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Forbidden Subgraphs and the Hamiltonian Theme

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ABSTRACT

Let F be the unique graph with degree sequence 1, 1, 1, 3, 3, 3. We show that every connected graph G that contains no induced subgraph isomorphic to $K_{1,3}$ or F is traceable. Moreover, if G is 2-connected then G is hamiltonian.

1. Introduction.

In this article we consider finite simple graphs without loops or multiple edges. A graph is *connected* if each pair of vertices is joined by a path, while a graph is *n-connected* if the removal of fewer than n vertices results in a connected graph. The *distance* d(x,y) between vertices x and y of a connected graph G is the least number of edges in an x-y path. If S is a set of vertices, the *distance from the vertex* x *to* S is $d(x,S) = \min\{d(x,s) \mid s \in S\}$. The *diameter*, diam G, of a connected graph G is the maximum distance between two vertices of G. If S is a subset of the vertex set V(G) of a graph G,

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then the *subgraph induced by* S is denoted by $\langle S \rangle$. The *neighborhood,* N(x), of a vertex x is the set of all vertices adjacent to x. A graph G is *locally connected* if $\langle N(x) \rangle$ is connected for each $x \in V(G)$. The graph G is *traceable (hamiltonian)* if it contains a path (cycle) through all its vertices. Such a path (cycle) is called a hamiltonian path (cycle).

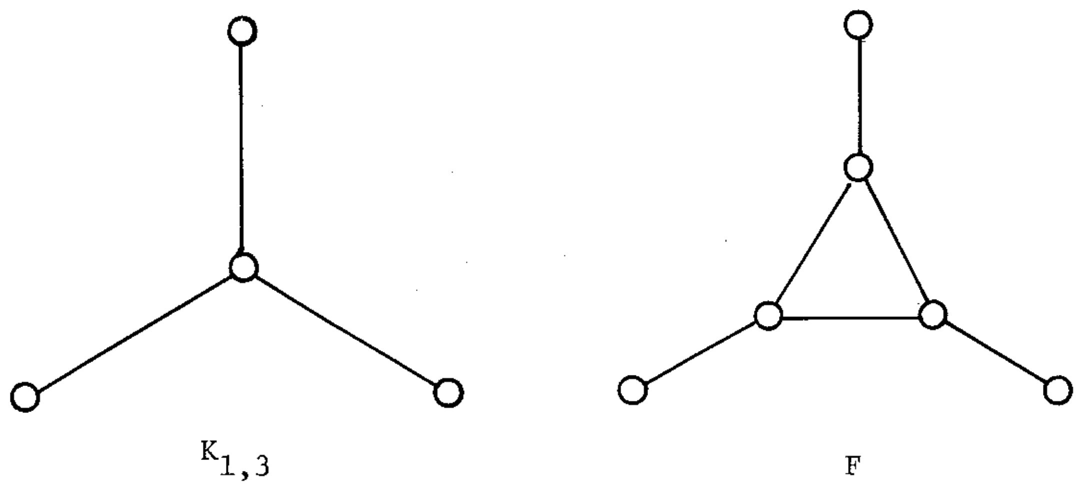


Figure 1.

Let c(G) denote the number of components of the graph G. A graph is *1-tough* if $c(G - S) \leq |S|$ for every nonempty proper subset S of V(G). The *complement* of the graph G, denoted $\overline{G}$, is the graph with vertex set V(G) and e is an edge of $\overline{G}$ if and only if e is not an edge of G.

The literature abounds with results concerning traceable and hamiltonian graphs. Recent studies have related the idea of forbidden subgraphs with other properties to obtain sufficient conditions for a graph to be hamiltonian. The object of this paper is to investigate the graphs $K_{1,3}$ and F (cf. Figure 1) and their relation to the hamiltonian theme.

Theorem A. (Oberly and Sumner [4]). A connected, locally connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ is hamiltonian.

Theorem B. ([2]). A 2-connected graph with diameter at most 2 that contains no induced subgraph isomorphic to $K_{1,3}$ is hamiltonian.

Theorem C. (Jung [3], cf. Bermond [1]). If G is 1-tough then either G is hamiltonian, or its complement $\overline{G}$ contains the graph $\overline{F}$ as a subgraph.

In terms of forbidden induced subgraphs Theorem C has the following natural Corollary.

Corollary D. If G is 2-connected and contains no hamiltonian cycle, then G has an induced subgraph isomorphic to $K_{1,3}$ or to a spanning subgraph of F.

In this paper we prove the following theorems.

Theorem 1. A connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or F is traceable.

Theorem 2. A 2-connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or F is hamiltonian.

Before beginning the proof of Theorem 1, a few observations will be helpful.

2. Observations.

Throughout the next two sections G denotes a connected graph of diameter d with no induced subgraphs isomorphic to $K_{1,3}$ or F. It was shown in [2] that if $d \leq 2$ then G is traceable; hence, we assume that $d \geq 3$. Let $P : v_0, v_1, \ldots, v_{d-1}, v_d$ be a path of length $d = d(v_0, v_d)$. Define the following subsets of V(G):

$$U_i = \{x \notin V(P) \mid xv_i, xv_{i+1} \in E(G) \text{ and } xv_{i-1}, xv_{i+2} \notin E(G)\}$$

for $0 \leq i \leq d-1$, and

$$V_j = \{x \notin V(P) \mid xv_j, xv_{j+1}, xv_{j+2} \in E(G) \text{ and } xv_{j-1}, xv_{j+3} \notin E(G)\}$$
for $0 \leq j \leq d-2$.

Also define

$$A_0 = \{x \notin V(P) \mid xv_0 \in E(G) \text{ and } xv_1 \notin E(G)\},$$
$$A_d = \{x \notin V(P) \mid xv_d \in E(G) \text{ and } xv_{d-1} \notin E(G)\}.$$

If $u, w \in U_i (0 \leq i \leq d-2)$ and uw is not an edge of G then $\langle\{u,w,v_{i+1},v_{i+2}\}\rangle \cong K_{1,3}$. Thus $\langle U_i\rangle (0 \leq i \leq d-2)$ is complete. If $i = d-1$ then $\langle\{u,w,v_{d-1},v_{d-2}\}\rangle \cong K_{1,3}$ unless uw is an edge of G. Thus $\langle U_{d-1}\rangle$ is complete as well. A similar argument shows $\langle V_j\rangle$ is complete $(0 \leq j \leq d-2)$.

If $a \in U_i$ and $b \in V_i (1 \leq i \leq d-2)$ and $ab \notin E(G)$ then $\langle\{a,b,v_i,v_{i-1}\}\rangle \cong K_{1,3}$; thus $\langle U_i \cup V_i\rangle$ is complete $(1 \leq i \leq d-2)$. Similarly $\langle U_{i+1} \cup V_i\rangle$ is complete $(0 \leq i \leq d-3)$. We have shown:

(A). <u>The graphs</u> $\langle U_i\rangle (0 \leq i \leq d-1)$, $\langle V_j\rangle (0 \leq j \leq d-2)$, $\langle U_i \cup V_i\rangle$ $(1 \leq i \leq d-2)$, <u>and</u> $\langle U_{i+1} \cup V_i\rangle (0 \leq i \leq d-3)$ <u>are complete</u>.

Suppose $x \in V(G)$ and x is adjacent to some $v_i \in V(P)$ $(1 \leq i \leq d-1)$. Then $\langle\{x,v_i,v_{i-1},v_{i+1}\}\rangle \cong K_{1,3}$ unless one of the edges xv_{i-1}, xv_{i+1} or $v_{i-1}v_{i+1}$ is in G. If $v_{i-1}v_{i+1}$ is in G then $d(v_0,v_d) < d$, a contradiction. Hence, at least one of xv_{i-1} and xv_{i+1} is in G, that is, x is adjacent to at least two consecutive vertices of P. Since $d(v_0,v_d) = d$, no vertex is adjacent to four vertices of P and by (A) all adjacencies to P must be consecutive. Thus the sets $U_i (0 \leq i \leq d-1)$ and $V_j (0 \leq j \leq d-2)$ are all distinct. Thus:

(B). <u>Any vertex of</u> G <u>adjacent to a vertex of</u> P <u>lies in exactly one of</u> $U_i (0 \leq i \leq d-1)$, $V_j (0 \leq j \leq d-2)$ <u>or</u> $A_k (k = 0,d)$.

Now suppose the vertex x is not adjacent to a vertex of P, but is adjacent to y where $yv_i (2 \leq i \leq d-2)$ is an edge of G. By (B), the vertex y is adjacent to v_{i-1} or

v_{i+1}. Say $yv_{i-1} \in E(G)$ (a similar argument applies if $yv_{i+1} \in E(G)$). If yv_{i+1} is in G then $\langle\{x,y,v_{i+1},v_{i-1}\}\rangle \cong K_{1,3}$ unless at least one of xv_{i-1} or xv_{i+1} is in G. If yv_{i+1} is not in G, then $\langle\{x,y,v_{i-2},v_{i-1},v_i,v_{i+1}\}\rangle \cong F$ unless $xv_j \in E(G)$ for some $j \in \{i-2, i-1, i, i+1\}$. If $y \in V_0$, then $\langle\{x,y,v_0,v_2\}\rangle \cong K_{1,3}$ unless at least one of xv_0 and xv_2 is in G. In any case, x must be adjacent to a vertex of P, a contradiction. We have shown:

(C). <u>Every vertex that is adjacent to a vertex in</u> $U_i (1 \leq i \leq d-2)$ <u>or</u> $V_j (0 \leq j \leq d-2)$ <u>is contained in one of</u> $U_k (0 \leq k \leq d-1)$, $V_\ell (0 \leq \ell \leq d-2)$, A_0 <u>or</u> A_d.

Let $X = \{x \in V(G) \mid d(x,V(P)) > 1\}$. Clearly $V(G)$ is partitioned by X, A_0, A_d, $\bigcup_{i=0}^{d-1} U_i$, $\bigcup_{j=0}^{d-2} V_j$ and $V(P)$. The next two observations concern the structure of X.

Note by (C), if $x \in X$ then x is not adjacent to any vertex of $U_i (1 \leq i \leq d-2)$ or $V_j (0 \leq j \leq d-2)$. Let $d(x,V(P)) = 2$. Then there exists y in A_0, U_0, A_d or U_{d-1} such that $xy \in E(G)$. Since $y \in A_0 \cup U_0$ or $y \in A_d \cup U_{d-1}$, we may assume without loss of generality $y \in A_0 \cup U_0$. If $y \in U_0$ and $A_0 \neq \emptyset$ then for all $a \in A_0$, $xa \in E(G)$; otherwise $\langle\{v_0,v_1,v_2,x,y,a\}\rangle \cong F$ implying $ay \in E(G)$ and $\langle\{y,v_1,a,x\}\rangle \cong K_{1,3}$.

Suppose $X \neq \emptyset$.

Case 1. If $A_0 \neq \emptyset$ then define the following sets:

$$S_1 = \{x \in X \mid \text{ for some } y \in A_0, xy \in E(G)\},$$

$$S_i = \{x \in X \mid \text{ for some } y \in S_{i-1}, xy \in E(G)\} - \bigcup_{k=1}^{i-1} S_k \quad (1 < i).$$

Case 2. If $A_0 = \emptyset$ then define the following sets:

$$T_1 = \{x \in X \mid \text{ for some } y \in U_0, xy \in E(G)\},$$

$$T_i = \{x \in X \mid \text{ for some } y \in T_{i-1}, xy \in E(G)\} - \bigcup_{k=1}^{i-1} T_k \quad (1 < i).$$

(D). Let $X \neq \emptyset$. If $A_0 \neq \emptyset$ then $S_1 \neq \emptyset$ and if $A_0 = \emptyset$ then $T_1 \neq \emptyset$.

Choose any two vertices $x, y \in A_0$. Then $\langle\{v_0, v_1, x, y\}\rangle$ shows that xy is in G. Hence, $\langle A_0\rangle$ and, similarly, $\langle A_d\rangle$ are complete.

Let $x, y \in S_1$. By definition of S_1 there exist $x', y' \in A_0$ such that xx' and yy' are in G. If $x' = y'$ then $\langle\{v_0, x', x, y\}\rangle$ implies that x is adjacent to y. If $x' \neq y'$ then $xy \in E(G)$ for otherwise $\langle\{v_0, v_1, x', y', x, y\}\rangle \cong F$. We conclude that $\langle S_1\rangle$ is complete. Also, observe that if $A_0 = \emptyset$ then a similar argument shows that $\langle T_1\rangle$ is complete.

We shall show that $\langle S_i\rangle$ is complete for each i. Assume that $\langle S_j\rangle$ is complete for all $1 \leq j \leq k$, and let $x, y \in S_{k+1}$. By definition there exist $x', y' \in S_k$ such that $xx', yy' \in E(G)$. Also, there are $x'', y'' \in S_{k-1}$ such that $x'x'', y'y'' \in E(G)$ (if $k = 1$ then $x'', y'' \in A_0$). If $x' = y'$ then $\langle\{x'', x', x, y\}\rangle \cong K_{1,3}$ unless x is adjacent to y. If $x' \neq y'$ then, since $\langle S_k\rangle$ is complete, x' is adjacent to y'. Consider $\langle\{x'', x', y', x\}\rangle$; it follows that $x''y'$ or xy' is in G. In the former case, by our choice of x'' there is a vertex p such that $px'' \in E(G)$ and p is not adjacent to any of x', y', x, y. Now $\langle\{p, x'', x', y', x, y\}\rangle$ shows that x is adjacent to y or, as in the former case, xy' is in G. The graph $\langle\{x, y, y', y''\}\rangle$ implies that x is adjacent to y. Thus we have:

(E). Let $X \neq \emptyset$. If $A_0 \neq \emptyset$ then $\langle S_i\rangle$ is complete for all i. If $A_0 = \emptyset$ then $\langle T_i\rangle$ is complete for all i.

Let $x_1 \in S_1$. Since $d(x_1, v_d) \leq d$ and $d(v_0, v_d) = d$, it follows from (C) that there exists a vertex x_i in some S_i such that x_i is adjacent to z for some $z \in U_{d-1} \cup A_d$. Choose i maximum with this property. If $i \geq 3$, let $x_{i-1} \in S_{i-1}$ be adjacent to x_i, and $x_{i-2} \in S_{i-2}$ be adjacent to x_{i-1}. In the case that $i = 2$, let $x_{i-2} \in A_0$ and in the case that $i = 1$, let $x_{i-1} \in A_0$ and $x_{i-2} = v_0$.

Suppose that $S_{i+1} \neq \emptyset$, say $x_{i+1} \in S_{i+1}$. Since x_{i+1} is not adjacent to z, $\langle\{x_{i-1},x_i,x_{i+1},z\}\rangle$ implies z is adjacent to x_{i-1}. Consideration of $\langle\{x_{i-2},x_{i-1},x_i,x_{i+1},z,v_d\}\rangle$ yields an adjacency between x_{i-2} and z, since x_{i-2}, x_{i-1}, x_i, and x_{i+1} are not adjacent to v_d. Now a contradiction arises from $\langle\{x_{i-2},x_i,z,v_d\}\rangle$. Therefore, $S_j = \emptyset$ for all $j \geq i + 1$.

(F). Let i be the maximum such that $S_i \neq \emptyset$. Then there are $x \in S_i$ and $z \in A_d$, or $z \in U_{d-1}$ if $A_d = \emptyset$, such that x is adjacent to z. If $A_0 = \emptyset$ then the preceding statement holds with T_i replacing S_i.

Let $Y = \{y \in X | y \notin \cup S_k\}$. Suppose $Y \neq \emptyset$. Since G is connected there exist $y \in Y$ and $y' \in U_{d-1} \cup A_d$ such that $yy' \in E(G)$. Choose x_i and z as guaranteed by (F). By an argument similar to that establishing (D), z and y' are both in A_d, or both in U_{d-1} when $A_d = \emptyset$. If $z = y'$ then $\langle\{x_i,y,z,v_d\}\rangle$ implies that y is adjacent to x_i, which contradicts $y \in Y$. If $z \neq y'$ then $\langle\{x_i,y,y',z,v_d,v_{d-1}\}\rangle$ leads to a contradiction when $A_d \neq \emptyset$, while $\langle\{x_i,y,y',z,v_{d-2},v_{d-1}\}\rangle$ gives a contradiction if $A_d = \emptyset$.

(G). If $A_0 \neq \emptyset$ then $X = \cup S_k$. If $A_0 = \emptyset$ then $X = \cup T_k$.

3. Proof of Theorem 1.

For $C, D \subseteq V(G)$ write $C \sim D$ whenever there exist vertices $c \in C$ and $d \in D$ such that $cd \in E(G)$. Let $C_1, C_2, \ldots, C_n$ be a partition of $V(G)$ satisfying: $\langle C_i\rangle$ is complete and $C_i \sim C_{i+1}$. Then the sequence $C_1, C_2, \ldots, C_n$ is used to denote a hamiltonian path of G in which the vertices of C_i are traced consecutively and precede the vertices of C_{i+1} in the hamiltonian path. Also, if $C_i = \{v\}$ we write v in place of C_i.

Let i be chosen as in (F).

If $A_0 = \emptyset$ and $A_d \neq \emptyset$ then $U_0, v_0, V_0, U_1, v_1, \ldots, v_{d-2}, V_{d-2}, v_{d-1}, U_{d-1}, v_d, A_d, T_i, T_{i-1}, \ldots, T_1$ represents a hamiltonian path of G. If $A_0 = \emptyset$ and $A_d = \emptyset$ then $U_0, v_0, V_0, U_1, v_1, \ldots, v_{d-2}, V_{d-2}, v_{d-1}, v_d, U_{d-1}, T_i, T_{i-1}, \ldots, T_1$ is a hamiltonian path. Let $A_0 \neq \emptyset$ and $A_d \neq \emptyset$. If one of $V_{d-2} = \emptyset$, $U_{d-1} = \emptyset$, $V_{d-2} \sim U_{d-1}$ holds then $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, V_{d-2}, U_{d-1}, v_d, A_d, S_i, S_{i-1}, \ldots, S_1$ represents a hamiltonian path. If none of the three conditions hold then consider

(a). $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, V_{d-2}, v_d, U_{d-1}, A_d, S_i, S_{i-1}, \ldots, S_1$;

(b). $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, U_{d-1}, V_{d-2}, A_d, S_i, S_{i-1}, \ldots, S_1$.

If $V_{d-2} \sim A_d$ then (b) yields a hamiltonian path. If $U_{d-1} \sim A_d$ then (a) gives a hamiltonian path. An induced $K_{1,3}$ occurs if neither $V_{d-2} \sim A_d$ nor $U_{d-1} \sim A_d$. Finally, if $A_0 \neq \emptyset$ and $A_d = \emptyset$ then $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, V_{d-2}, v_d, U_{d-1}, S_i, S_{i-1}, \ldots, S_1$ is a hamiltonian path.

Note that we are able to trace G, under the appropriate conditions as listed above, even when subsets of $V(G)$ are empty. Also observe that whenever $X \neq \emptyset$, that is, $S_1 \neq \emptyset$ or $T_1 \neq \emptyset$, then G is in fact hamiltonian.

The graph G is traceable and the proof of Theorem 1 is complete. ∎

Clearly, connectedness is a necessary hypothesis in Theorem 1. Also it is easy to construct nontraceable graphs containing an induced $K_{1,3}$ or F; of course, these can be contained in traceable graphs.

4. Proof of Theorem 2.

Let G be a 2-connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or F. Fix a pair $v_0, v_d \in V(G)$ such that $d(v_0,v_d) = d = \text{diam } G$. The proof of Theorem 1 allows us to assume that every $v_0 - v_d$ path P of length d possesses property (*):

$$X = \{x \notin P \mid d(P,x) > 1\} = \emptyset . \qquad (*)$$

Also, by Theorem B, we may assume that $d \geq 3$.

Let us suppose that $d > 3$. We claim that there are $v_0 - v_d$ paths P and Q such that P has length d, $V(P) \cap V(Q) = \{v_0,v_d\}$ and $G(Q) = \langle (V(G) - V(Q)) \cup \{v_0,v_d\}\rangle$ is connected with neither v_0 nor v_d a cutvertex of $G(Q)$.

For each $v_0 - v_d$ path P^* of length d let U_i $(0 \leq i \leq d-1)$, V_i $(0 \leq i \leq d-2)$, A_0, and A_d be defined as in Section 2. Relabel the sequence of sets $A_0, U_0, V_0, \ldots, U_i, V_i, \ldots, U_{d-2}, V_{d-2}, U_{d-1}, A_d$ by $W_0, W_1, W_2, \ldots, W_{2i+1}, W_{2i+2}, \ldots, W_{2d-3}, W_{2d-2}, W_{2d-1}, W_{2d}$. Let $D(P^*)$ be the collection of paths $R^*: v_0, x_1, \ldots, x_k$ such that $x_i \in W_{j_i}$ and $j_k = \max\{j_1,j_2,\ldots,j_k\}$. Choose $R \in \bigcup_{P^*} D(P^*)$ such that j_k is maximum. Say $R \in D(P)$ where $P: v_0, v_1, \ldots, v_d$ and $R: v_0, x_1, \ldots, x_k$. We show that $2d-3 \leq j_k \leq 2d$. Otherwise, one of the following cases holds.

Case 1. $j_k = 0$. Then $x_k \in A_0$. This is impossible as G is 2-connected.

Case 2. $j_k = 2t+1$ $(0 \leq t \leq d-3)$. Then $x_k \in U_t$. If $y \in V_t$ then, by (A), x_k is adjacent to y, contradicting our choice of R. Thus, $V_t = \emptyset$. As v_{t+1} is not a cutvertex, there exist $x, y \in V(G)$ such that $xy \in E(G)$ and one of the following holds:

(i). $x \in U_t$, $y \in U_{t+1}$, (ii). $x \in U_t$, $y \in V_{t+1}$,
(iii). $x \in U_t$, $y \in U_{t+2}$, (iv). $x \in V_{t-1}$, $y \in U_{t+1}$,
(v). $x \in U_{t-1}$, $y \in U_{t+1}$, (vi). $x \in A_0$, $y \in U_1 \cup U_2$ $(t = 0)$.

(Observe that there must be an edge xy for some $x \in \bigcup_{i=0}^{2t+1} W_i$, $y \in \bigcup_{i=2t+2}^{2d} W_i$. All possibilities, other than (i) – (vi), either contradict $d(v_0,v_d) = d$, or give rise to an induced $K_{1,3}$ or F when $t = 0$ or $t = d-3$.)

If one of (i) – (iv) holds then $x = x_k$ or, by (A), xx_k is in G. Then one of $v_0, x_1, \ldots, x_k, x, y$ (if $x \notin V(R)$) or $v_0, x_1, \ldots, x_i, y$ (if $x = x_i$) contradicts the choice of R.

Suppose that (v) holds. Examining $\langle\{v_{t+1}, v_{t+2}, v_{t+3}, x, x_k, y\}\rangle$ shows that xx_k or x_ky is in G. It is obvious that either edge gives a contradiction.

If (vi) holds then v_0, x, y contradicts the choice of R.

Case 3. $j_k = 2t + 2$ $(0 \le t \le d-3)$. Then $x_k \in V_t$. As before, (A) implies that $U_{t+1} = \emptyset$.

Assume first that $V_{t+1} = \emptyset$. Since v_{t+2} is not a cut-vertex of G and $d(v_0,v_d) = d$ there must exist an edge xy such that

(i). $x \in U_t \cup V_t$, $y \in U_{t+2}$, (ii). $x \in U_{d-3}$, $y \in A_d$ $(t = d-3)$,
(iii). $x \in V_{d-3}$, $y \in A_d$ $(t = d-3)$, (iv). $x \in A_0$, $y \in U_2$ $(t = 0)$.

If (i) holds then one of the paths $v_0, x_1, \ldots, x_k, x, y$ (if $x \notin V(R)$) and $v_0, x_1, \ldots, x_i, y$ (if $x = x_i$) contradicts the choice of R. If (ii) holds then $\langle\{v_{d-4}, v_{d-3}, v_{d-2}, v_{d-1}, x, y\}\rangle \cong F$. We argue in a similar way if (iv) holds. In the event (iii) holds, then $\langle\{x, v_{d-3}, v_{d-1}, y\}\rangle \cong K_{1,3}$.

Let $z \in V_{t+1}$. Consider the $v_0 - v_d$ path

$$P' : v_0, v_1, \ldots, v_{t+1}, z, v_{t+3}, \ldots, v_d$$

of length d. Define the sets U'_i, V'_i for the path P'. Note that $U'_i = U_i$ $(i \neq t+1, t+2)$, $V'_i = V_i$ $(i \neq t, t+1, t+2)$, $x_k \in U'_t$ and $v_{t+2} \in V'_{t+1}$. Now x_k is adjacent to v_{t+2}, so $\langle \{v_0, x_1, \ldots, x_k, v_{t+2}\} \rangle$ contains a path contradicting our choice of R.

As a consequence of these cases, $x_k \in U_{d-2}, V_{d-2}, U_{d-1}$, or A_d. Let us show that $x_k \in U_{d-2}$ leads to a contradiction. If $x_k \in U_{d-2}$ then $V_{d-2} = \emptyset$. If both A_d and U_{d-1} were empty then v_{d-1} would be a cutvertex of G, an impossibility. If $A_d = \emptyset$ and $U_{d-1} \neq \emptyset$ then, because v_{d-1} is not a cut-vertex, we obtain, as in case 2, a path terminating in U_{d-1}. This contradicts our choice of R. So assume $A_d \neq \emptyset$. As G is 2-connected there exist $y \in A_d$ and x in one of A_0, U_i, or V_i such that x is adjacent to y. If $x \in U_i$ $(1 \leq i \leq d-3)$ then $\langle \{x, y, v_{i-1}, v_i, v_{i+1}, v_{i+2}\} \rangle \cong F$; if $x \in A_0 \cup U_0$ then $d(v_0, v_d) < 4$; if $y \in V_i$ $(0 \leq i \leq d-3)$ then $\langle \{x, y, v_i, v_{i+2}\} \rangle \cong K_{1,3}$. Hence, $A_d \sim (U_{d-2} \cup U_{d-1})$. If $A_d \not\sim U_{d-2}$ then, again arguing as in case 2, we obtain a path terminating in U_{d-1}, contrary to $x_k \in U_{d-2}$. Thus, $A_d \sim U_{d-2}$ which again invalidates our choice of R.

We conclude that $x_k \in U_{d-1}, V_{d-2}$, or A_d. Let Q be the path obtained by adjoining v_d to R. Then $V(Q) \cap V(P) = \{v_0, v_d\}$ and, because of property (*), $G(Q)$ is connected. It remains to show that Q may be chosen so that neither v_0 nor v_d are cutvertices of $G(Q)$.

Suppose that v_0 is a cutvertex of $G(Q)$. As a result of property (*), $\langle V(G(Q)) - \{v_0\} \rangle$ has two components, one being $\langle V(G(Q)) \cap A_0 \rangle$. Since G is 2-connected, some $x \in A_0$ and $y \in V(Q)$ are adjacent in G. Choose i maximum such that x_i is adjacent to a vertex of A_0 and let

$V(Q') = \{v_0, x_i, x_{i+1}, \ldots, x_k, v_d\} \cup A_0$. Then v_0 is not a cutvertex of $G(Q')$. A similar argument allows us to adjust Q' if v_d is a cutvertex of $G(Q')$.

The initial claim has been shown. That G is hamiltonian follows upon showing that $G(Q)$ contains a hamiltonian $v_0 - v_d$ path. For convenience, let U_i (respectively, V_i, A_0, A_d) denote $U_i - V(Q)$ (respectively, $V_i - V(Q), A_0 - V(Q), A_d - V(Q)$).

In order to be brief we list several observations.

(1). If $A_0 \neq \emptyset$ then $A_0 \sim (U_0 \cup U_1 \cup V_0)$.

This holds because v_0 is not a cutvertex of $G(Q)$ and $A_0 \sim U_i$ $(3 \leq i \leq d-1)$ contradicts $d(v_0, v_d) = d > 4$, $A_0 \sim U_2$ gives rise to F as an induced subgraph of G, and $A_0 \sim V_i$ $(1 \leq i \leq d-2)$ yields $K_{1,3}$.

(2). If $U_0 \neq \emptyset$ then $A_0 \sim U_0$, or $A_0 = \emptyset$.

Let $z \in A_0$ and $x \in U_1 \cup V_0$ be adjacent, and let $x_0 \in U_0$. Then $\langle\{z, x, v_1, v_2, v_3, x_0\}\rangle$ implies zx_0 or xx_0 is in $G(Q)$. If xx_0 is in $G(Q)$ then $\langle\{x_0, x, z, v_2\}\rangle \cong K_{1,3}$ unless zx_0 is an edge. Thus $A_0 \sim U_0$.

(3). If $A_0 \neq \emptyset, U_0 = \emptyset$ then (a) $A_0 \sim V_0$ or (b) $A_0 \sim U_1$.

Let (1)', (2)', (3)' denote the corresponding facts concerning $A_d, U_{d-1}, U_{d-2}, V_{d-2}$.

(4). If $U_0, V_0, V_1 \neq \emptyset$ then either (c) $U_0 \sim V_0$, (d) $U_0 \sim V_1$, or (e) $V_0 \sim V_1$.

Examining $\langle\{v_1, x_0, y_0, y_1\}\rangle$ for $x_0 \in U_0, y_0 \in V_0, y_1 \in V_1$ shows that (4) holds.

Suppose $U_0 \neq \emptyset, U_{d-1} \neq \emptyset$. By (2), (2)' and in accordance with which of (4), (c), (d), or (e) holds trace $G(Q)$ as follows:

(c). $v_0, A_0, U_0, V_0, v_1, U_1, V_1, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$

(d). $v_0, A_0, U_0, V_1, v_1, V_0, U_1, v_2, U_2, V_2, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$;

(e). $v_0, A_0, U_0, v_1, V_0, U_1, V_1, v_2, U_2, V_2, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$.

(These represent hamiltonian paths whether or not A_0 is empty. Also, if one or more of V_0 or V_1 is empty then the appropriate one of (c) or (d) still yields a hamiltonian path.)

Suppose $U_0 = \emptyset, U_{d-1} \neq \emptyset$. Apply (2)' and whichever of (3a) or (3b) holds to trace $G(Q)$:

(3a). $v_0, A_0, V_0, v_1, U_1, V_1, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$;

(3b). $v_0, A_0, U_1, V_0, v_1, V_1, v_2, U_2, V_2, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$.

Suppose $U_0 = \emptyset, U_{d-1} = \emptyset$. Then one of (3a), (3b) holds and one of (3a)', (3b)' holds. We trace $G(Q)$ as follows:

(3a), (3a)'. $v_0, A_0, V_0, v_1, U_1, V_1, \ldots, v_{d-2}, U_{d-2}, v_{d-1}, V_{d-2}, A_d, v_d$;

(3a), (3b)'. $v_0, A_0, V_0, v_1, U_1, V_1, \ldots, v_{d-2}, V_{d-2}, v_{d-1}, U_{d-2}, A_d, v_d$;

(3b), (3b)'. $v_0, A_0, U_1, V_0, v_1, V_1, v_2, U_2, V_2, \ldots, v_{d-2}, V_{d-2}, v_{d-1}, U_{d-2}, A_d, v_d$.

It now remains to show that if diam $G = d = 3$ then G is hamiltonian. For diameter 3 the preceding approach leads to a prohibitive number of cases. We employ an alternative technique.

By Theorem A there is a $v \in V(G)$ such that $\langle N(v) \rangle$ is disconnected. Since G contains no induced $K_{1,3}$ then $N(v) = A(v) \cup B(v)$ where $A(v) \neq \emptyset \neq B(v)$, $A(v) \cap B(v) = \emptyset$ and both $\langle A(v) \rangle$ and $\langle B(v) \rangle$ are complete. Let

$$C(v) = \{x \in V(G) \mid d(x,A(v)) = 1, d(x,v) = 2\} \text{ and}$$

$$D(v) = \{x \in V(G) \mid d(x,B(v)) = 1, d(x,v) = 2\}.$$

Since G is 2-connected, $C(v) \cup D(v) \neq \emptyset$.

Case 1. Assume that $C(v) \not\subseteq D(v)$ and $D(v) \not\subseteq C(v)$. We wish to show $\langle C(v) - D(v) \rangle$ is complete. Let $c, c' \in C(v) - D(v)$

and choose $a, a' \in A(v)$ such that ac and $a'c'$ are edges of G. If $a = a'$ then $\langle\{a,c,c',v\}\rangle$ implies cc' is in G. If $a \neq a'$ then with any $b \in B(v)$ we use $\langle\{a,a',c,c',v,b\}\rangle$ to conclude that cc' is in G. Hence, $\langle C(v) - D(v)\rangle$ and similarly $\langle D(v) - C(v)\rangle$ are nonempty complete graphs. A similar argument also shows that $C \cap D$ can be partitioned into sets C' and D' such that $\langle C_0\rangle$ and $\langle D_0\rangle$ are complete, where $C_0 = (C - D) \cup C'$ and $D_0 = (D - C) \cup D'$.

Observe that if $V(G) = \{v\} \cup A(v) \cup B(v) \cup C(v) \cup D(v)$ then $C_0 \sim D_0$ as diam $G = 3$. In this case,

$$v, A, C_0, D_0, B, v$$

represents a hamiltonian cycle. Thus we may assume that there are vertices at a distance 3 from v. Let

$$E(v) = \{x \in V(G) \mid d(x,C(v)) = 1, d(x,v) = 3\} \text{ and}$$
$$F(v) = \{x \in V(G) \mid d(x,D(v)) = 1, d(x,v) = 3\}.$$

Observe that since G contains no induced $K_{1,3}$, (**) there is no element of $E(v) \cup F(v)$ adjacent to an element of $C(v) \cap D(v)$.

Suppose that $E(v) \neq \emptyset$ and $F(v) = \emptyset$ (note a similar argument will hold if $E(v) = \emptyset$ and $F(v) \neq \emptyset$). Let $d \in D(v) - C(v)$ and $e \in E(v)$. If $d(d,e) = 2$ then there is $c \in C(v)$ such that ec and dc are edges in G. As already observed, $c \in C(v) - D(v)$ and $\langle\{a,c,d,e\}\rangle \cong K_{1,3}$ for any $a \in A$ adjacent to c. Therefore, $d(d,e) = 3$ and there are vertices $c \in C(v) - D(v)$ and $x \in C(v) \cup D(v)$ such that e, c, x, d is a path. Again choosing $a \in A$, with $ac \in E(G)$, we have ax is in G. Now $\langle\{a,c,d,e,v,x\}\rangle \cong F$ and hence we conclude that $E(v) \neq \emptyset \neq F(v)$.

It is straightforward to show that $\langle E(v)\rangle$ and $\langle F(v)\rangle$ are complete. If $E(v) \neq F(v)$, then by (**) any path from $E(v)$ to $F(v)$ which contains elements of $C(v) \cup D(v)$ has length at least 4. Thus, $E(v) \sim F(v)$ and in fact, we may assume that $E(v) \not\subseteq F(v)$ and $(E(v) - F(v)) \sim F(v)$. Therefore,

$$v , A(v) , C_0 , E(v) - F(v) , F(v) , D_0 , B(v) , v$$

represents a hamiltonian cycle.

Case 2. Assume that $C(v) = D(v)$. In this case $E(v) \cup F(v) = \emptyset$ and $V(G) = \{v\} \cup A(v) \cup B(v) \cup C(v)$. If diam $G = 2$ then G is hamiltonian by Theorem B. Thus, we may assume that there exist c and $c' \in C(v)$ such that $d(c,c') = 3$. We shall show that Case 1, with c replacing v, now applies.

Since c is not adjacent to c' in G, no $a \in A(v)$ is adjacent to both c and c'. If there is an $a \in A(v)$ adjacent to neither, then by choosing $b , b' \in B(v)$ with bc and $b'c'$ in G we have that $\langle \{v,b,b',a,c,c'\} \rangle \cong F$. Thus, $A(v)$ is partitioned as A and A' ($A \neq \emptyset$ and $A' \neq \emptyset$) with $a \in A$ if and only if ac is in G and $a \in A'$ if and only if ac' is in G. Similarly, $B(v)$ can be partitioned into nonempty sets B and B'. So $\langle N(c) \rangle$ is disconnected and we define $A(c) , B(c) , C(c) , D(c)$ with $A \subseteq A(c)$ and $B \subseteq B(c)$.

Let $y \in A'$. Then $d(c,y) = 2$ and $d(y,A(c)) = 1$ so $y \in C(c)$. If $y \in D(c)$ then by the definition of $D(c)$ there is some $z \in B(c)$ such that yz is in G. Now $z \notin A(v)$ as then $z \in A \subseteq A(c)$; moreover, $z \notin B(v)$ for otherwise yz is not in G. Therefore, $z \in C(v)$. Since $\langle \{v,y,z,c'\} \rangle \ncong K_{1,3}$ then zc' is in G. But $z \in B(c)$ implies zc is in G and hence $d(c,c') = 2$. This is a contradiction and we now conclude that $C(c) \nsubseteq D(c)$ and similarly $D(c) \nsubseteq C(c)$. Thus Case 1 applies with c replacing v.

Case 3. Assume that $D(v) \subset C(v)$ and that there exists $x \in V(G)$ such that $d(v,x) = 3$.

Then $E(v)$, as defined in Case 1, is nonempty and by (**) satisfies $E(v) = \{x \in V(G) | d(x,C(v) - D(v)) = 1, d(x,v) = 3\}$. Also, v is not a cutvertex of G so $D(v) \neq \emptyset$. As in Case 1

$\langle C(v) - D(v)\rangle$ is complete. In fact, $\langle D(v)\rangle$ is also complete (Observe that if d is not adjacent to d' in $\langle D(v)\rangle$ then there exists $a' \in A(v)$ not adjacent to d and $b \in B(v)$ adjacent to d. Now there exists $e \in E(v)$, $c \in C(v) - D(v)$ and $a \in A(v)$ such that e, c, a, d is a path and c is adjacent to d. (This is true because if c is adjacent to d, any vertex $a \in A(v)$ adjacent to c must be adjacent to d, otherwise $\langle\{c,a,d,e\}\rangle \cong K_{1,3}$. If c is not adjacent to d there exists a path e, c, c', d with ec' not in G, but $\langle\{c,a,e,c'\}\rangle$ implies ac' is in G and $\langle\{c,a,c',v,e,d\}\rangle$ gives a contradiction.) Finally, $\langle\{a,c,d,e,a',b\}\rangle$ gives a contradiction.

Since $d(D(v), E(v)) \leq 3$, there is a $c \in C(v) - D(v)$ adjacent to a vertex of $D(v)$ (otherwise there is a $K_{1,3}$ centered in $A(v)$). Since G is 2-connected there exists $c', c'' \in C(v) - D(v)$ (note (**) holds) such that both c' and c'' are adjacent to vertices of $E(v)$ and $c \neq c''$. Then

$v, B(v), D(v), c, E(v), c'', C(v) - (D(v) \cup \{c,c''\}), A(v), v$

(if $c = c'$) or

$v, B(v), D(v), c, c', E(v), c'', C(v) - (D(v) \cup \{c,c',c''\})$, $A(v), v$ (if $c \neq c'$) represent hamiltonian cycles in G.

Case 4. Suppose $D(v) \subset C(v)$ and $d(v,x) \leq 2$ for all $x \in V(G)$. For simplicity let $A = A(v)$, $B = B(v)$, etc. Then $V(G) = \{v\} \cup A \cup B \cup C$. Observe $\langle C - D\rangle$ is complete (as in Case 1).

Subcase A. Suppose $\langle D\rangle$ is complete. If $C - D \sim D$ then $v, A, C - D, D, B, v$ represents a hamiltonian cycle. If $C - D \not\sim D$, then since G is 2-connected, there exists at leas two vertices in A, say a_1 and a_2, with adjacencies in $C - D$ (and these adjacencies are distinct unless $|C - D| = 1$) Further, a_1 and a_2 have no adjacencies in D or an induced

$K_{1,3}$ would result. Since $D \subset C$, there exists $a \in A$ such that $a \sim D$, thus, $a \neq a_1$ or a_2. Then

$$v, a_1, C - D, a_2, A - \{a, a_1, a_2\}, a, D, B, v$$

represents a hamiltonian cycle.

Subcase B. Suppose $\langle D \rangle$ is not complete. Choose nonadjacent $d_1, d_2 \in D$. We note d_1 and d_2 have no common adjacencies in A or B (for an induced $K_{1,3}$ would result). Further, each $a \in A$ is adjacent to exactly one of d_1 and d_2 (since otherwise, for $b_1, b_2 \in B$ such that $b_1d_1, b_2d_2 \in E(G)$, $\langle \{a, v, b_1, b_2, d_1, d_2\} \rangle \cong F$).

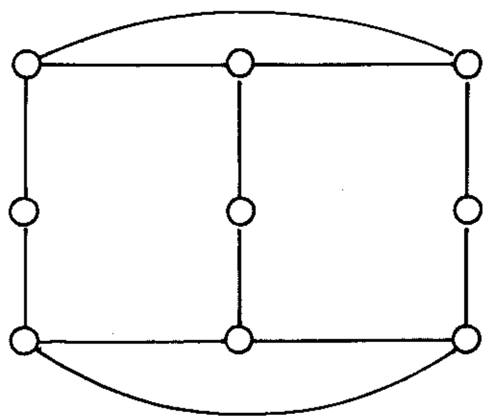
Fix $a \in A$ and define $D_1 = \{x \in D \mid ax \in E(G)\}$ and $D_2 = \{x \in D \mid ax \notin E(G)\}$. Clearly $\langle D_1 \rangle$ is complete or a $K_{1,3}$ centered at a would exist. Further, $\langle D_2 \rangle$ is complete for otherwise there exists nonadjacent $d, d' \in D_2$. Then if d and d' have a common adjacency in B, a $K_{1,3}$ exists; while if $b, b' \in B$ such that db and $d'b'$ are edges of G, then $\langle \{a, v, b, b', d, d'\} \rangle \cong F$. Thus, either case produces a contradiction, and $\langle D_2 \rangle$ is complete.

Recall that each vertex in $C - D$ has an adjacency in A. Let $c \in C - D$ such that $ca' \in E(G)$ (some $a' \in A$). If $a'd_1 \in E(G)$ then cd_1 is an edge of G or a $K_{1,3}$ would exist. If $a' \not\sim d_1$, then there exists $a'' \in A$ such that $a''d_1 \in E(G)$. Further, choose $b \in B$ such that $bd_2 \in E(G)$. Then by considering $\langle \{c, a', a'', d_1, v, b\} \rangle$ we see $bd_1 \in E(G)$ or $cd_1 \in E(G)$. But bd_1 contradicts the fact that d_1 and d_2 have no common adjacencies in B. As c was arbitrary in $C - D$, d_1 is adjacent to each vertex in $C - D$. A similar argument shows d_2 is adjacent to each vertex in $C - D$. Now

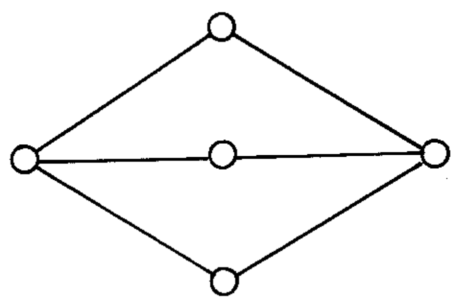
$$v, B, D_1 - \{d_1\}, d_1, C - D, d_2, D_2 - \{d_2\}, A, v$$

represents a hamiltonian cycle in G. This completes Case 4 and the proof of the Theorem. ∎

In Figure 2 we display various examples that demonstrate the independence of Theorem A and Theorem 2, as well as the need to forbid both induced subgraphs in Theorem 2.

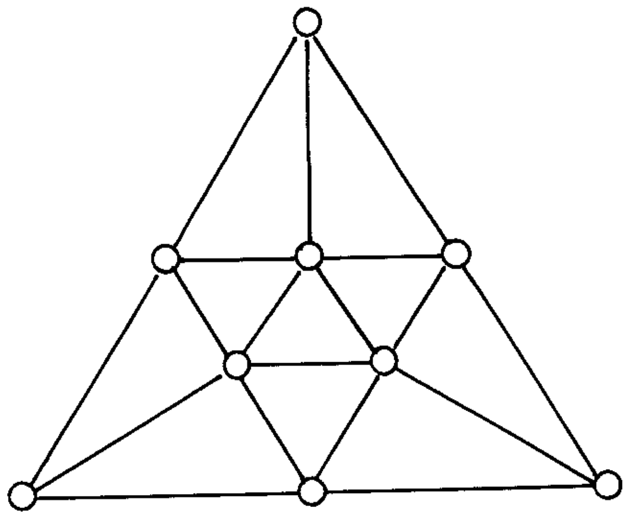


A 2-connected nonhamiltonian graphs containing no induced $K_{1,3}$ (but containing F).

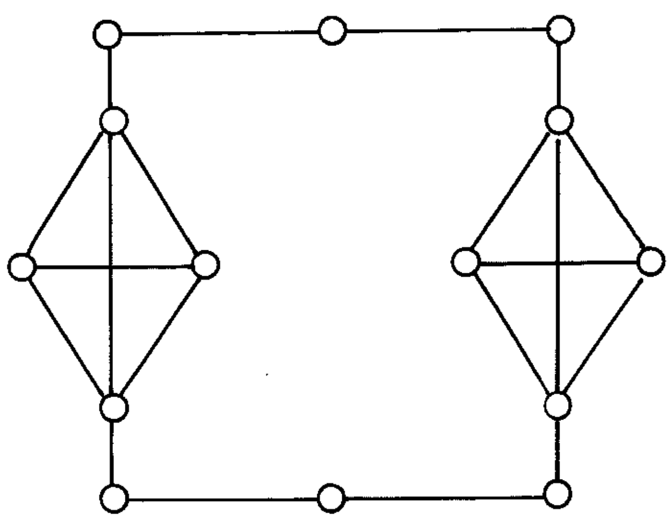


A 2-connected nonhamiltonian graph containing no induced F (but containing $K_{1,3}$).

Figure 2a.



A hamiltonian graph that is locally connected, containing no induced $K_{1,3}$ and containing F.



A hamiltonian graph, containing no induced $K_{1,3}$ or F, that is not locally connected.

Figure 2b.

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