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POTENTIAL FORBIDDEN TRIPLES IMPLYING  
HAMILTONICITY: FOR SUFFICIENTLY  
LARGE GRAPHS

RALPH J. FAUDREE

*University of Memphis, Memphis, TN 38152, USA*

RONALD J. GOULD

*Emory University, Atlanta, GA 30322, USA*

AND

MICHAEL S. JACOBSON

*University of Colorado at Denver*  
*Denver, CO 80217, USA*

**Abstract**

In [2], Brousek characterizes all triples of connected graphs,  $G_1, G_2, G_3$ , with  $G_i = K_{1,3}$  for some  $i = 1, 2$ , or  $3$ , such that all  $G_1G_2G_3$ -free graphs contain a hamiltonian cycle. In [8], Faudree, Gould, Jacobson and Lesniak consider the problem of finding triples of graphs  $G_1, G_2, G_3$ , none of which is a  $K_{1,s}$ ,  $s \geq 3$  such that  $G_1G_2G_3$ -free graphs of sufficiently large order contain a hamiltonian cycle. In [6], a characterization was given of all triples  $G_1, G_2, G_3$  with none being  $K_{1,3}$ , such that all  $G_1G_2G_3$ -free graphs are hamiltonian. This result, together with the triples given by Brousek, completely characterize the forbidden triples  $G_1, G_2, G_3$  such that all  $G_1G_2G_3$ -free graphs are hamiltonian. In this paper we consider the question of which triples (including  $K_{1,s}$ ,  $s \geq 3$ ) of forbidden subgraphs potentially imply all sufficiently large graphs are hamiltonian. For  $s \geq 4$  we characterize these families.

**Keywords:** hamiltonian, forbidden subgraph, claw-free, induced subgraph.

**2000 Mathematics Subject Classification:** 05C45.

## 1. INTRODUCTION

The problem of recognizing graph properties based on forbidden connected subgraphs has received considerable attention. A wide variety of properties and forbidden families have been studied. In particular, the property of being hamiltonian has been widely studied. A series of results culminated in the characterization of the pairs of forbidden subgraphs which imply all graphs free of these pairs of graphs are hamiltonian by Bedrossian [1]. In his proof, Bedrossian used a small order nonhamiltonian graph to eliminate some cases. Faudree and Gould [5] extended the collection to characterize the forbidden pairs which imply all sufficiently large ( $n \geq 10$  suffices) graphs are hamiltonian.

Since the only single forbidden subgraph that implies a graph is hamiltonian is  $P_3$  (the path on 3 vertices) and it forces the graph to be complete, the problem of all single or pairs of forbidden subgraphs implying hamiltonicity has been completely characterized, both for all graphs and for all sufficiently large graphs.

An interesting feature of both characterizations for pairs is that the claw,  $K_{1,3}$ , must be one of the graphs in each pair. This led naturally to the question: If we consider triples of forbidden subgraphs implying hamiltonicity, must the claw always be one of the graphs in the triple? This question was answered negatively in [8]. There, all triples containing no  $K_{1,t}$ ,  $t \geq 3$  which imply all sufficiently large graphs are hamiltonian were given. Brousek [2] gave the collection of all triples which include the claw that imply all 2-connected graphs are hamiltonian.

We follow the notation of [4]. In addition, we say a graph  $H$  is  $G_1G_2G_3$ -free if  $H$  does not contain  $G_i$ ,  $i = 1, 2, 3$  as an induced subgraph. In [6], a characterization was given of all triples  $G_1, G_2, G_3$  with none being  $K_{1,3}$ , such that all  $G_1G_2G_3$ -free graphs are hamiltonian. Thus, the remaining case is, for sufficiently large graphs, to determine the possible triples where  $G_1 = K_{1,s}$ , with  $s \geq 3$ .

The purpose of this paper is to study those triples which include  $K_{1,s}$ ,  $s \geq 3$  such that all 2-connected graphs of sufficiently large order and free of such triples are hamiltonian. For  $s \geq 4$  we characterize these triples. For  $s = 3$  we present a list of triples which potentially imply hamiltonicity. The triples containing  $K_{1,3}$  will be further studied in [7].

Given a cycle with an implied orientation, we write  $x^+$  and  $x^-$  for the successor and predecessor of  $x$  on the cycle, respectively. Further, by  $[x, y]$

we mean the subpath of  $C$  beginning at  $x$  and ending at  $y$  and following the orientation of  $C$ . We also use the notation  $H \leq G$  to mean that  $H$  is an induced subgraph of  $G$ .

For the remainder of this paper we will assume  $G_1, G_2$  and  $G_3$  are connected. We define the graph  $C(i, j, k)$  (see Figure 1 for  $C(2, 2, 1)$ ) to be the graph obtained by identifying the end vertex of paths of lengths  $i, j$  and  $k$ , respectively. This graph may be thought of as a form of generalized claw as  $K_{1,3} = C(1, 1, 1)$ . Define the graphs  $Z_i(m)$  and  $J_i(m)$  to be the complete graph on  $m$  vertices ( $m \geq 3$ ) with a path of length  $i$  or  $i$  edges joined to a single vertex of the  $K_m$ , respectively (see Figure 1 for  $Z_1(m)$  and  $J_2(m)$ ). Note that  $Z_1 = Z_1(3)$  is the notation common in the literature. The book  $B_n$  is obtained by identifying an edge from each of  $n$  copies of  $K_3$  (see Figure 1 for  $B_2$ ).

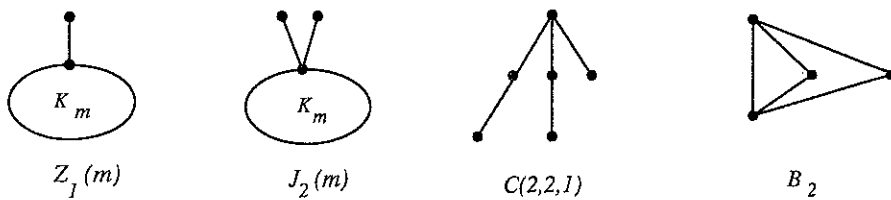


Figure 1. Common forbidden graphs.

Let  $C_3 = K_3$  and  $P_n$  be a path on  $n$  vertices. Let the family  $N(i, j, k)$  be obtained by identifying an endvertex of each of  $P_{i+1}, P_{j+1}$  and  $P_{k+1}$  with distinct vertices of a  $K_3$ . We follow the standard that  $i \geq j \geq k$ . In particular, we denote the net  $N = N(1, 1, 1)$  (see Figure 2), while other special cases have been commonly denoted in the literature as  $Z_3 = N(3, 0, 0)$ ,  $B = N(1, 1, 0)$  and  $W = N(2, 1, 0)$ . We further define the graph family  $N(G_1, G_2, G_3)$  to be those graphs obtained by identifying a distinct vertex of  $K_3$  with a distinct vertex of  $G_1, G_2$  and  $G_3$  respectively. If the vertex of  $G_i$  to be identified is important, we specify it as in the definition of  $N(i, j, k)$ . In particular, if  $G_i = Z_1(m)$ , for some  $i$ , then the vertex being identified from  $Z_1(m)$  will always be the vertex of degree one. For our purposes, the graphs  $G_i$  ( $i = 1, 2, 3$ ) will always be one of  $K_n, P_n$ , or  $Z_1(m)$ , and hence, there will be no ambiguity in the graph constructed.

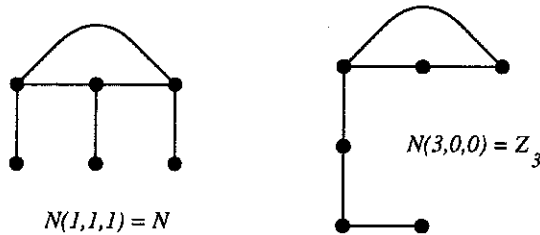


Figure 2. More common forbidden graphs.

We will need the following characterization of forbidden pairs from [5].

**Theorem 1.1.** *Let  $R$  and  $S$  be connected graphs ( $R, S \neq P_3$ ) and  $G$  a 2-connected graph of order  $n \geq 10$ . Then  $G$  is  $(R, S)$ -free implies  $G$  is hamiltonian if, and only if,  $R = K_{1,3}$  and  $S$  is an induced subgraph of one of  $N(1, 1, 1), N(3, 0, 0), N(2, 1, 0)$  or  $P_6$ .*

2. TRIPLES INCLUDING  $K_{1,s}, s \geq 4$

In this section, we characterize those triples  $G_1, G_2, G_3$ , one of which is  $K_{1,s}, (s \geq 4)$  such that  $G_1G_2G_3$ -free graphs of sufficiently large order are hamiltonian. We begin by showing certain triples containing  $K_{1,s}$  do imply hamiltonicity.

**Theorem 2.1.** *If  $G$  is a 2-connected  $K_{1,s}P_4J_2(m)$ -free graph ( $s \geq 4$  and fixed,  $m \geq 3$  and fixed) of sufficiently large order  $n$ , then  $G$  is hamiltonian.*

**Proof.** Observe first that there must be a vertex of degree at least  $\sqrt{n-1}$ , for otherwise  $G$  would have diameter at least four and an induced  $P_4$  would result.

Using the neighborhood of such a vertex, for  $n$  sufficiently large, since  $G$  contains no induced  $K_{1,s}$ , by Ramsey's Theorem,  $G$  contains a  $K_{l'}$  (where  $l' = l'(n) > ms$ ). Select a largest clique  $K_l$  in  $G$ . Note that there are no vertices at distance 2 from this clique, for if there were, an induced  $P_4$  is easily found. Thus, every vertex not in  $K_l$  is adjacent to vertices in  $K_l$ .

Let  $S = V(G) - V(K_l)$  and

$$S_L = \{v \in S \mid 1 \leq \deg_{K_l}(v) < l - (m - 2)\} \text{ and}$$

$$S_B = S - S_L.$$

Let  $x, y \in S_L$  and suppose that  $x$  and  $y$  are not adjacent. Further, without loss of generality, suppose that  $\deg_{K_l}(x) \leq \deg_{K_l}(y)$ . If the neighborhood  $N_{K_l}(x) \not\subseteq N_{K_l}(y)$ , then there exist vertices  $w_1 \in N_{K_l}(x) - N_{K_l}(y)$  and  $w_2 \in N_{K_l}(y) - N_{K_l}(x)$  such that  $w_1, x, w_2, y$  is an induced  $P_4$ , a contradiction. But now,  $x$  and  $y$  must have at least one common neighbor in  $K_l$  and a  $J_2(m)$  results. Hence, the induced graph on  $S_L, \langle S_L \rangle$ , must be complete.

Now in  $\langle S_B \rangle$  we select a longest path  $P_1$ . If  $P_1$  is not all of  $S_B$ , we select a longest path in  $\langle S_B - V(P_1) \rangle$ , and continue this process until all of  $S_B$  is covered by these paths. It is easy to see there are at most  $s - 1$  such paths, for otherwise, due to the degree condition on  $S_B$ , there would be a vertex of  $K_l$  common to the neighborhoods of all the final vertices of these paths and  $K_{1,s}$  would result.

Now for each path  $P_i, i = 1, \dots, t$  ( $t < s$ ) created above and for some spanning path of  $\langle S_L \rangle$ , we match the  $2(t + 1)$  end vertices of these paths to  $2(t + 1)$  distinct vertices of  $K_l$ . Note that in the special case that  $V(\langle S_L \rangle)$  has only one neighbor in  $K_l$ , the fact  $G$  is 2-connected implies  $V(\langle S_L \rangle)$  has a neighbor in  $S_B$ . Include that neighbor in  $S_L$  and proceed as above. Hence,  $G$  is clearly hamiltonian, completing the proof of the Theorem. ■

**Theorem 2.2.** *If  $G$  is a 2-connected  $K_{1,s}P_4B_2$ -free graph ( $s \geq 4$ ) of sufficiently large order  $n$ , then  $G$  is hamiltonian.*

**Proof.** From Theorem 3 in [8],  $G$  being 2-connected  $P_4B_2K_{2, \lceil \frac{n+1}{2} \rceil}$ -free implies  $G$  is hamiltonian and  $K_{1,s} \leq K_{2, \lceil \frac{n+1}{2} \rceil}$ , if  $s \leq \lceil \frac{n+1}{2} \rceil$ , and so the result follows. ■

**Theorem 2.3.** *If  $G$  is a 2-connected  $K_{1,s}P_rZ_1(m)$ -free graph (with  $r \geq 5, s \geq 4, m \geq 3$  fixed) of sufficiently large order  $n$ , then  $G$  is hamiltonian.*

**Proof.** As before,  $G$  contains a vertex of degree at least  $n^{\frac{1}{r}}$  or  $P_r$  would be an induced subgraph of  $G$ . By Ramsey's Theorem, since  $K_{1,s} \not\leq G$ , we see  $G$  contains  $K_{l'}$  for  $l' > sm$  and  $l' = l'(n)$ . Choose a largest clique  $K_l$  in  $G$ .

Since  $G$  is 2-connected, there exists  $x \in V(G) - V(K_l)$  with  $x$  adjacent to vertices of  $K_l$ . Note that  $x$  must be nonadjacent to at most  $m - 2$  vertices of  $K_l$ , for otherwise a  $Z_1(m)$  results.

If there exists a vertex  $y$  at distance 2 from  $K_l$  through  $x$ , since  $l > sm$ , then an  $m$ -clique including  $x$  along with  $y$  forms a  $Z_1(m)$ , again a contradiction. Thus, every vertex of  $S = V - V(K_l)$  must have adjacencies in  $K_l$ . Further,  $S_L$  (defined as before) is empty, hence  $S_B = S$ .

As before, choose a system of longest paths  $P_i, i = 1, \dots, t$ , that covers  $S$ . If  $t \geq s$ , since  $l > s(m-2)$  we would find  $K_{1,s}$  in  $G$ , a contradiction.

Thus, since the end vertices of these  $t < s$  paths all have high degree ( $\geq l - (m-2)$ ) to  $K_l$  and  $l > s(m-2)$ , we can match the end vertices of each of these paths to  $2t$  distinct vertices of  $K_l$  and thus,  $G$  is clearly hamiltonian. ■

Note, Theorem 2.3 also holds when  $r = 4$ , however this triple follows from Theorem 2.1.

**Theorem 2.4.** *If  $G$  is a 2-connected  $K_{1,s}C(l, 1, 1)Z_1$ -free ( $l, s$  fixed,  $l \geq 2$ ,  $s \geq 4$ ) graph of sufficiently large order  $n$ , then  $G$  is hamiltonian.*

*Proof.* Suppose  $G$  is not hamiltonian. Then, from our previous result, we know that  $G$  contains a long induced path. Choose  $P = P_r$  with  $r > ls$  to be a longest induced path in  $G$ . Since  $V(P) \neq V(G)$  and  $G$  is 2-connected, there exists a vertex  $x \notin V(P)$  adjacent to a vertex on  $P$ . Say  $x$  is adjacent to  $v$  (where  $v$  is not an end vertex of  $P$ ). If  $x$  is also adjacent to  $v^+$ , then since  $P$  is an induced path, we see that  $Z_1$  results unless  $x$  is adjacent to the entire path. But if  $x$  is adjacent to all of  $P$ , since  $r > ls$ , a  $K_{1,s}$  would result.

Now we note that if  $x$  has no adjacencies within  $l$  vertices of  $v$  (on either side), then  $C(l, 1, 1)$  results. Hence,  $x$  must have an adjacency within every  $l$  vertices of any other adjacency on  $P$ . But  $r > ls$ , so again  $K_{1,s} \leq G$ . The only remaining possibility is that  $x$  must be adjacent to both end vertices of  $P$ .

Now suppose  $y$  is at distance 2 from  $P$  through  $x$ . Then we immediately find  $C(l, 1, 1) \leq G$ . Hence, all vertices of  $V(G) - V(P)$  are at distance one from  $P$  and therefore are adjacent to only the end vertices of  $P$ .

Suppose  $x$  and  $y$  are two vertices at distance one from  $P$ . If  $xy \notin E(G)$ , then  $C(l, 1, 1)$  is found using either end vertex, say  $w$ , of  $P$  along with  $x, y$  and an  $l$  vertex segment of  $P$  following  $w$ . Thus,  $xy \in E(G)$  and now  $\langle x, y, w, w^+ \rangle \cong Z_1$ , a contradiction. ■

In order to complete the characterization of triples containing  $K_{1,s}$  with  $s \geq 4$ , we need the families of graphs in Figure 3. For convenience, the graph  $H_2 = F_1$  (see Figure 4).

We now show that the triples shown to imply hamiltonicity in Theorems 2.1 - 2.4 form a complete list.

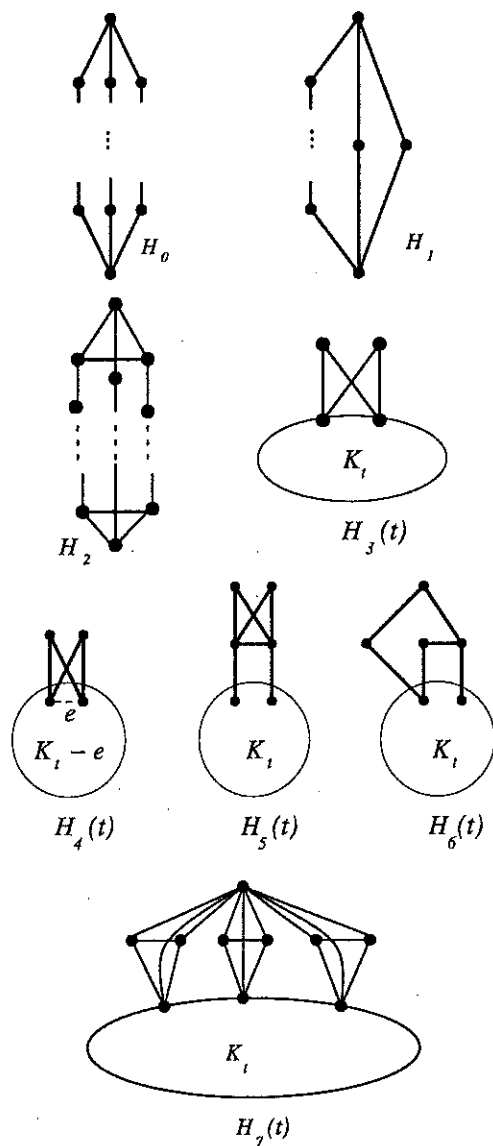


Figure 3. More nonhamiltonian graphs.

**Theorem 2.5.** *If  $G$  is a 2-connected graph of sufficiently large order which is  $G_1G_2G_3$ -free where  $G_1G_2G_3$  are one of the following triples:*

- (a)  $K_{1,s}, P_4, J_2(m)$ ;  $s \geq 4, m \geq 3$ ,
- (b)  $K_{1,s}, P_4, B_2$ ;  $s \geq 4$ ,

- (c)  $K_{1,s}, P_r, Z_1(m)$ ;  $r \geq 5, s \geq 4, m \geq 3$ ,  
 (d)  $K_{1,s}, C(l, 1, 1), Z_1(3) = Z_1$ ;  $l \geq 2, s \geq 4$

or  $G_1G_2G_3$  is a triple of induced subgraphs of one of these triples, then  $G$  is hamiltonian. Furthermore, these are the only possible triples that contain  $K_{1,s}, s \geq 4$ .

**Proof.** We know each of these triples implies hamiltonicity by Theorems 2.1 – 2.4. Thus, we need only show there are no other possibilities.

Since the graphs  $H_0-H_7$  of Figure 3 are all  $K_{1,s}$ -free ( $s \geq 4$ ) non-hamiltonian, we may assume without loss of generality  $G_2 \leq H_0$ . Thus,  $P_4 \leq G_2 \leq C(i, j, k)$ . Further, since  $P_4 \not\leq H_3$  and  $P_4 \not\leq H_4$ , we see that  $G_3 \leq H_3$  and  $G_3 \leq H_4$ . This implies that  $K_r \leq G_3 \leq J_2(m)$ , for  $r \geq 3$  and some  $m \geq 3$ , or else  $G_3 \leq B_2$ .

Since in either case  $K_3 \leq G_3$  and  $G_3 \not\leq H_1$  then  $G_2 \leq H_1$ . Hence, as  $G_2 \leq H_0$ , we see that  $G_2 \leq C(l, 1, 1)$ , for some  $l \geq 2$ . Thus, either  $G_2$  is a path  $P_k, k \geq 4$ , or  $G_2 = C(l, 1, 1)$ , that is  $P_k \leq G_2 \leq C(l, 1, 1)$ .

*Case 1.* Suppose  $G_2 = P_r, r \geq 6$ .

Since  $P_6 \not\leq H_4, P_6 \not\leq H_5$  and  $P_6 \not\leq H_6$ , then  $G_3 \leq H_4, G_3 \leq H_5$  and  $G_3 \leq H_6$ . But then,  $G_3 \leq Z_1(m)$  for some  $m \geq 3$ . This yields triple (c), when  $r \geq 6$ .

*Case 2.* Suppose  $G_2 = P_5$ .

Note  $H_5$  is  $K_{1,s}P_5J_2(m)$ -free, where  $s \geq 4$ . Thus, the triple  $K_{1,s}, P_5, J_2(m)$  is excluded from consideration. Next consider  $H_7$ , which is  $K_{1,4}P_5B_2$ -free, excluding this triple from consideration. Now consider  $H_4, H_5$  which are  $K_{1,4}, P_5$ -free. This implies  $G_3$  is a subgraph of both  $H_4$  and  $H_5$ , hence  $G_3 \leq Z_1(m), m \geq 3$ . This completes case (c).

*Case 3.* Suppose  $G_2 = P_4$ .

Since  $H_3$  and  $H_4$  are  $K_{1,s}P_4$ -free, we see that  $G_3 \leq H_3$  and  $G_3 \leq H_4$ . Thus,  $G_3 \leq J_2(m)$  for some  $m \geq 3$  or  $G_3 \leq B_2$ . Hence, we obtain the triples of (a) and (b).

*Case 4.* Suppose  $G_2 = C(l, 1, 1), l \geq 2$ .

Now  $G_2 \not\leq H_2, G_2 \not\leq H_3$  and  $G_3 \not\leq H_4$  thus,  $G_3 \leq H_2, G_3 \leq H_3$  and  $G_3 \leq H_4$ . Hence, using  $H_2$ , we see that  $K_3 \leq G_3$  and thus,  $\omega(G_3) = 3$ . But then, using  $H_2$  and  $H_3$  or  $H_4$ , we see that  $G_3 \leq Z_1$ , and we obtain family (d). ■



3. DETERMINING FAMILIES OF TRIPLES INCLUDING  $K_{1,3}$

In this section the graphs of Figures 4, 5 and 6 represent families of  $K_{1,3}$ -free nonhamiltonian graphs. Note that  $F_1 = H_2$ . For  $i = 2, 3, 5, 6, 7, 8, 9$  we denote by  $F_i(t)$  the graph from the family  $F_i$  for fixed  $t$ , ( $t \geq 3$  for  $i = 2, 3$  and  $t \geq 1$  for  $i = 5, 6, \dots, 9$  respectively). Note that in  $F_i(t)$ ,  $i = 5, \dots, 9$ , the vertices at distance one from the  $K_t$  are in fact adjacent to all vertices of the  $K_t$ .

Let  $\mathcal{A}$  be the collection of triples  $G_1G_2G_3$  with  $G_1 = K_{1,3}$  so that 2-connected  $G_1G_2G_3$ -free graphs of sufficiently large order are hamiltonian. We use the families of graphs of Figures 4, 5 and 6 to arrive at a restricted class of triples which contains  $\mathcal{A}$ . Due to the size of this class, we continue the study of these triples in [7]. Note that the case that no  $G_i$ ,  $i = 1, 2, 3$ , is equal to a star was characterized in [8].

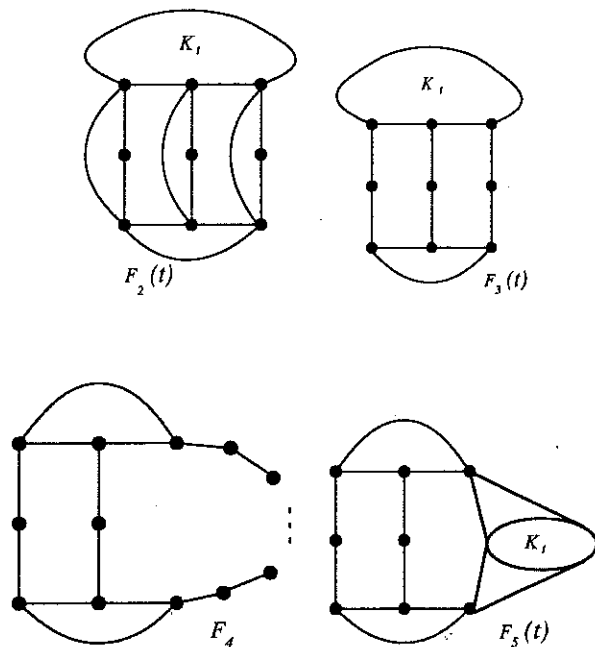


Figure 4. Forbidden families  $F_1$  through  $F_5$ .

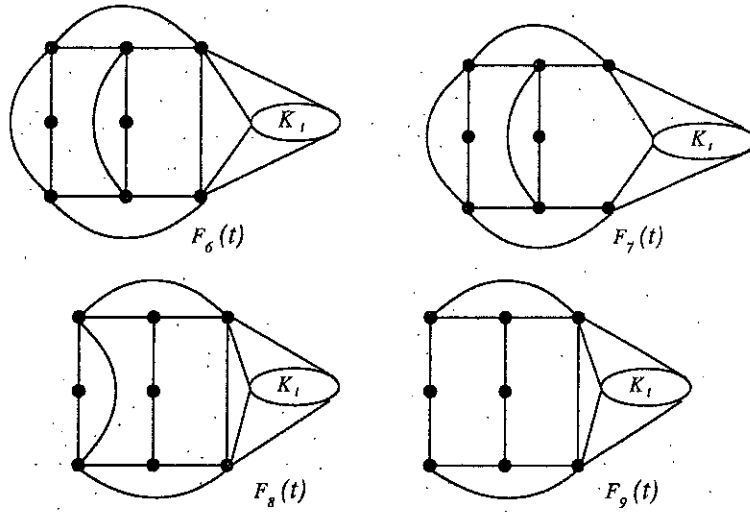


Figure 5. Forbidden families  $F_6$  to  $F_9$ .

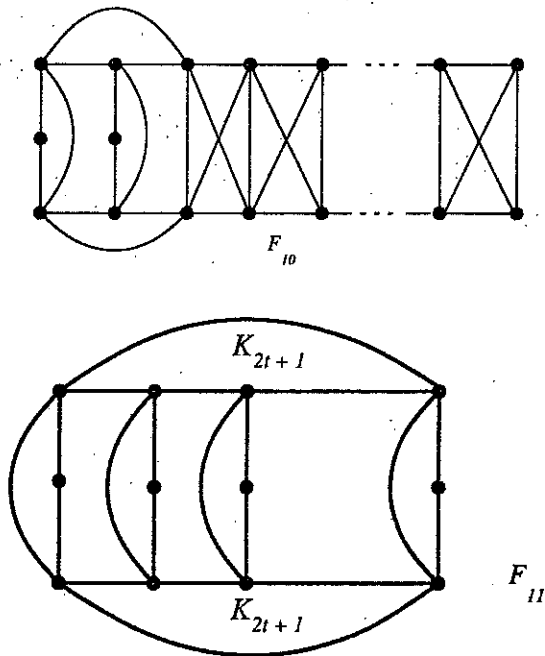


Figure 6. Forbidden families  $F_{10}$  and  $F_{11}$ .

Without loss of generality, we may assume  $G_2 \leq F_1$ . This implies  $G_2 \leq N(i, j, k), i \geq j \geq k \geq 0$ , where possibly  $G_2 = P_l, l \geq 4$ . If  $l \leq 6$ , then  $K_{1,3}P_l$  implies  $G$  is hamiltonian. Now, based on the different structures of  $G_2$ , we determine the possibilities for  $G_3$ . First we present three Lemmas which will help expedite the cases. Throughout this section we consider only 2-connected  $G_1G_2G_3$ -free graphs  $G$ .

**Lemma 3.1.** *If  $G_3$  is an induced subgraph of all of the graphs in  $\{F_2, F_3, F_6\}$  then either*

- (a)  $G_3 \leq G_1$  where  $G$  is  $K_{1,3}G_1$ -free implies  $G$  is hamiltonian or
- (b) the clique number  $\omega(G_3) \geq 4$ .

**Proof.** If  $\omega(G_3) \leq 2$ , then by the cycle structure of  $F_2$  and  $F_3$ ,  $G_3$  must be a path. Since there are no induced  $K_{1,3}$  and  $F_2$  contains no induced  $P_7$ , it follows that  $G_3 \leq P_6$ . But  $K_{1,3}P_6$ -free graphs are hamiltonian by Theorem 1.1.

If  $\omega(G_3) = 3$ , then  $G_3$  contains at most one  $K_3$ , since the distance between two distinct  $K_3$  in  $F_2$  is at most one and it is more than one in  $F_3$ . Also note that there are no cycles other than  $K_3$  in  $G_3$ , since  $F_2$  has only 4-cycles as other induced cycles, while  $F_3$  has only 6-cycles as other induced cycles. Thus,  $G_3 \leq N(i, j, k)$  where  $i, j, k \geq 0$ .

If  $i, j, k > 0$ , then  $G_3 \leq N(2, 1, 1)$  by  $F_2$  or  $F_3$  and by  $F_6$  it follows that  $G_3 \leq N(1, 1, 1)$ , hence we are again done by Theorem 1.1. If  $k = 0$  and  $i, j > 0$ , then by  $F_3, j = 1$  and by  $F_6, i \leq 2$ . Thus,  $G_3 \leq N(2, 1, 0)$  and we are done by Theorem 1.1. If  $j = k = 0$  and  $i > 0$ , then  $F_2$  implies that  $i \leq 3$  and so  $G_3 \leq N(3, 0, 0)$  and we are again done by Theorem 1.1. Thus, either  $\omega(G_3) \geq 4$  or we have a pair of graphs implying  $G$  is hamiltonian. ■

**Lemma 3.2.** *If  $G$  is a 2-connected non-hamiltonian  $K_{1,3}G_3$ -free graph of sufficiently large order  $n$  and  $G_3$  is an induced subgraph of each of the graphs of  $\{F_2, F_3, F_5, F_6\}$  or  $\{F_2, F_3, F_6, F_7\}$ , then  $G_3 \leq Z_3(m), m \geq 4$ .*

**Proof.** By Lemma 3.1,  $\omega(G_3) \geq 4$ . Since  $G_3$  is an induced subgraph of  $F_5$  and  $F_6$  (or  $F_6$  and  $F_7$ ) containing a  $K_4$ , it follows that  $G_3 \leq Z_t(m)$ , with  $m \geq 4$  and  $G_3 \leq F_2$  implies that  $t \leq 3$ . ■

**Lemma 3.3.** *If  $G$  is a 2-connected non-hamiltonian  $K_{1,3}G_3$ -free graph of sufficiently large order  $n$  and  $G_3$  is an induced subgraph of each of the graphs in  $\{F_2, F_3, F_5, F_6, F_{10}\}$ , then  $G_3 \leq Z_2(4)$ .*

*Proof.* By Lemma 3.1,  $\omega(G_3) \geq 4$ , and since  $G_3 \leq F_{10}$ , we see that  $\omega(G_3) \leq 4$ , so  $\omega(G_3) = 4$ . Lemma 3.2 now implies that  $G_3 \leq Z_3(4)$  and by considering  $F_{10}$  it follows that  $G_3 \leq Z_2(4)$ . ■

For Propositions 3.1 – 3.7 of this Section, we assume that  $G_2 = N(i, j, k)$  for certain values of  $i \geq j \geq k$  and  $G_1 = K_{1,3}$ .

**Proposition 3.1.** *If  $k \geq 2$ , then  $K_{1,3}G_3$  implies  $G$  is hamiltonian.*

*Proof.* If  $G$  is  $K_{1,3}$ -free and non-hamiltonian and  $k \geq 2$ , then we have that  $G_2 \geq N(2, 2, 2)$ . Since  $F_2 - F_{10}$  are all  $K_{1,3}N(2, 2, 2)$ -free,  $G_3$  must be an induced subgraph of each of them. But then  $F_4$  implies  $\omega(G_3) \leq 3$ . Now by Lemma 3.1 we are done. ■

Thus, we next need to consider the cases where  $k = 1$  or  $k = 0$ .

**Proposition 3.2.** *Suppose  $k = 1$  and  $j > 1$ . Then,*

- (a)  $G_3 \leq Z_2(4)$  when  $j \geq 3$  and
- (b)  $G_3 \leq Z_3(m)$ , with  $m \geq 4$ , when  $j = 2$ .

*Proof.* Since each of  $F_2, F_3, F_5, F_6$  are  $K_{1,3}N(i, 3, 1)G_3$ -free, if  $j \geq 3$ , apply Lemma 3.3 and if  $j = 2$ , apply Lemma 3.2. ■

The graph  $H_2(l_1, l_2, l_3)$  ( $l_i \geq 3$  for  $i = 1, 2, 3$ ) is two copies of  $K_3$  with corresponding vertices joined by  $P_i$ 's whose endvertices are identified with the corresponding vertices of the two copies of  $K_3$ . Note that this graph is just one particular member of the family  $H_2 = F_1$ .

**Proposition 3.3.** *Suppose  $k = j = 1$ , then*

- (a) *If  $i \geq 4$ , then  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ .*
- (b) *If  $i = 3$ , then  $G_3 \leq Z_3(m)$ ,  $m \geq 4$  or  $G_3 \leq N(K_m, K_3, P_1)$ ,  $m \geq 4$  or  $G_3 \leq N(K_3, P_2, P_2)$ .*
- (c) *If  $i = 2$ , then  $G_3 \leq F_6(m)$ .*
- (d) *If  $i = 1$ , then  $G_2 = N(1, 1, 1)$  and  $K_{1,3}N(1, 1, 1)$ -free implies hamiltonicity.*

*Proof.* Suppose  $i \geq 4$ . Since  $F_2, F_3, F_5$  and  $F_6$  are all  $K_{1,3}N(4, 1, 1)$ -free, by Lemma 3.1,  $\omega(G_3) \geq 4$ , and then Lemma 3.2 implies  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ .

If  $i = 3$ , we note that  $F_2, F_5, F_6$  and  $F_7$  are all  $K_{1,3}N(3, 1, 1)$ -free. Suppose  $\omega(G_3) = 3$  and  $G_3$  contains more than one  $K_3$ . Then  $F_2$  and  $F_5$  imply  $G_3$  contains only two  $K_3$  and these two  $K_3$  share a vertex. Thus,  $G_3 \leq N(K_3, 1, 1)$ .

Suppose  $\omega(G_3) \geq 4$ . By considering  $F_6$  and  $F_7$  we see that at most one vertex, say  $w$ , of the large clique may have adjacencies outside the clique. If  $w$  has one adjacency outside the clique, then  $F_2$  and  $F_5$  imply  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ . If  $w$  has more than one adjacency outside the clique, then  $F_2$  implies the degree outside the clique is exactly two and those two vertices must be adjacent. The family  $F_2$  implies there can be only one of these two with additional adjacencies. Then  $F_5$  and  $F_7$  imply the extension beyond these two vertices can be at most one edge from one vertex, hence  $G_3 \leq N(K_m, 1, 0)$ ,  $m \geq 4$ .

If  $i = 2$ , since  $F_6$  is  $K_{1,3}N(2, 1, 1)$ -free, we conclude that  $G_3 \leq F_6(m)$ . If  $i = 1$ , apply Theorem 1.1. ■

**Proposition 3.4.** *Suppose  $k = 0$  and  $j \geq 3$ , then  $G_3 \leq Z_2(4)$ .*

*Proof.* If  $j \geq 3$ , the families of graphs  $F_2, F_3, F_5, F_6$  and  $F_{10}$  are all  $K_{1,3}N(3, 3, 0)$ -free, so by Lemma 3.1,  $\omega(G_3) \geq 4$  and using family  $F_{10}$  and Lemma 3.3, it follows that  $\omega(G_3) = 4$ , and thus,  $G_3 \leq Z_2(4)$ . ■

**Proposition 3.5.** *Suppose  $k = 0$  and  $j = 2$ , then*

- (a) *If  $i \geq 3$ , then  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ .*
- (b) *If  $i = 2$ , then  $G_3 \leq P_7$  or*

$$G_3 = C_6 \text{ if } \omega(G_3) = 2, \text{ or}$$

$$G_3 \leq H_2(3, 3, 3) \text{ or}$$

$$G_3 \leq N(4, 0, 0), \text{ if } \omega(G_3) = 3 \text{ or}$$

$$G_3 \leq Z_4(m), \text{ with } m \geq 4 \text{ if } \omega(G_3) \geq 4.$$

*Proof.* (a) If  $j = 2$  and  $i \geq 3$ , again  $F_2, F_3, F_5, F_6$  and  $F_7$  are  $K_{1,3}N(3, 2, 0)$ -free, so by Lemma 3.1,  $\omega(G_3) \geq 4$  and by Lemma 3.2, we see that  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ .

(b) If  $j = 2$  and  $i = 2$ , then only families  $F_3$  and  $F_5$  are  $N(2, 2, 0)$ -free. First suppose that  $\omega(G_3) = 2$ . Then we see that  $G_3 \leq P_7$  or  $G_3 = C_6$ . Suppose  $\omega(G_3) = 3$ . Now if  $G_3$  contains two  $K_3$ , then from  $F_3$  we see they

are disjoint and we get that  $G_3 \leq H_2(3, 3, 3)$ . If  $G_3$  contains only one  $K_3$ , then  $F_3$  implies  $G_3 \leq N(4, 0, 0)$  or  $G_3 \leq N(3, 1, 0)$ , or  $G_3 \leq N(2, 1, 1)$ . But then note that  $N(3, 1, 0)$  and  $N(2, 1, 1)$  are subgraphs of  $H_2(3, 3, 3)$ . Finally, if  $\omega(G_3) \geq 4$ , then  $F_3$  imply  $G_3 \leq Z_4(m)$ . ■

**Proposition 3.6.** *Suppose  $k = 0$  and  $j = 1$ , then*

- (a) *If  $i \geq 4$ , then  $G_3 \leq P_6$  if  $\omega(G_3) = 2$  or  
 $G_3 \leq Z_3(m)$  if  $\omega(G_3) \geq 3$ .*
- (b) *If  $i = 3$ , then  $G_3 \leq P_6$  if  $\omega(G_3) = 2$ , or  
 $G_3 \leq N(K_m, K_3, P_2)$  or  
 $G_3 \leq N(Z_1(m), P_3, P_1)$  if  $\omega(G_3) \geq 3$ .*
- (c) *If  $1 \leq i \leq 2$ , then  $G_2 \leq N(2, 1, 0)$ , and  $K_{1,3}N(2, 1, 0)$ -free implies hamiltonicity.*

**Proof.**

- (a) If  $j = 1$  and  $i \geq 4$ , families  $F_6, F_7, F_8, F_9$  and  $F_{11}$  are  $K_{1,3}N(4, 1, 0)$ -free and so  $G_3 \leq P_6$  if  $\omega(G_3) = 2$ . If  $\omega(G_3) \geq 3$ , by examining the largest common subgraphs of  $F_6, F_7, F_8, F_9$ , and  $F_{11}$ , we see that  $G_3 \leq Z_3(m)$ .
- (b) If  $j = 1$  and  $i = 3$ , families  $F_6, F_7, F_8$  are  $K_{1,3}N(3, 1, 0)$ -free and so  $G_3 \leq P_6$  if  $\omega(G_3) = 2$ . By examining the largest common subgraphs of  $F_6, F_7, F_8$ , the other graphs are immediate.
- (c) If  $j = 1$  and  $i = 2$ , we note that all  $K_{1,3}N(2, 1, 0)$ -free graphs are hamiltonian by Theorem 1.1. ■

**Proposition 3.7.** *Suppose  $k = 0$  and  $j = 0$ , then*

- (a) *If  $i \geq 5$ , then  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ .*
- (b) *If  $i = 4$ , then  $G_3 \leq F_2(3)$ .*
- (c) *If  $0 \leq i \leq 3$ , then  $G_2 \leq Z_3$  and  $K_{1,3}G_2$  is sufficient to imply hamiltonicity.*

**Proof.**

- (a) If  $j = 0$  and  $i \geq 5$ , then  $F_2, F_3, F_6$  and  $F_7$  are all  $K_{1,3}N(5, 0, 0)$ -free and so by Lemma 3.2,  $G_3 \leq Z_3(m)$ ,  $m \geq 4$ .

- (b) If  $j = 0$  and  $i = 4$ ,  $G_3 \leq F_2$ , as  $F_2$  and  $F_{11}$  are the only  $K_{1,3}N(4, 0, 0)$ -free families.
- (c) If  $j = 0$  and  $i = 3$ , then all  $K_{1,3}N(3, 0, 0)$ -free graphs of order  $n \geq 10$  are hamiltonian by Theorem 1.1.

All other cases for  $i$  lead directly to  $G_2$  being one of the graphs of Theorem 1.1 and hence, no new triples result. ■

We next consider the situation when  $G_2 = P_l$ , for  $l \geq 7$ .

**Theorem 3.1.** *Suppose  $G_2 = P_l$ ,  $l \geq 7$ .*

- (a) *If  $l = 7$ , then  $G_3 \leq F_2(3)$  or*  

$$G_3 \leq N(K_m, K_3, P_1) \text{ or}$$

$$G_3 \leq N(Z_1, 1, 0).$$
- (b) *If  $l \geq 8$ , then  $G_3 \leq Z_3(m)$ , where  $m \geq 4$ .*

**Proof.** If  $l = 7$ , an argument similar to earlier ones involving the number of copies of  $K_3$  in  $G_3$  produces the result. If  $l \geq 8$ , then since  $F_2, F_3, F_5, F_6$  and  $F_7$  must contain  $G_3$ , applying Lemma 3.2 we obtain the result. ■

We end this section by summarizing the potential triples determined in this section.

In 2-connected Claw, $N(i, j, k)$ , $G_3$ -Free with $i \geq j \geq k \geq 1$	
$i, j, k$	Possible Maximal Third Graph(s) $G_3$
$k \geq 2$	No new triples
$k = 1, j \geq 3$	$Z_2(4)$
$k = 1, j = 2$	$Z_3(m), m \geq 4$
$k = j = 1, i \geq 4$	$Z_3(m), m \geq 4$
$k = j = 1, i = 3$	$Z_3(m), N(K_m, P_2, P_1), m \geq 4, N(K_3, P_2, P_2)$
$k = j = 1, i = 2$	$F_2(m), m \geq 4$
$k = j = 1, i = 1$	No new triples

In 2-connected Claw, $N(i, j, 0)$ , $G_3$ -Free with $i \geq j$	
$i, j, 0$	Possible Maximal Third Graph(s) $G_3$
$j \geq 3$	$Z_2(4)$
$j = 2, i \geq 3$	$Z_3(m), m \geq 4$
$j = 2, i = 2$	if $\omega(G_3) = 2$ : $P_7, C_6$
$j = 2, i = 2$	if $\omega(G_3) = 3$ : $H_2(3, 3, 3), N(4, 0, 0)$
$j = 2, i = 2$	if $\omega(G_3) \geq 4$ : $Z_4(m), m \geq 4$
$j = 1, i \geq 4$	if $\omega(G_3) = 2$ : No new triples
$j = 1, i \geq 4$	if $\omega(G_3) \geq 3$ : $N(Z_1(m), P_2, P_1)$
$j = 1, i = 3$	if $\omega(G_3) = 2$ : no new triples
$j = 1, i = 3$	if $\omega(G_3) \geq 3$ : $N(K_m, K_3, P_2), N(Z_1(m), P_3, P_1)$
$j = 1, 1 \leq i \leq 2$	No new triples
$j = 0, i \geq 5$	$Z_3(m), m \geq 4$
$j = 0, i = 4$	$F_2(3)$
$j = 0, 0 \leq i \leq 3$	No new triples

In 2-connected Claw, $P_t, G_3$ -Free	
$t$	Possible Third Graph(s) $G_3$
$t \geq 8$	$Z_3(m), m \geq 4$
$t = 7$	$F_2(3), N(K_m, K_3, P_1), N(Z_1, P_2, P_1)$
$t \leq 6$	No new triples

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