the number of loops at w. Additionally, for a given subgraph H' of G, let $N_{H'}(x)$ denote the set of vertices in H' that are adjacent to x in G. This set is the *neighborhood* of x in H'. If F_1 and F_2 are two subgraphs of H, then $E_H(F_1, F_2)$ will represent the set of edges having one end-vertex in F_1 and the other end vertex in F_2 and $e_H(F_1, F_2) = |E_H(F_1, F_2)|$.

Now consider a connected multigraph H, possibly with loops. Let

$$\eta(H) = max_{X \subset V(G)} e(X, V(G) - X)$$

denote the maximum size of a bipartite subgraph in H, or in other words, the size of a maximum edge cut in H.

Our main result will be: Given a connected multigraph H, possibly with loops, if G is sufficiently large and $\delta(G) \geq \frac{n+\eta(H)-2}{2}$, then G is H-linked. To see that this minimum degree is needed, suppose that the multigraph H has $\eta(H)$ as the maximum size of a bipartite subgraph. Also suppose that this cut determines a partition of $V(H)$ into sets X and Y. Now suppose that G is formed from two complete graphs G_1 and G_2, each of order m, that intersect on $\eta(H) - 1$ vertices. If the set S chosen as the image of $V(H)$ under f_1 is such that the vertices of X lie in $G_1 - G_2$ and the vertices of Y lie in $G_2 - G_1$, then clearly $G_1 \cap G_2$ is not large enough to allow an embedding of H. Further, $\delta(G) = m - 1$. Since $|V(G)| = 2m - \eta(H) + 1$, we see that $\delta(G) = \frac{n+\eta(H)-3}{2}$. Thus, the minimum degree condition is necessary.

2. Main Result

For convenience we let $\eta = \eta(H)$. We now state our main result.

Theorem 2.1. *Let H be a connected multigraph on k vertices. If G is a graph of sufficiently large order n and $\delta(G) \geq \frac{n+\eta-2}{2}$, then G is H-linked. Furthermore, on any path between ground vertices in G, there will be at most two intermediate vertices.*

Proof. Let G be as above and let $S \subset V(G)$ be the image of f_1, that is, the ground vertices in G. For convenience, until the end of the proof, we will remove any loops from H. Our goal is to show that we can construct the necessary subdivision of H in G, using paths with at most two intermediate vertices between ground vertices. After this, we will build the paths corresponding to loops. These will also contain at most three intermediate vertices.

Clearly, by the minimum degree condition, any two nonadjacent vertices $x, y \in G$ satisfy $|N_G(x) \cap N_G(y)| \geq \eta$. We first note that for any two vertices $x, y \in S$ such that $f_1^{-1}(x) f_1^{-1}(y) \in E(H)$ and x and y are adjacent in G, then x and y already have the desired path between them.

We now define the auxiliary graph L as follows: Let $V(L) = S$ and let $E(L)$ consist of all edges xy where $x, y \in S$ are such that $f_1^{-1}(x) f_1^{-1}(y) \in E(H)$, x and y are not adjacent in G and

$$|N_{G-S}(x) \cap N_{G-S}(y)| \leq 3|E(H)| - 1. \tag{1}$$

Our goal will be to link in G these pairs from L with at most two intermediate vertices. We will then show we can link the remaining nonadjacent pairs with single vertices. □

Claim 2.1.1. *The graph L is bipartite.*

Proof. Suppose not and let C be the shortest odd cycle in L, and assume that for some integer $t > 1$ the vertices of C are, in order,

$$x_1, y_1, \ldots, x_{t-1}, y_{t-1}, x_t.$$

By our construction of L, the common neighborhood in G of y_1 with either x_1 or x_2 is at most $3|E(H)| < 3k^2$. This, combined with the fact that $\delta(G) > \frac{n}{2}$ implies that

$$|N_G(x_1) \cap N_G(x_2)| = (1 - o(1))\frac{n}{2}. \tag{2}$$

Since there are at most $k (= o(n))$ vertices in C, the reader can verify that the neighborhood intersections of $x_1, \ldots, x_t$ and $y_1, \ldots, y_{t-1}$ both have order $\frac{n}{2}$. However, this contradicts the fact that $x_1 x_t$ is an edge in L, implying that no such odd cycle C exists. □

Let X and Y be the partite sets of L. We will impose this partition on S and H when necessary. Let $Z_X = \{x \in X : d_L(x) = 0\}$ and $Z_Y = \{y \in Y : d_L(y) = 0\}$. Further, let

$$U_X = \{u \in \bigcap_{x \in X - Z_X} N_{G-S}(x) : u \notin (\bigcup_{y \in Y - Z_Y} N_{G-S}(y))\},$$

that is, U_X is the set of common neighbors in $G - S$ of all vertices in $X - Z_X$, that are also not neighbors of any vertex in $Y - Z_Y$. Similarly, let

$$V_Y = \{v \in \bigcap_{y \in Y - Z_Y} N_{G-S}(y) : v \notin (\bigcup_{x \in X - Z_X} N_{G-S}(x))\}.$$

Lemma 2.1. $|U_X| = (1 - o(1))\frac{n}{2^{(k/2)+1}}$ *and* $|U_Y| = (1 - o(1))\frac{n}{2^{(k/2)+1}}$.

Proof. Let $C_1 = X_1 \cup Y_1$ and (if it exists) $C_2 = X_2 \cup Y_2$ be nontrivial components of L. Let $N_i^X := \bigcap_{x \in X_i} N_{G-S}(x)$, and let $N_i^Y := \bigcap_{y \in Y_i} N_{G-S}(y)$. As in the proof of Claim 2.1.1, $|N_1^X| \geq (1-o(1))\frac{n}{2}$, and $|N_1^Y| \geq (1-o(1))\frac{n}{2}$. Also, note that N_1^X and N_1^Y are almost disjoint. Now choose vertices $x \in X_2$ and $y \in Y_2$ such that $xy \in E(L)$. The vertices of X_2 within G have a large common neighborhood in $G - S$, and the same is true for the vertices of Y_2 within G. Thus, either

$$(|N(x) \cap N_1^X| \geq (1-o(1))\frac{|N_1^X|}{2} \text{ and } |N(y) \cap N_1^Y| \geq (1-o(1))\frac{|N_1^Y|}{2});$$

or,

$$(|N(x) \cap N_1^Y| \geq (1-o(1))\frac{|N_1^Y|}{2} \text{ and } |N(y) \cap N_1^X| \geq (1-o(1))\frac{|N_1^X|}{2}).$$

If the latter case is true, reverse the labels of X_2 and Y_2. These observations, combined with (2), imply that all of the vertices in $X_1 \cup X_2$ have a common neighborhood within $G - S$ of order $(1-o(1))\frac{n}{4}$, and all of the vertices in $Y_1 \cup Y_2$ have a common neighborhood within $G - S$ of order $(1-o(1))\frac{n}{4}$. Inductively, it is easy to see that if $C_1, \ldots, C_r$ are the nontrivial components of L, then

$$|\bigcap_{i=1}^{r} U_{G-S}(X_i)| \geq (1-o(1))\frac{n}{2^{r+1}} \geq (1-o(1))\frac{n}{2^{(k/2)+1}}$$

and

$$|\bigcap_{i=1}^{r} U_{G-S}(Y_i)| \geq (1-o(1))\frac{n}{2^{r+1}} \geq (1-o(1))\frac{n}{2^{(k/2)+1}},$$

where $U_{G-S}(X_i)$ represents the common neighborhood in $G - S$ of the vertices in X_i, and $U_{G-S}(Y_i)$ represents the common neighborhood in $G - S$ of the vertices in Y_i.

Let N_X and N_Y denote the common neighborhoods of the vertices in $X - Z_X$ and $Y - Z_Y$ respectively. Note that $N_X - S = \cap_{i=1} U_{G-s}(X - i)$. We wish to show almost none of the vertices in N_X are adjacent to any vertex in $Y - Z_Y$ and, similarly, almost none of the vertices in N_Y are adjacent to any vertex in $X - Z_X$.

We can accomplish this by showing that for any x in $X - Z_X$ and all but at most $O(k^4)$ vertices v in $N_{G-S}(x)$, v is not adjacent to any vertex in $Y - Z_Y$ that lies in the same component of L as x. Consider any edge xy in some component C of L. By the definition of L, we know that in G, x and y have at most $3|E(H)| < 3k^2$ vertices in their common neighborhood. As above, the vertices of $C \cap X$ have at least $(1-o(1))\frac{n}{2}$ other vertices in their common intersection. As each vertex in $C \cap X$ is adjacent in L to at least one vertex from Y, and possibly all of them, there are at least

$$(1-o(1))\frac{n}{2} - 3k^2|X||Y| \geq (1-o(1))\frac{n}{2} - 3k^4 = (1-o(1))\frac{n}{2} \tag{3}$$

vertices in the common neighborhood of $C \cap X$ that are not adjacent to any of the vertices in $C \cap Y$.

Thus, for any non-trivial component C_i of L, nearly all of the vertices in N_i^X are not adjacent to $C_i \cap Y$. Therefore, at least

$$(1 - o(1))|N_X| \geq (1 - o(1))\frac{n}{2^{(k/2)+1}} \tag{4}$$

vertices in N_X are not adjacent to any vertex in $Y - Z_Y$ and hence are in U_X. The proof is similar for V_Y. □

Lemma 2.2. *Let $u \in U_X$, $v \in V_Y$, $x \in X$ and $y \in Y$ and let z be the number of vertices of degree 0 in L. Then $d(u, N_{G-S}(y)) + d(v, N_{G-S}(x)) \geq \eta - z$.*

Proof. Let u be a vertex of U_X. Let z_X and z_Y be the number of vertices of degree 0 in L lying in the sets X and Y, respectively. Then

$$\begin{aligned} d(u, N_{G-S}(y)) &= d(u) - d(u, S) - d(u, G - S - N_{G-S}(y)) \\ &\geq \frac{n + \eta - 2}{2} - d(u, X) - d(u, Y) - (n - k - 1 - d_{G-S}(y)). \end{aligned}$$

However by our definitions of U_X and V_Y, u is not adjacent to any vertices of Y with nonzero degree in L. Thus,

$$\begin{aligned} d(u, N_{G-S}(y)) &\geq \frac{n + \eta - 2}{2} - |X| - z_Y - n + k + 1 + d_{G-S}(y) \\ &\geq \frac{n + \eta - 2}{2} - |X| - z_Y - n + k + 1 + d(y) - d(y, X) - d(y, Y) \\ &\geq n + \eta - 2 - |X| - z_Y - n + k + 1 - d(y, X) - |Y| + 1 \\ &\geq \eta + k - |X| - |Y| - z_Y - d(y, X) \\ &= \eta - d(y, X) - z_Y. \end{aligned}$$

Similarly, $d(v, N_{G-S}(y)) \geq \eta - d(x, Y) - z_X$. Therefore,

$$\begin{aligned} d(u, N_{G-S}(x)) + d(v, N_{G-S}(y)) &\geq 2\eta - d(x, Y) - d(y, X) - z_X - z_Y \\ &\geq \eta - z. \end{aligned}$$

□

Lemma 2.3. *Let M be a connected graph and let $W,\ Z$ be a vertex partition of $V(M)$. Then for any $F \subseteq V(M)$,*

$$e(W - F, Z - F) \leq \eta(M) - |F|.$$

We proceed by induction. First, assume that $F = \{v\}$. Without loss of generality, we may assume that v is in W. Let W' denote $W - \{v\}$.

If the result fails, then $e(W', Z) = \eta(M)$. As M is connected, either W' or Z must contain some x that is adjacent to v in M, say W'. Then, $W', Z \cup \{v\}$ is a partition of the vertices of M, but

$$e(W', Z \cup \{v\}) > e(W', Z) = \eta(M),$$

a contradiction.

Now, assume that $|F| = p$ and assume that there is some partition of $V(M)$ into sets W and Z such that $e(W - F, Z - F) \geq \eta(G) - p + 1$. Since M is connected, there is some vertex v in F such that v is adjacent to, without loss of generality, some w in W. If we denote $F - \{v\}$ by F', the induction hypothesis implies that there is no partition of $V(M - F')$ into sets W' and Z' such that $e(W' - F', Z' - F') \geq \eta(M) - p + 2$. However, this is clearly contradicted by choosing $W' = W$ and $Z' = Z \cup \{v\}$. Hence the lemma holds. □

This implies that if there are exactly j isolated vertices in L, then L has at most $\eta(H) - j$ edges. Now we can proceed to patch the edges of H represented in L.

Assume that we have constructed as many paths in G representing edges from L as possible using two intermediate vertices, and then constructed as many remaining paths as we could using one intermediate vertex. Let xy be an edge in L that has not yet been mapped, and let u be in U_X and v be in V_Y. Applying Lemma 2.3 to the degree sum in Lemma 2.2 we have that

$$|N_{G-S}(u) \cap N(y)| + |N_{G-S}(v) \cap N(x)| \geq |E(L)|. \tag{5}$$

Let $x'y'$ be some edge in L with x' in X and y' in Y that has already been mapped. We consider two cases.

Case 1. The path representing $x'y'$ has been constructed with one intermediate vertex.

Let w be the intermediate vertex used and assume that both u and v are adjacent to w in G. Recall that by our choice of u and the definition of U_X, ux' is an edge in G. Then, $x'uwy'$ would patch the edge $x'y'$ and increase the number of patched edges using 2 intermediate vertices, a contradiction. Thus, there is at most one edge from u or v to w.

Case 2. The path representing $x'y'$ has been constructed with two intermediate vertices.

Let these intermediate vertices be u' and v'. One may assume that they lie in U_X and V_Y respectively. We wish to show here, as in Case 1, that there can be at most one edge from u or v to u' and v' that is accounted for in (5).

Note that (5) counts only edges from u to the neighborhood of y and from v to the neighborhood of x. Since u' is not adjacent to any y in Y and v' is not adjacent to any x in X, the only possible way to account for two edge from (5) is if $u'v$ and $v'u$ were both edges in G. However, then we could use the paths $xu'vy$ and $x'uv'y$ to construct an additional path, contradicting our maximality assumption.

Thus, we have used at most one edge from the degree count in (5) for each path already constructed. As there are at least $\eta(H) - j$ such edges, and at most $\eta(H) - j - 1$ paths already constructed, either u has an unused common neighbor with y or v has an unused common neighbor with x, allowing us to construct the path representing xy. Hence, by our maximality assumption, we can construct all of the paths for edges represented in L.

Now that we have joined all of the pairs of vertices that correspond to edges in L, it remains to show that we can join those pairs of vertices that correspond to the remaining edges. Let $T \subset E(H)$ represent those edges already linked in G such that $E(L) \subset T$. As above, assume that we have used at most 2 intermediate vertices to link each pair.

Let x and y be vertices in S corresponding to an edge in $E(H) - T$. As xy is not in L, (1) implies that there are at least $3|E(H)|$ vertices in G adjacent to both x and y. At this point, we have linked at most $|E(H)| - 1$ pairs of vertices in T with at most 2 intermediate vertices each. Together with the $k - 2$ other vertices in S, there are at most

$$2(|E(H)| - 1) + k - 2 < 3|E(H)| \tag{6}$$

vertices accounted for thus far. Hence, there is some vertex w adjacent to both x and y that is not in S and is not being used to link any pair of vertices in S that correspond to an edge in H. We may therefore link x and y in G with the path xwy.

At the start of the proof, we removed all loops from H. Thus far, we have managed to embed this subgraph of H. Now, assume that H had loops $l_1, \ldots, l_m$ where m may be arbitrarily large with respect to k. We will patch these loops using at most three intermediate vertices. Consider some loop l_{i+1} adjacent to some vertex x in H, and assume that we have already dealt with $l_1, \ldots, l_i$. There are at most

$$3k^2 + k + 3i = o(n)$$

vertices from the neighborhood of x already in use in our embedding. Call this set of vertices $V(x)$. If there is a pair of adjacent vertices in $N(x) \setminus V(x)$, we can easily embed l_{i+1}. If there is not, however, $N(x) \setminus V(x)$ is an independent set in G of order at least $(1 - o(1))\frac{n}{2}$. In this case, our minimum degree condition implies that for any pair of vertices a, b in $N(x) \setminus V(x)$ there is some vertex p in $V(G) \setminus N(x)$ that is not being utilized in our embedding and is adjacent to both a and b. If so, we will construct the path representing l_{i+1} with the path apb. $\square$

3. Remarks

As stated, this result holds for n sufficiently large. If H does not have exceedingly many loops relative to k, then we require the order of n to exceed $2^{(k/2)+1}$ as in Lemma 2.1. It is possible, however, that H may have arbitrarily many loops, in which case, our bound for n would be based on $|E(H)|$. A. Kostochka and G. Yu [4] have provided a linear bound on n for the case that H is a connected loopless multigraph with minimum degree at least 2.

Once a graph G is known to be H-linked, it then becomes interesting to know under what conditions an H-subdivision can be extended to a spanning H-subdivision. This question is addressed in [3] and provides a generalization of a number of well-known results.

Acknowledgements. The authors would like to thank the referees for their helpful comments.

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Received: June 24, 2004
Final version received: December 5, 2005