Graph Minors and Linkages

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Abstract: Bollobás and Thomason showed that every $22k$-connected graph is k-linked. Their result used a dense graph minor. In this paper, we investigate the ties between small graph minors and linkages. In particular, we show that a 6-connected graph with a K_9^- minor is 3-linked. Further, we show that a 7-connected graph with a K_9^- minor is $(2,5)$-linked. Finally, we show that a graph of order n and size at least $7n-29$ contains a K_9^{--} minor.
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1. INTRODUCTION

All graphs considered in this paper are simple graphs, that is, finite graphs without multiple edges or loops. For any graph G, we will use $|G|$ and $||G||$ to denote the number of vertices and the number of edges of G, respectively. Let H be a connected subgraph of a graph G, then let G/H denote the graph obtained by contracting all vertices of H to a vertex and let $G[H]=G[V(H)]$ denote the subgraph induced by the vertex set of H in G. Let $N(v)$ denote the set of vertices in G, which are adjacent to v and set $N[v]=N(v)\cup\{v\}$. In this paper, K_n always stands for the complete graph with n vertices, K_n^- denotes a subgraph of K_n with exactly one edge deleted, and K_n^{-i} denotes a subgraph of K_n with exactly $i(\geq 2)$ edges deleted. When $i=2$, we sometimes use K_n^{--} for K_n^{-2}.

Let $s_1, s_2, \ldots, s_k$ be k positive integers. A graph G is said to be $(s_1, s_2, \ldots, s_k)$-*linked* if it has at least $\sum_{i=1}^k s_i$ vertices and for any k disjoint vertex sets $S_1, S_2, \ldots, S_k$ with $|S_i|=s_i$, G contains vertex-disjoint connected subgraphs F_1, $F_2, \ldots, F_k$ such that $S_i \subseteq V(F_i)$. The case $s_1=s_2=\cdots=s_k=2$ has been studied extensively. A $(2,2,\ldots,2)$-linked graph is called k-*linked*, that is, for any $2k$ distinct vertices x_1, y_1, x_2, $y_2, \ldots, x_k$, and y_k there exist k vertex-disjoint paths P_1, $P_2, \ldots, P_k$ such that P_i joins x_i and y_i, $1\leq i\leq k$.

A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and/or vertices and contracting edges. An H-minor of G is a minor isomorphic to H. A *subdivision* of a graph is obtained by replacing some of its edges by paths so that the paths are pairwise internally disjoint. Clearly, if G contains a subdivision of H then G has H as a minor, but the converse is not necessarily true.

Linkages, subdivisions, and minors have been related in a number of results. For example, Larman and Mani [12] and Jung [5] noticed that if $\kappa(G)\geq 2k$ and if G contains a subdivision of K_{3k} then G is k-linked. Mader [15] showed that a graph contains a subdivision of K_{3k} if its connectivity is sufficiently large. Robertson and Seymour [17] showed that the observation of Larman and Mani and of Jung remains true under the very much weaker condition that G has K_{3k} as a minor. Instead of considering K_{3k} minors, Bollobás and Thomason [1] considered graphs containing a dense graph as a minor. Using this idea, they showed that every $22k$-connected graph is k-linked, thus confirming the long-standing belief that linear connectivity would suffice.

Jung [10] showed that every 4-connected non-planar graph is 2-linked. Thomassen [21] and Seymour [19] gave a characterization of graphs, which are not 2-linked. Chakravarti and Robertson also proved a variation of the result on 2-linked graphs [16]. Our main purpose is to develop more ties between small graph minors and graph linkages. To do so, we study graphs containing dense minors on 9 vertices. In particular, the following results are obtained.

Theorem 1.1. *If a 6-connected graph G has K_9^- as a minor, then G is 3-linked.*

Yu [23] completely characterized graphs G which do not contain two vertex-disjoint connected subgraphs F_1 and F_2 such that $S_1 \subseteq V(F_1)$ and $S_2 \subseteq V(F_2)$ for two disjoint vertex sets S_1 and S_2 with $|S_1| = 2$ and $|S_2| = 3$. Consequently, he proved that every 8-connected graph is $(2,3)$-linked. We will prove the following theorem.

Theorem 1.2. *If a 7-connected graph G has K_9^- as a minor, then G is $(2,5)$-linked.*

Note that in [2], we consider several additional questions of this type. Finally, we show the following.

Theorem 1.3. *If G is a graph on $n \geq 9$ vertices with at least $7n - 29$ edges, then G has K_9^{--} as a minor.*

We do not feel Theorem 1.3 is best possible. Hence, we make the following conjecture.

Conjecture 1.4. *If G is a graph on n vertices with at least $6n - 20$ edges, then G has K_9^{--} as a minor.*

In addition, we make these related conjectures.

Conjecture 1.5. *If G is a graph on n vertices with at least $\frac{13n-47}{2}$ edges, then G has K_9^- as a minor.*

Conjecture 1.6. *If G is a graph on n vertices with at least $7n - 27$ edges, then G has K_9 as a minor with finitely many exceptions.*

Conjecture 1.7. *If G is a 6-connected graph with K_9^{--} as a minor, then G is 3-linked.*

Very recently, a proof of Conjecture 1.6 was announced by Thomas et al. [20]. Finally, we note another long-standing conjecture.

Conjecture 1.8. *Every 8-connected graph graph is 3-linked.*

We will give proofs of Theorems 1.1 and 1.2 in Section 2 and of Theorem 1.3 in Section 3.

We define $G + H$ be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, where G and H are two vertex disjoint graphs. We define $2G = G + G'$, where G' is isomorphic to G and $V(G') \cap V(G) = \emptyset$. Let G be a graph and A be a subset of $V(G)$. To avoid cumbersome notation, at times, we simply use A to denote the subgraph induced by A, that is $G[A]$, provided no confusion will arise.

2. LINKAGES

In this section, we will prove Theorem 1.1 and Theorem 1.2. We will use inductive arguments showing slightly stronger statements of each result. We will need the following definitions.

Definition 2.1. *Let $S, A, B \subseteq V(G)$ be sets of vertices in a graph G. Let $\ell = |A \cap B|$. If $S \subseteq A$, $V(G) = A \cup B$, and there are no edges between $A \backslash B$ and $B \backslash A$, then we call (A, B) an S-cut of size ℓ.*

Definition 2.2. *Let H be a minor of a connected graph G. Let $C_1, C_2, \ldots, C_{|H|}$ be a partition of $V(G)$, such that each $G[C_i]$ is connected, and contraction of the C_i's yields H. Let $S \subseteq V(G)$. An S-cut (A, B) of G is called an S^H-cut if $C_i \subseteq B \backslash A$ for some $1 \leq i \leq |H|$.*

A. Proof of Theorem 1.1

Now we shall prove the following result, which is stronger than Theorem 1.1.

Theorem 2.1. *Let G be a graph, and let $S = \{x_1, x_2, y_1, y_2, z_1, z_2\} \subset V(G)$ be a set of 6 vertices. Let G^* be the graph obtained from G by adding all missing edges in $G[S]$. Suppose that there is a partition $C_1, C_2, \ldots, C_9$ of $V(G)$, such that each $G^*[C_i]$ is connected, and contraction of the C_i's in G^* yields $H = K_9^-$. Further suppose that G^* has no S^H-cut of size smaller than 6. Then there are three vertex disjoint paths in G connecting (x_1, x_2), (y_1, y_2), and (z_1, z_2), respectively.*

Proof. Suppose the statement is false, and G is a counterexample with the minimum number of edges. Let $S, C_1, C_2, \ldots C_9$ be as in the theorem, and suppose the desired linkage cannot be found. By the choice of G, we know that $G[S]$ contains no edges.

Claim 2.1.1. *The subgraphs $G[C_i]$ $(i = 1, 2, \ldots, 9)$ contain no edges.*

Suppose to the contrary that for some i, $G[C_i]$ contains an edge. Without loss of generality, we may assume that $uv \in E(C_1)$, and since $G[S]$ is empty, $v \notin S$. By the choice of G, there has to be an S^H-cut (A, B) of size 6 in G^* with $u, v \in A \cap B$, otherwise the contraction of uv would yield a smaller counterexample.

A simple count shows that at least four of the nine C_i sets contain no vertices of $A \cap B$. By symmetry, we may assume that $C_i \cap A \cap B \neq \emptyset$ for $1 \leq i \leq k$, and $C_i \cap A \cap B = \emptyset$ for $i > k$, where k is an integer with $1 \leq k \leq 5$. As $S \subseteq A$, and $G^*[C_i]$ is connected, we know that $C_i \subseteq B \backslash A$ or $C_i \subseteq A \backslash B$ for each $i > k$. By the definition of S^H-cuts, we know that $C_i \subseteq B \setminus A$ for at least one $i > k$, hence it is, in fact, true that $C_i \subseteq B \setminus A$ for all $i > k$, otherwise the C_i would not contract to a K_9^- in G^*.

Since there is no S^H-cut of size less than 6 in G^*, there does not exist a cut of size less than 6 in A separating S and $A \cap B$. By Menger's Theorem, there are 6 vertex disjoint paths from S to $A \cap B$ in $G[A]$. Label the vertices of $S' = A \cap B$ with $x_1', x_2', y_1', y_2', z_1', z_2'$ according to the starting vertices in S of these paths. Let $C_i' = C_i \cap B$ for $1 \leq i \leq 9$. $G[B]$, S', $C_1', \ldots, C_9'$ satisfy all the conditions of the statement, and $G[B]$ is smaller than G as there is at least one vertex in $S \backslash B$ (note that $v \notin S$).

By the choice of G, we can find three vertex disjoint paths in $G[B]$ connecting (x_1', x_2'), (y_1', y_2'), and (z_1', z_2'), respectively. This, together with the six paths in $G[A]$, produces three vertex disjoint paths in G connecting (x_1, x_2), (y_1, y_2), and (z_1, z_2), respectively, a contradiction. This shows that $G[C_i]$ $(i = 1, \ldots, 9)$ contain no edges. ■

Note that this implies that for each $1 \leq i \leq 9$, $C_i \subseteq S$ or $|C_i| = 1$. Therefore, $9 \leq |V(G)| \leq 14$. We will finish the proof by an analysis broken into cases according to $|V(G)|$. We may always assume that $|C_i| \geq |C_j|$ for $1 \leq i < j \leq 9$.

Case 2.1.1. *Suppose* $|V(G)| = 9$.

Note that in this case $|C_i| = 1$ for all $1 \leq i \leq 9$. Let $V(G) \backslash S = \{v_1, v_2, v_3\}$. Since the paths in the following sets $\{x_1 v_1 x_2, y_1 v_2 y_2, z_1 v_3 z_2\}$ and $\{x_1 v_2 x_2, y_1 v_3 y_2, z_1 v_1 z_2\}$ are edge disjoint, respectively, one of these sets is the desired set of vertex-disjoint paths, a contradiction.

Case 2.1.2. *Suppose* $|V(G)| = 10$.

In this case $|C_1| = 2$. Let $V(G) \backslash S = \{v_1, v_2, v_3, v_4\}$.

First suppose that $C_1 = \{x_1, x_2\}$ (the cases $C_1 = \{y_1, y_2\}$ and $C_1 = \{z_1, z_2\}$ are analogous). There exists a matching from C_1 into $V(G) \backslash S$, otherwise there is an S^H-cut smaller than 6 in G^*. We may assume that $\{x_1 v_1, x_2 v_2\}$ is such a matching. If $v_1 v_2 \in E(G)$, then one of $\{x_1 v_1 v_2 x_2, y_1 v_3 y_2, z_1 v_4 z_2\}$ and $\{x_1 v_1 v_2 x_2, y_1 v_4 y_2, z_1 v_3 z_2\}$ is the desired set of vertex-disjoint paths, a contradiction. Thus, we may assume that $v_1 v_2 \notin E(G)$. As G^* contracts to a K_9^-, v_3 has a neighbor in C_1, hence we may assume that $x_1 v_3 \in E(G)$. But now $\{x_1 v_3 v_2 x_2, y_1 v_1 y_2, z_1 v_4 z_2\}$ is the desired set of vertex-disjoint paths, a contradiction.

Now suppose that $C_1 = \{x_1, y_1\}$ (again the other cases are handled by a similar argument). There exists a matching from C_1 into $V(G) \backslash S$. We may assume that $\{x_1 v_1, y_1 v_2\}$ is such a matching. At most, one of the edges in a path in $\{x_1 v_1 x_2, y_1 v_2 y_2, z_1 v_3 z_2\}$ is missing, but then this edge can be replaced by a path of length 2 through v_4 to produce the desired set of vertex disjoint paths, a contradiction completing this case.

Case 2.1.3. *Suppose* $|V(G)| = 11$.

Let $V(G) \backslash S = \{v_1, v_2, v_3, v_4, v_5\}$.

First suppose that $|C_1| = 3$. We may assume that $x_1, y_1 \notin C_1$. Now $G^*[x_1, y_1, v_1, v_2, v_3, v_4, v_5]$ is a K_7 or a K_7^-, and therefore 3-linked. We can find a matching from $\{x_2, y_2, z_1, z_2\}$ into $\{v_2, v_3, v_4, v_5\}$, otherwise there is an S^H-cut smaller than 6 in G^*. Without loss of generality, suppose the matching is x_2v_2, y_2v_3, z_1v_4, z_2v_5. We can now connect the paths in the necessary manner inside $G^*[x_1, y_1, v_1, v_2, v_3, v_4, v_5]$, since this graph is 3-linked. Note that the edge x_1y_1 is not used in this path system, so this is, in fact, a path system in G, a contradiction.

Now suppose that $|C_1| = |C_2| = 2$. If $x_1, y_1 \notin C_1 \cup C_2$, the same argument as above applies. By symmetry we may assume that $C_1 \cup C_2 = \{y_1, y_2, z_1, z_2\}$. If $x_jv_k \notin E(G)$ for some $1 \le j \le 2$ and some $1 \le k \le 5$, say $x_1v_1 \notin E(G)$, then $G[x_2, v_1, v_2, v_3, v_4, v_5]$ is a K_6 and thus 3-linked, and a very similar argument can be used to find the paths. Thus, we may assume that $x_jv_k \in E(G)$ for $1 \le j \le 2$ and $1 \le k \le 5$. There is a matching from $\{y_1, y_2, z_1, z_2\}$ into $\{v_1, v_2, v_3, v_4, v_5\}$, say $y_1v_1, y_2v_2, z_1v_3, z_2v_4 \in E(G)$. If $v_1v_2, v_3v_4 \in E(G)$, then $\{x_1v_5x_2, y_1v_1v_2y_2, z_1v_3v_4z_2\}$ is the desired set of vertex disjoint paths, a contradiction. Hence, we may assume that $v_1v_2 \notin E(G)$. As G^* contracts to a K_9^-, v_5 is adjacent to both C_1 and C_2. If $v_5y_1 \in E(G)$ (and similarly if $v_5y_2 \in E(G)$), then $\{x_1v_1x_2, y_1v_5v_2y_2, z_1v_3v_4z_2\}$ is the desired set of vertex disjoint paths. Hence, $v_5z_1, v_5z_2 \in E(G)$. But then $\{x_1v_4x_2, y_1v_1v_3v_2y_2, z_1v_5z_2\}$ are the desired paths and this contradiction completes this case.

Case 2.1.4. *Suppose* $|V(G)| = 12$.

Let $V(G)\backslash S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. If $|C_1| \ge 3$, then $|C_3| = 1$ and $G[C_3 \cup \{v_1, v_2, v_3, v_4, v_5, v_6\}]$ is a K_7 or a K_7^- and the same argument as in Case 2.1.3 applies. Hence, we may assume that $|C_1| = |C_2| = |C_3| = 2$.

There is a matching from S into $V(G)\backslash S$, say $\{x_1v_1, x_2v_2, y_1v_3, y_2v_4, z_1v_5, z_2v_6\}$ is such a matching. One of the edges v_1v_2, v_3v_4, v_5v_6 is missing, otherwise the three paths are easy to find. This implies that every v_i has at least three neighbors in S, one in each of C_1, C_2, and C_3. Further, each vertex in S has at least two neighbors in $V(G)\backslash S$, otherwise G is not minimal.

Suppose that $x_2v_1 \in E(G)$. Then, similar to our earlier arguments, either $\{x_1v_1x_2,\ y_1v_3v_4y_2,\ z_1v_5v_2v_6z_2\}$ or $\{x_1v_1x_2,\ y_1v_3v_2v_4y_2,\ z_1v_5v_6z_2\}$ is the desired path system, a contradiction. So, assume that $x_2v_1 \notin E(G)$. By similar arguments, we may conclude that $x_1v_2, y_1v_4, y_2v_3, z_1v_6, z_2v_5 \notin E(G)$.

Suppose that $x_1v_3, x_2v_3 \in E(G)$. If $y_1v_1 \in E(G)$ or $y_1v_2 \in E(G)$, or $y_1v_4 \in E(G)$, a path system can easily be found. So, we may assume $y_1v_1, y_1v_2, y_1v_4 \notin E(G)$. Thus, $y_1v_5 \in E(G)$ or $y_1v_6 \in E(G)$, by symmetry we may assume $y_1v_5 \in E(G)$. If $z_1v_1 \in E(G)$, then $\{x_1v_3x_2, y_1v_5v_4y_2, z_1v_1v_6z_2\}$ is a path system, a contradiction. Thus $z_1v_1 \notin E(G)$. Similarly, $z_1v_2 \notin E(G)$. As v_1 and v_2 have at least three neighbors in S, we have $y_2v_1, y_2v_2, z_2v_1, z_2v_2 \in E(G)$. If $z_1v_4 \in E(G)$, then $\{x_1v_3x_2, y_1v_5v_1y_2, z_1v_4v_6z_2\}$ is a path system, a contradiction. Thus, $z_1v_4 \notin E(G)$, and $z_1v_3 \in E(G)$ as z_1 has at least two neighbors in $V(G)\backslash S$. If $x_1v_4 \in E(G)$, then $\{x_1v_4v_2x_2, y_1v_5v_1y_2, z_1v_3v_6z_2\}$ is a path system, a contradiction. Thus, $x_1v_4 \notin E(G)$, and similarly $x_2v_4 \notin E(G)$. But now the only possible neighbors of v_4 in S are y_2 and z_2, a contradiction establishing that x_1v_3 and x_2v_3 cannot both be edges.

By symmetrical arguments, we can establish that $N(x_1) \cap N(x_2) = N(y_1) \cap N(y_2) = N(z_1) \cap N(z_2) = \emptyset$. Therefore, every v_i has exactly three neighbors in S.

By symmetry, we may assume that $v_1v_2 \notin E(G)$ and $N(v_1) = \{x_1, y_1, z_1\}$. If $x_1v_3 \in E(G)$, then $\{x_1v_3v_2x_2, y_1v_1v_4y_2, z_1v_5v_6z_2\}$ is a path system, a contradiction. Thus, $x_1v_3 \notin E(G)$ and hence $x_2v_3 \in E(G)$.

If $y_1v_2 \in E(G)$, then $\{x_1v_1v_3x_2, y_1v_2v_4y_2, z_1v_5v_6z_2\}$ is a path system, a contradiction. Thus, $y_1v_2 \notin E(G)$ and hence $y_2v_2 \in E(G)$.

If $x_2v_4 \in E(G)$, then $\{x_1v_1v_4x_2, y_1v_3v_2y_2, z_1v_5v_6z_2\}$ is a path system, a contradiction. Thus, $x_2v_4 \notin E(G)$ and hence $x_1v_4 \in E(G)$.

If $y_2v_5 \in E(G)$, then $\{x_1v_4v_2x_2, y_1v_3v_5y_2, z_1v_1v_6z_2\}$ is a path system, a contradiction. Thus, $y_2v_5 \notin E(G)$ and hence $y_1v_5 \in E(G)$. But now, $\{x_1v_4v_3x_2, y_1v_5v_2y_2, z_1v_1v_6z_2\}$ is a path system, the final contradiction finishing the case $|V(G)| = 12$.

Case 2.1.5. *Suppose* $|V(G)| > 12$.

Let $V(G) \setminus S \supseteq \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Then $G[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ is a K_7 or a K_7^-, and therefore 3-linked. The path system can easily be found by finding a matching from S to six vertices of $V(G)\setminus S$, establishing this last case and completing the proof of the theorem.

B. Proof of Theorem 1.2

Again, we will prove a slightly stronger statement.

Theorem 2.2. *Let G be a graph, and let* $S = \{x_1, x_2, y_1, y_2, y_3, y_4, y_5\} \subset V(G)$ *be a set of 7 vertices. Let* G^* *be the graph obtained from G by adding all missing edges in* $G[S]$. *Suppose that there is a partition* $C_1, C_2, \ldots, C_9$ *of* $V(G)$, *such that each* $G^*[C_i]$ *is connected, and contraction of the* C_i'*s in* G^* *yields* $H = K_9^-$. *Further suppose that* G^* *has no* S^H*-cut of size smaller than 7. Then there are two vertex disjoint connected subgraphs in G containing* $\{x_1, x_2\}$ *and* $\{y_1, y_2, y_3, y_4, y_5\}$, *respectively.*

Proof. Suppose the statement is false and G is a counterexample with the minimum number of edges. Let $S, C_1, C_2, \ldots, C_9$ be as in the theorem, and suppose the desired subgraphs cannot be found. By the choice of G, we know that $G[S]$ contains no edges.

Claim 2.2.1. *The subgraphs* $G[C_i]$ *contain no edges.*

Suppose the result fails to hold. Without loss of generality, we may assume that $uv \in E(C_1)$, and $v \notin S$. By the choice of G, there has to be an S^H-cut (A, B) of size 7 with $u, v \in A \cap B$, otherwise the contraction of uv would yield a smaller counterexample.

A simple count shows that at least three of the nine C_i sets contain no vertices of $A \cap B$. By symmetry, we may assume that $C_i \cap A \cap B \neq \emptyset$ for $1 \leq i \leq k$, and

$C_i \cap A \cap B = \emptyset$ for $i > k$, where k is an integer with $1 \le k \le 6$. As $S \subseteq A$, and $G^*[C_i]$ is connected, we know that $C_i \subseteq B \setminus A$ or $C_i \subseteq A \setminus B$ for each $i > k$. Since $C_i \subseteq B \setminus A$ for at least one $i > k$, it is, in fact, true that $C_i \subseteq B \setminus A$ for all $i > k$, otherwise the C_i would not contract to a K_9^- in G^*.

Since there is no S^H-cut of size less than 7 in G^*, there are 7 vertex disjoint paths from S to $A \cap B$ in $G[A]$. Label the vertices of $S' = A \cap B$ with x_1', x_2', $y_1', y_2', y_3', y_4', y_5'$ according to the starting vertices of these paths. Let $C_i' = C_i \cap B$ for $1 \le i \le 9$. $G[B]$, S', $C_1', \ldots, C_9'$ satisfy all the conditions of the statement, and $G[B]$ is smaller than G as there is at least one vertex in $S \backslash B$ (note that $v \notin S$).

By the choice of G, we can find two vertex disjoint connected subgraphs in $G[B]$ containing $\{x_1', x_2'\}$ and $\{y_1', y_2', y_3', y_4', y_5'\}$, respectively. This, together with the seven paths in $G[A]$, produces the desired subgraphs in G, a contradiction, completing the claim. ■

Note that this implies that for each $1 \le i \le 9$, $C_i \subseteq S$ or $|C_i| = 1$. Therefore, $9 \le |V(G)| \le 15$ and we can assume that $|V(C_i)| \ge |V(C_j)|$ for $1 \le i < j \le 9$. We will finish the proof by an analysis broken up into cases according to $|V(G)|$.

Case 2.2.1. *Suppose* $|V(G)| = 9$.

Note that in this case $|C_i| = 1$ for all $1 \le i \le 9$. Let $V(G) \backslash S = \{v_1, v_2\}$. Then one of $G[x_1, x_2, v_1], G[y_1, y_2, y_3, y_4, y_5, v_2]$ and $G[x_1, x_2, v_2], G[y_1, y_2, y_3, y_4, y_5, v_1]$ is the desired set of connected subgraphs, a contradiction.

For all other cases note that every vertex in S has at least two neighbors in $V(G) \backslash S$. Suppose the contrary, say y_1, has at most one neighbor in $V(G) \backslash S$. If y_1 has no neighbors in $V(G) \backslash S$, then $(A = S, B = V(G) \setminus \{y_1\})$ is an S^H-cut of size 6. On the other hand, if y_1 has exactly one neighbor in $V(G) \backslash S$, say $y_1 v_1 \in E(G)$, then $C_i \backslash \{y_1\} \ne \emptyset$ for all $1 \le i \le 9$ since $|V(G) \backslash S| \ge 3$, and $G \backslash \{y_1\}$ with $S' = (S \backslash \{y_1\}) \cup \{v_1\}$ would be a smaller counterexample, contradicting the minimality of $E(G)$.

Case 2.2.2. *Suppose* $|V(G)| = 10$.

Now $|C_1| = 2$. Let $V(G) \backslash S = \{v_1, v_2, v_3\}$. We know that $N(x_1) \cap N(x_2) \cap V(G) \backslash S \ne \emptyset$, since $|N(x_1) \cap (V(G) \setminus S)| \ge 2$ and $|N(x_2) \cap (V(G) \backslash S)| \ge 2$. We may assume that $x_1 v_1, x_2 v_1 \in E(G)$. Every y_i is connected to at least one of v_2 and v_3. All we need to show in order to find a contradiction is that $G[y_1, y_2, y_3, y_4, y_5, v_2, v_3]$ is connected. If $v_2 v_3 \in E(G)$, this is clear. Otherwise, observe that $|C_i| = 1$ for $2 \le i \le 9$, and thus there is a y_j with $y_j v_2, y_j v_3 \in E(G)$.

Case 2.2.3. *Suppose* $|V(G)| = 11$.

Let $V(G) \backslash S = \{v_1, v_2, v_3, v_4\}$. If $N(x_1) \cap N(x_2) \cap (V(G) \backslash S) \ne \emptyset$, say $x_1 v_1$, $x_2 v_1 \in E(G)$, then $G[x_1, x_2, v_1]$ and $G[y_1, y_2, y_3, y_4, y_5, v_2, v_3, v_4]$ are connected subgraphs. Thus, suppose that $N(x_1) \cap N(x_2) \cap (V(G) \backslash S) = \emptyset$, say $N(x_1) = \{v_1, v_2\}$ and $N(x_2) = \{v_3, v_4\}$. Note that this implies that neither x_1 nor x_2 is in a

C_i by itself, so at least three of the $y_i's$ have at least three neighbors in $V(G)\backslash S$, at least two of the $y_i's$ are connected to all four vertices in $V(G)\backslash S$.

By symmetry, we may assume that $v_1v_3, v_1v_4, v_2v_3 \in E(G)$ (potentially $v_2v_4 \notin E(G)$). As there are at most two vertices in $\{y_1, y_2, y_3, y_4, y_5\}$ with less than three neighbors in $V(G) \setminus S$, we can pick $1 \leq j < k \leq 4$ such that $G[x_1, x_2, v_j, v_k]$ is connected, and such that every y_i has a neighbor in $\{v_1, v_2, v_3, v_4\}\backslash\{v_j, v_k\}$. But now $G[V(G) \setminus \{x_1, x_2, v_j, v_k\}]$ is connected, a contradiction.

Case 2.2.4. *Suppose $n = |V(G)| \geq 12$.*

Let $V(G)\backslash S = \{v_1, v_2, v_3, \ldots, v_{n-7}\}$. If $N(x_1) \cap N(x_2) \neq \emptyset$, say $x_1v_1, x_2v_1 \in E(G)$, then $G[x_1, x_2, v_1]$ and $G[y_1, y_2, y_3, y_4, y_5, v_2, v_3, \ldots, v_{n-7}]$ are connected subgraphs. Thus, suppose that $N(x_1) \cap N(x_2) = \emptyset$.

Suppose that $|N(x_1)| = |N(x_2)| = 2$, say $N(x_1) = \{v_1, v_2\}$ and $N(x_2) = \{v_3, v_4\}$. By symmetry, we may assume that $v_1v_3, v_1v_4, v_2v_3 \in E(G)$ (potentially $v_2v_4 \notin E(G)$). If every y_i has a neighbor in $\{v_1, v_2, v_3, \ldots, v_{n-7}\}\backslash\{v_1, v_3\}$, then $G[x_1, x_2, v_1, v_3]$ and $G[y_1, y_2, y_3, y_4, y_5, v_2, v_4, v_5, \ldots, v_{n-7}]$ are connected subgraphs. Therefore, there is an y_i with $N(y_i) = \{v_1, v_3\}$, say $i = 1$. Similarly, we may assume that $N(y_2) = \{v_1, v_4\}$ and $N(y_3) = \{v_2, v_3\}$. But now $(A = S \cup \{v_1, v_2, v_3, v_4\}, B = \{y_4, y_5, v_1, v_2, \ldots, v_{n-7}\})$ is an S^H-cut of size 6 in G^*, a contradiction.

Now suppose that $|N(x_1) \cup N(x_2)| \geq 5$, say $N(x_1) \supseteq \{v_1, v_2\}$ and $N(x_2) \supseteq \{v_3, v_4, v_5\}$. By symmetry, we may assume that $v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4 \in E(G)$ (potentially $v_2v_5 \notin E(G)$). By similar arguments as above, $N(y_1) = \{v_1, v_3\}$, $N(y_2) = \{v_1, v_4\}$, $N(y_3) = \{v_1, v_5\}$, $N(y_4) = \{v_2, v_3\}$, and $N(y_5) = \{v_2, v_4\}$. Further, we actually have $N(x_1) = \{v_1, v_2\}$ and $N(x_2) = \{v_3, v_4, v_5\}$.

If $k = 12$, then four of the C_i, consist of vertices in S, and hence $|N(u)| \geq 4$ for some $u \in S$, a contradiction. If $k > 12$, then $(A = S \cup \{v_1, v_2, v_3, v_4, v_5\}, B = \{v_1, v_2, \ldots, v_{n-7}\})$ is an S^H-cut of size 5 in G^*, a contradiction, completing the proof. ■

3. GRAPH SIZE AND MINORS

The center piece of studying graph minors is the following conjecture due to Hadwiger [4].

Conjecture 3.1. *For all $k \geq 1$, every k-chromatic graph has a K_k minor.*

For $k = 1, 2, 3$, it is easy to prove, and for $k = 4$, Hadwiger [4] and Dirac [3] proved it independently. In 1937, Wagner [22] proved that the case $k = 5$ is equivalent to the Four Color Theorem. Robertson, Seymour, and Thomas [18] proved that a minimal counterexample to the case $k = 6$ is a graph G, which has a vertex v such that $G\backslash\{v\}$ is planar. Hence, the case $k = 6$ of Hadwiger's conjecture holds. For $k = 7$, Kawarabayashi and Toft [11] proved that any 7-

chromatic graph has either K_7 or $K_{4,4}$ as a minor. Jakobsen [6] proved that every 7-chromatic graph has a K_7^{--} as a minor.

To study extremal graphs, for any positive integer k, let $g(k)$ be the least value such that every graph on n vertices and $g(k)n$ edges contains K_k as a minor. Mader [15] showed that $g(k)$ exists and is at most 2^{k-3}. In fact, Mader [14] proved that $g(k) \leq 8k\log_2(k)$ and that $g(k) = k-2$ for $k \leq 7$. Jørgensen [9] proved that every graph G with $||G|| \geq 6|G| - 20$ has K_8 as a minor or G is a special graph. We will prove Theorem 1.3 in this section. We first state the following related results.

Theorem 3.2 [14]. *For any $k \leq 7$, every graph with $|G| \geq k$ vertices and $||G|| \geq (k-2)|G| - (k-1)(k-2)/2 + 1$ contains K_k as a minor.*

Theorem 3.3 [6]. *Every graph G with $|G| \geq 7$ and $||G|| \geq 4|G| - 8$ contains K_7^{-2} as a minor.*

Theorem 3.4 [8]. *Every graph G with $|G| \geq 7$ and $||G|| \geq (9|G| - 23)/2$ contains K_7^- as a minor or a special graph with 8 vertices.*

Theorem 3.5 [7]. *Every graph G with $|G| \geq 8$ and $||G|| \geq 5|G| - 14$ has K_8^{-2} as a minor.*

Theorem 3.6 [9]. *Every graph G with $|G| \geq 8$ and $||G|| \geq 6|G| - 20$ has K_8 as a minor, unless G belongs to a special class of graphs with $||G|| = 6|G| - 20$ and $|G| = 5m$ for some integer $m \geq 2$.*

Let t be a positive integer and H be a graph. For any $A \subseteq V(H)$, let $DE(A)$ denote the set of edges of H dominated by A. Define

$$\gamma_t(H) = \max_{A \subseteq V(H)} \{|DE(A)| \ : \ |A| = t\}.$$

Clearly, $\gamma_1(H)$ is the maximum degree of H. Let $\overline{H}$ denote the complement of H and define that $\gamma'_t(H) = \gamma_t(\overline{H})$. A vertex set $S \subseteq N(v)$ is called a *v-saturated cut* if $S \cup \{v\}$ is a cut of G. A v-saturated cut S is *minimal* if there is no v-saturated cut, which is a proper subset of S.

A. Proof of Theorem 1.3

If $|G| = 9$, we have that $||G|| \geq 7 \times 9 - 29 = 34$, which implies that G is a K_9^{--}.

Assume that $|G| = n > 9$, and Theorem 1.3 is true for any graph of order less than n (but ≥ 9), and G does not have K_9^{--} as a minor. Let $\delta(G)$ denote the minimum degree of a graph G, v be a vertex of G such that $d(v) = \delta(G)$. Set $H = G[N(v)]$ and $h = |H| = d(v)$. Since G does not have K_9^{--} as a minor, no subgraph of G has K_9^{--} as a minor. In particular, $G' = G \setminus \{v\}$ does not have K_9^{--} as a minor. Thus, $||G'|| < 7|G'| - 29$, which implies that $\delta \geq 8$. On the other hand, if $\delta \geq 14$, then it is readily seen that $||G'|| \geq 7|G'| - 14$, thus G' has K_9^{--} as a minor and hence, so does G, a contradiction. Thus, we have that

$$8 \leq d(v) \leq 13.$$

Claim 3.1.1. *$\delta(H) \geq 7$ and $\delta(G) \geq 9$.*

Proof. Suppose to the contrary, there is a vertex $u \in N(v)$ such that $d_H(u) = |N(u) \cap N(v)| \leq 6$. Then, G/uv, the graph obtained from G by contracting the edge uv, has $|G| - 1$ vertices and

$$||G/uv|| \geq ||G|| - 7 \geq 7|G| - 29 - 7 = 7|G/xy| - 29.$$

By our assumption, G/uv has K_9^{--} as a minor and so does G, a contradiction. Since H is not K_8, the fact that $\delta(G) \geq 9$ is clear as $\delta(H) \geq 7$. ■

Claim 3.1.2. *$||H|| \leq 5h - 15$, and if $G \setminus N[v]$ is disconnected then there is a v-saturated cut B such that $B \neq N(v)$.*

Proof. Suppose the claim is false, then by Theorem 3.5, H has K_8^{--} as a minor. Thus, G has K_9^{--} as a minor since v is adjacent to every vertex of H, contrary to the assumption.

Now, suppose $N(v)$ is the only v-saturated cut. Then each vertex in $N(v)$ has a neighbor in every component of $G \setminus N[v]$ (and there are at least two such components). Since $\delta(H) \geq 7$ and $h = d(v) \leq 13$, we see that $||H|| \geq 4h - 8$. By Theorem 3.3, H has a K_7^{-2} as a minor, which implies G has a K_9^{--} as a minor, contrary to the assumption. ■

Claim 3.1.3. *We have that $h \geq 10$. Further, equality holds only if $G \setminus N[v]$ is disconnected and any neighbor of x and any neighbor of y are not in the same component for any two nonadjacent vertices x, $y \in N(v)$.*

Proof. By Claim 3.1.1, $||H|| \geq 7h/2$. Combining it with Claim 3.1.2, we have that

$$7h/2 \leq 5h - 15,$$

and thus, $h \geq 10$. Suppose $G \setminus N[v]$ is connected. Let $x, y \in N(v)$ be two nonadjacent vertices such that both are adjacent to the same component of $G \setminus N[v]$. Contracting this component to vertex x, we see that the resulting graph in H still cannot have K_8^{--} as a minor, otherwise G would have K_9^{--} as a minor. Hence, we have that

$$7h/2 + 1 \leq 5h - 15,$$

which implies that $h \geq 11$. ■

Claim 3.1.4. *Let B be a minimal v-saturated cut. Then,*

$$||G[B]|| \leq 6b - 24 - 2\gamma_1'(G[B]),$$

where $b = |G[B]|$.

Proof. Since $B \cup \{v\}$ is a cut of G, let G_1 and G_2 be two induced subgraphs of G such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = B \cup \{v\}$. By the minimality of B, we have that all vertices of B are adjacent to every component in $G \backslash (B \cup \{v\})$. Note that v may or may not have this property. Let x_1 be a vertex of B such that $d_{\overline{G[B]}}(x_1) = \gamma_1'(G[B])$. Contracting all components of $G_2 \backslash (B \cup \{v\})$ to x_1, we obtain a graph G_1^*. Clearly,

$$|G_1^*| = |G_1| \quad \textit{and} \quad ||G_1^*|| = ||G_1|| + \gamma_1'(G[B]).$$

Since G does not have a K_9^{--} as a minor, G_1^* does not have a K_9^{--} as a minor. Thus,

$$||G_1^*|| \leq 7|G_1^*| - 30.$$

Thus, we have that

$$||G_1|| \leq 7|G_1| - 30 - \gamma_1'(G[B]).$$

Similarly, we can show that

$$||G_2|| \leq 7|G_2| - 30 - \gamma_1'(G[B]).$$

Thus,

$$\begin{aligned} 7|G| - 29 &\leq ||G|| = ||G_1|| + ||G_2|| - ||G[B \cup \{v\}]|| \\ &\leq 7|G_1| - 30 - \gamma_1'(G[B]) + 7|G_2| - 30 - \gamma_1(G[B]) - ||G[B]|| - b \\ &= 7(|G| + b + 1) - 60 - 2\gamma_1'(G[B]) - ||G[B]|| - b \\ &= 7|G| + 6b - 53 - 2\gamma_1'(G[B]) - ||G[B]||. \end{aligned}$$

Thus, Claim 3.1.4 is proved. ■

Claim 3.1.5. *Let B be a graph induced by a minimal v-saturated cut. Then, $b = |B| \geq 5$ and $\gamma_2'(B) \geq 5$, with the exception that $b = 7$ or 8 and $\overline{B}$ is a 2-regular graph. In any case, we have that $\gamma_2'(B) \geq 4$ and $\gamma_3'(B) \geq 5$.*

Proof. The inequality $b \geq 5$ directly follows from Claim 3.1.4, since

$$0 \leq ||B|| \leq 6b - 24 - 2\gamma_1'(B).$$

Note that $\gamma_2'(B) \geq 5$ if $\gamma_1'(B) \geq 4$ and $||\overline{B}|| \geq 5$. By the fact that $||B|| + ||\overline{B}|| = b(b-1)/2$ and from Claim 3.1.4, we have that $||\overline{B}|| \geq 5$ if $\gamma_1'(B) \geq 4$. Thus, we assume that $\gamma_1'(B) \leq 3$.

Suppose that $\gamma'_1(B) = 3$ and $\gamma'_2(B) < 5$. Let x be the vertex such that $d_{\overline{B}}(x) = 3$. Then, the maximum degree of $\overline{B} \setminus \{x\}$ is at most 1. Thus,

$$||\overline{B}|| \leq 3 + (b-1)/2 \leq (b+5)/2.$$

Applying that $\gamma'_1(B) = 3$ to Claim 3.1.4, we have that

$$||\overline{B}|| = b(b-1)/2 - ||B|| \geq b(b-1)/2 - (6b - 24 - 6) \geq \frac{1}{2}(b^2 - 13b + 60).$$

However, the equation

$$(b+5)/2 \geq \frac{1}{2}(b^2 - 13b + 60)$$

does not have a solution. Thus, $\gamma'_1(B) \leq 2$.

Suppose that $b = 5$. In this case, we have that $||B|| + ||\overline{B}|| = 10$ and $||B|| \leq 6 - 2\gamma'_1(B) \leq 6$. Thus, $||\overline{B}|| \geq 4$, so $\gamma'_1(B) \geq 2$, which, in turn, implies that $||B|| \leq 2$. But then, $\gamma'_2(B) \geq 5$, proving the claim in this case.

Suppose now that $b = 6$. Then we have that $||B|| + ||\overline{B}|| = 15$ and $||B|| \leq 12 - 2\gamma'_1(B)$. Thus, $||\overline{B}|| \geq 3$ and so $\gamma'_1(B) \geq 1$. This, in turn, implies that $||B|| \leq 10$. Hence, $||\overline{B}|| \geq 5$, and so, $\gamma'_1(B) \geq 2$. This, in turn, implies that $||B|| \leq 8$. Now $||\overline{B}|| \geq 7$, which implies that $\gamma'_1(B) \geq 3$, a contradiction.

Since G does not have K_9^{--} as a minor, B does not contain K_7 as a subgraph. Thus, $\gamma'_1(B) \geq 1$ for $b \geq 7$.

Now suppose that $b = 7$. Then we have that $||B|| + ||\overline{B}|| = 21$ and $||B|| \leq 18 - 2\gamma'_1(B) \leq 16$. Thus, $||\overline{B}|| \geq 5$, so $\gamma'_1(B) \geq 2$, which, in turn, implies that $||B|| \leq 14$. Thus, $||\overline{B}|| \geq 7$. Since $\gamma'_1(B) \leq 2$ and $b = 7$, $\overline{B}$ is a 2-regular graph.

Suppose next that $b = 8$. Then $||B|| + ||\overline{B}|| = 28$ and $||B|| \leq 24 - 2\gamma'_1(B) \leq 22$, so that $||\overline{B}|| \geq 6$. Thus, $\gamma'_1(G) \geq 2$, which, in turn, implies that $||B|| \leq 20$. But since $\gamma'_1(B) \leq 2$ and $b = 8$, $\overline{B}$ is a 2-regular graph.

Now let D_1 and D_2 be two components of $G - (V(B) \cup \{v\})$ such that $D_2 \cap N(v) \neq \emptyset$.

If B has K_6 as a minor, contracting D_1 and D_2 along with using v yields a K_9^{--}. Thus, we may assume that B does not have K_6 as a minor. Using Theorem 3.2 for the case $k = 6$, we have that

$$||B|| \leq 4b - 10.$$

Suppose that $b \geq 9$. In this case, we have that

$$||\overline{B}|| \geq (b(b-1)/2) - 4b + 10 = (b-2)(b-9)/2 + 1 + b,$$

which implies $||\overline{B}|| \geq b + 1$. Hence, $\gamma'_1(B) \geq 3$ for $b \geq 9$, a contradiction. ■

Since H does not contain K_8^{--} as a minor, $||H|| \leq 5h - 15$. We define $\theta = 5h - 14 - ||H||$. Let $C_1, C_2, \ldots, C_m$ be the components of $G \setminus N[v]$ and $B_i = G[N(C_i) \cap N(v)]$ for each $i = 1, 2, \ldots, m$. Note that $B_i = B_j$ may happen for different i and j.

Claim 3.1.6.

$$\theta \leq \begin{cases} 4 & \textit{if } h = 10, 11, 12 \textit{ and}, \\ 5 & \textit{if } h = 13. \end{cases}$$

Further, the second equality holds only when all except one vertex in H have degree 7 and the exception has degree 8.

Proof. Since the minimum degree of H is at least 7, we have that $5h - 14 - \theta \geq ||H|| \geq \lceil 7h/2 \rceil$. It is readily seen that Claim 3.1.6 holds by solving the inequality. ■

Let $u \in N(v)$ such that $d_H(u) = 7$. Let $H^* = G[V(H) \cup \{v\}] \setminus \{u\}$. Then, $|H^*| = h$ and

$$||H^*|| \geq 7h/2 - 7 + h = 9h/2 - 7.$$

Using the fact $h \leq 13$, we see that $||H^*|| \geq 5h - 14$, which together with Theorem 3.5 implies that H^* contains K_8^{--} as a minor. Note, every vertex of H^* is either adjacent to u or to one of the C_i, since $d(v)$ is minimum degree of G. Now, since G does not have K_9^{--} as a minor, the following claim holds.

Claim 3.1.7. $m \geq 2$.

Claim 3.1.8. *There exists an i, $1 \leq i \leq m$ such that $\gamma_2'(B_i) < \theta$.*

Proof. Suppose, to the contrary, that $\gamma_2'(B_i) \geq \theta$ for all i. We now show that there exist a vertex x in B_1 and a vertex y in B_2 such that $|N_{\overline{B_1}}(x) \cup N_{\overline{B_2}}(y)| \geq \theta$. Let x_i and y_i be two vertices in B_i such that $\{x_i, y_i\}$ dominates at least θ edges in $\overline{B_i}$ for $i = 1, 2$. Then

$$|N_{\overline{B_i}}(x_i) \cup N_{\overline{B_i}}(y_i)| \geq \theta,$$

and without loss of generality, assume $d_{\overline{B_i}}(x_i) \geq d_{\overline{B_i}}(y_i)$. We may further assume that $d_{\overline{B_1}}(x_1) \geq d_{\overline{B_2}}(x_2)$. If $d_{\overline{B_1}}(x_1) > \theta/2$ or $x_1x_2 \notin E(\overline{B_1})$ or $x_1x_2 \notin E(\overline{B_2})$, then $x = x_1$ and $y = x_2$ are a pair of desired vertices. Thus,

$$d_{\overline{B_1}}(x_1) = d_{\overline{B_2}}(x_2) = \theta/2,$$

which give that

$$d_{\overline{B_1}}(y_1) = d_{\overline{B_2}}(y_2) = \theta/2.$$

In particular, we have that either $\theta = 2$ or $\theta = 4$, since $\theta \leq 5$. Further, we have $x_1x_2 \in E(\overline{B_1}) \cap E(\overline{B_2})$. Similarly, we have that x_1y_2, y_1x_2, and $y_1y_2 \in E(\overline{B_1}) \cap E(\overline{B_2})$. Thus, $\theta = 4$ and

$$N_{\overline{B_2}}(y_1) = N_{\overline{B_1}}(y_1).$$

Hence, $x = x_1$ and $y = y_1$ are a pair of desired vertices.

Now contracting C_1 to x and C_2 to y, we get a new subgraph H_1 from $G[V(H \cup C_1 \cup C_2)]$ such that $|H_1| = |N(v)|$ and $||H_1|| \geq 5|H_1| - 14$, since $||H|| \geq 5h - 14 - \theta$. Thus, H_1 has K_8^{--} as a minor. This minor along with v shows that G has K_9^{--} as a minor, a contradiction. ∎

Combining Claims 3.1.5 and 3.1.8, we have the following: $4 \leq \gamma_2'(B_i) < \theta$ for some i. Thus, $\theta = 5$ and then by Claim 3.1.6 we obtain the following.

Claim 3.1.9. $h = d(v) = 13$ *and* $||H|| = (5h - 14) - 5$. *In particular, all vertices of H have degree* 7 *except one which has degree* 8.

Using Claim 3.1.5, we see that $\gamma_3'(B_i) \geq 5$. If $m \geq 3$, using an argument similar to before it is straightforward to show that there are vertices x_i in B_i $(i = 1, 2, 3)$ such that

$$|N_{\overline{B_1}}(x_1) \cup N_{\overline{B_2}}(x_2) \cup N_{\overline{B_3}}(x_3)| \geq 5.$$

Contracting C_i to x_i for $i = 1, 2, 3$ again produces a K_8^{--} minor in H from $G[V(H \cup C_1 \cup C_2 \cup C_3)]$, a contradiction. Thus we obtain the following.

Claim 3.1.10. $m = 2$.

Let B_i^* be a graph induced by a minimal v-saturated cut with $V(B_i^*) \subseteq V(B_i)$ for $i = 1,\ 2$. By Claim 3.1.5 and without loss of generality, assume that $\gamma_2'(B_1^*) = 4 < \theta = 5$. Hence, $7 \leq |B_1^*| \leq 8$ and $\overline{B_1^*}$ is a 2-regular graph.

Claim 3.1.11. $\gamma_2'(B_2^*) = 4$.

Proof. Suppose to the contrary that $\gamma_2'(B_2^*) \geq 5$. Then there exists $x_2 \in V(B_2^*)$ such that $d_{\overline{B_2^*}}(x_2) \geq 3$. Since $\overline{B_1^*}$ is 2-regular, there exists $x_1 \in V(B_1^*)$ such that $x_1x_2 \notin E(\overline{B_1^*})$. Now contracting C_1 to x_1 and C_2 to x_2, we again gain at least 5 edges. Then, as before, K_8^{--} would be a minor of H, a contradiction completing the proof of the claim. ∎

By Claims 3.1.5 and 3.1.11, $7 \leq |B_2^*| \leq 8$ and $\overline{B_2^*}$ is 2-regular.

Claim 3.1.12. $|V(B_1^*) \cap V(B_2^*)| = 1$, $|B_1^*| = |B_2^*| = 7$, $B_1^* = B_1$, *and* $B_2^* = B_2$.

Proof. Since $|B_1^*| \geq 7$ and $|B_2^*| \geq 7$ and $|V(B_1^*) \cup V(B_2^*)| \leq 13$, we have that $|V(B_1^*) \cap V(B_2^*)| \geq 1$. Suppose $|V(B_1^*) \cap V(B_2^*)| \geq 2$. Since all vertices in H have degree 7 except one, which has degree 8, there is a vertex $x \in V(B_1^*) \cap V(B_2^*)$

such that $d_H(x) = 7$. Then $d_{\overline{H}}(x) = 5$ as $h = 13$. Without loss of generality, assume $d_{\overline{B_1}}(x) \geq 3$. Since $\overline{B_2^*}$ is 2-regular and $|\overline{B_2^*}| \geq 7$, let $y \in \overline{B_2^*}$ such that y is not adjacent to x in $\overline{B_2}$. As before, contracting C_1 to x and C_2 to y leads to a contradiction.

The statement of $|B_1^*| = |B_2^*| = 7$ directly follows from the fact that $|V(B_1^*) \cap V(B_2^*)| = 1$ and $|B_1^* \cup B_2^*| \leq 13$. Further, $V(B_1^*) \cup V(B_2^*) = N(v)$. Let w be the vertex in $V(B_1^*) \cap V(B_2^*)$. Since $\overline{B_2^*}$ is 2-regular, B_2^* is 4-regular of order 7, hence hamiltonian. Therefore, $B_2^* \backslash \{w\}$ is connected. Thus, $N(C_1) \cap (V(B_2^*) \backslash \{w\}) = \emptyset$, for otherwise $G \backslash (V(B_1^*) \cup \{v\})$ is connected, a contradiction to the fact that B_1^* is a v-saturated cut. Thus, $B_1^* = B_1$. Similarly, $B_2^* = B_2$. ■

Let $x_1 \in V(B_1) \backslash V(B_2)$. Since $|V(B_1) \cup V(B_2)| \leq 13$ and $|B_1| = |B_2| = 7$, we see that $N(v) = V(B_1) \cup V(B_2)$. Since x_1 is adjacent to 4 vertices in B_1, we have $|N(x_1) \cap (V(B_2) \backslash \{w\})| = 3$. Let $y_1 \in V(B_2) \setminus \{w\}$ such that $x_1 y_1 \in E(G)$. Then, since $d_H(x_1) = 7$, we have that

$$|N(x_1) \cap (V(B_2) \backslash \{y_1, w\})| \leq 2.$$

Similarly, $|N(y_1) \cap (V(B_1) \backslash \{x_1, w\})| \leq 2$. Thus, $|(N_H(x_1) \cap N_H(y_1)) \backslash \{w\}| \leq 4$, and so $|N(x_1) \cap N(y_1) \cap N[v]| \leq 6$. Since $m = 2$, $N(x_1) \cap N(y_1) \cap (V(G) \backslash N[v]) = \emptyset$. Thus, $|N(x_1) \cap N(y_1)| \leq 6$. Now, as in the proof of Claim 3.1.1, G/x_1y_1 would contain a K_9^{--} minor, a contradiction, completing the proof. ■

Finally, we note that a similar proof technique can be used to show that a graph of order $n \geq 9$ with size at least $9n - 45$ contains a K_9 minor. Despite the fact this is not near the conjectured value, when combined with Theorem 1.1 it implies that 18-connected graphs are 3-linked.

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REFERENCES

[1] B. Bollobás and A. Thomason, Highly linked graphs, Combinatorica 16 (1996), 313–320.

[2] G. Chen, R. J. Gould, K. Kawarabayashi, F. Pfender, and B. Wei, On graph linkage problems, preprint.

[3] G. A. Dirac, Trennende Knotenpunktmengen und Reduzibilität abstrakter Graphen mit Anwendung auf das Vierfarbenproblem, J Reine Angew Math 204 (1960), 116–131.

[4] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrsschr Naturforsch Ges Zürich 88 (1943), 133–142.

[5] A. Huck, A sufficient condition for graphs to be weakly k-linked, Graphs and Combinatorics 7 (1991), 323–351.

[6] I. T. Jakobsen, A homomorphism theorem with an application to the conjecture of Hadwiger, Studia Sci Math Hung 6 (1971), 151–160.

[7] I. T. Jakobsen, On certain homomorphism properties of graphs I, Math Scand 31 (1972), 379–404.

[8] I. T. Jakobsen, On certain homomorphism properties of graphs II, Math Scand 52(2) (1983), 229–261.

[9] L. K. Jørgensen, Contractions to K_8, J Graph Theory 18(5) (1994), 431–448.

[10] H. A. Jung, Eine Verallgemeinerung des n-fachen zusammenhangs fur Graphen, Math Ann 187 (1970), 95–103.

[11] K. Kawarabayashi and B. Toft, Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, Combinatorica, accepted.

[12] D. G. Larman and P. Mani, On the existence of certain configurations within graphs and the 1-skeleons of polytopes. Proc Londo Math Soc 20 (1974), 144–160.

[13] W. Mader, Hinreichende Bedingungen fur die Existenz von Teilgraphen, die zu einem vollstämdogem Graphen homöomorph sind, Math Nachr 53 (1972), 145–150.

[14] W. Mader, Homomorphiesätze für Graphen, Math Ann 178 (1968), 154–168.

[15] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math Annalen 174 (1967), 265–268.

[16] N. Robertson and K. Chakravarti, Covering three edges with a bond in a nonseparable graph (Abstract), Combinatorics, 1979, Annals of Discrete Math. 8, M. Deza and I.G. Rosenberg, (Editors), North Holland, Amsterdam, 1980, 247p.

[17] N. Robertson and P. Seymour, Graph minors XIII, the disjoint paths problem, J Combin Theory Ser B 63(1) (1995), 65–110.

[18] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, Combinatorica 12 (1993), 279–361.

[19] P. Seymour, Disjoint paths in graphs, Discrete Math 29 (1980), 293–309.

[20] R. Thomas, private communication.

[21] C. Thomassen, 2-linked graphs, Eur J Combinatorics 1 (1980), 371–378.

[22] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math Ann 114 (1937), 570–590.

[23] X. Yu, private comminucation.