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LOCALLY SEMICOMPLETE DIGRAPHS WITH A FACTOR COMPOSED OF k CYCLES

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ABSTRACT. A digraph is locally semicomplete if for every vertex x, the set of in-neighbors as well as the set of out-neighbors of x induce semicomplete digraphs. Let D be a k-connected locally semicomplete digraph with $k \geq 3$ and $\overline{g}$ denote the length of a longest induced cycle of D. It is shown that if D has at least $7(k-1)\overline{g}$ vertices, then D has a factor composed of k cycles; furthermore, if D is semicomplete and with at least $5k+1$ vertices, then D has a factor composed of k cycles and one of the cycles is of length at most 5. Our results generalize those of [3] for tournaments to locally semicomplete digraphs.

1. Introduction

A subdigraph of a digraph D is called a *factor* if it contains all vertices of D. If a factor of D is composed of k vertex-disjoint cycles and each of the k cycles is of length at least 3, then we say that it is a *k-cycles-factor* or a *factor composed of k cycles.* Two cycles in a 2-cycles-factor of D are called *complementary cycles* in D.

Reid [8] proved that every 2-connected tournament on $n \geq 6$ vertices contains two complementary cycles of lengths 3 and $n-3$ respectively, unless it is isomorphic to a tournament on 7 vertices which contains no transitive subtournament on 4 vertices. With this statement as the basic

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step, Song [9] proved by induction that with the same exception, every 2-connected tournament on $n \geq 6$ vertices contains complementary cycles of all lengths k and $n-k$ for $k = 3, 4, \ldots, n-3$.

For the general case, Bollobás posed the following problem:

PROBLEM 1.1. *If k is a positive integer, what is the least integer $f(k)$ so that all but a finite number of $f(k)$-connected tournaments have a k-cycles-factor ?*

Recently, Chen, Gould and Li [3] proved that every k-connected tournament with at least $8k$ vertices has a k-cycles-factor.

In 1990, Bang-Jensen [1] introduced a very interesting generalization of tournaments – the class of locally semicomplete digraphs. A digraph is *semicomplete* if for any two distinct vertices, there is at least one arc between them. A digraph is *locally semicomplete* if for every vertex x, the set of in-neighbors as well as the set of out-neighbors of x induce semicomplete digraphs. A locally semicomplete digraph without a cycle of length 2 is called a *local tournament.*

It is clear that the class of locally semicomplete digraphs is a superclass of that of tournaments. The results about complementary cycles in 2-connected tournaments have been completely generalized to locally semicomplete digraphs in [5] and [6], respectively.

In [4], a similar problem to Problem 1.1 was posted for locally semicomplete digraphs, and another problem, similar to that of Song [9] for tournaments, is the following:

PROBLEM 1.2 ([4]). *Let k be a positive integer. What is the least integer $h(k)$ such that all but a finite number of $h(k)$-connected locally semicomplete digraphs D have a factor composed of k cycles of lengths $n_1, n_2, \ldots, n_k$ respectively, where $n_1, n_2, \ldots, n_k$ are any k integers each of which is not less than the length of a longest induced cycle of D ?*

Problem 1.2 has been completely solved in [6] for $k = 2$ and it was shown that $h(2) = 2$. As yet we have not seen any results about the general case for Problem 1.2. In this paper, we solve Problem 1.2 completely for a special case, when a locally semicomplete digraph is round-decomposable (see Corollary 4.2), and confirm the existence of a k-cycles-factor in some k-connected locally semicomplete digraphs. In particular, we show that every k-connected ($k \geq 2$) semicomplete digraph with at least $5k+1$ vertices has a factor composed of k cycles such that one of which is of length at most 5. Our results generalize and improve that of [3] for tournaments (see Corollary 4.8).

2. Terminology and preliminaries

We denote by $V(D)$ and $E(D)$ the vertex set and the arc set of a digraph D, respectively. The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$. In addition, $D - A = D\langle V(D) - A\rangle$.

If xy is an arc of D, then we say that x *dominates* y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \rightarrow B$. In addition, if $A \rightarrow B$, but there is no arc from B to A, then we say that A *strictly dominates* B, denoted by $A \Rightarrow B$.

The *outset* of a vertex $x \in V(D)$ is the set $N^+(x) = \{ y \mid xy \in E(D)\}$. Similarly, $N^-(x) = \{ y \mid yx \in E(D)\}$ is the *inset* of x. More generally, for a subdigraph A of D, we define its *outset* by $N^+(A) = \bigcup_{x\in V(A)} N^+(x) - A$ and its *inset* by $N^-(A) = \bigcup_{x\in V(A)} N^-(x) - A$. Every vertex of $N^+(A)$ is called an *out-neighbor* of A and every vertex of $N^-(A)$ is an *in-neighbor* of A.

The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *out-degree* and *indegree* of $x \in V(D)$, respectively. If $d^+(x) = d^-(x) = r$ holds for every vertex x of D, then we say that D is *r-regular*.

Paths and cycles in a digraph are always assumed to be directed. A cycle of length ℓ is called an ℓ-cycle. A digraph is said to be *connected*, if its underlying graph is connected.

A *strong component* H of D is a maximal subdigraph such that for any two vertices $x, y \in V(H)$, the subdigraph H contains a path from x to y and a path from y to x. The digraph D is *strong* or *strongly connected*, if it has only one strong component, and D is k-connected if $|V(D)| \geq k+1$ and for any set A of at most $k-1$ vertices, the subdigraph $D - A$ is strong.

If D is strong and S is a subset of $V(D)$ such that $D-S$ is not strong, then we say that S is a *separating set* of D. A separating set S of D is *minimal* if for any proper subset S' of S, the subdigraph $D - S'$ is strong.

Let R be a digraph on r vertices $v_1, v_2, \ldots, v_r$ and let $L_1, \ldots, L_r$ be a collection of digraphs. Then $R[L_1, \ldots, L_r]$ is the new digraph obtained from R by replacing each vertex v_i of R with L_i and adding an arc from every vertex of L_i to every vertex of L_j if and only if v_iv_j is an arc of R $(1 \leq i \neq j \leq r)$. Note that if we have $D = R[L_1, \ldots, L_r]$, then $R, L_1, \ldots, L_r$ are subdigraphs of D.

A digraph on n vertices is *round* if we can label its vertices $v_0, v_1, \ldots, v_{n-1}$ so that for each i, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \ldots, v_{i-1}\}$ (modulo n).

A locally semicomplete digraph D is *round-decomposable* if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[S_1, \ldots, S_r]$, where each S_i is a strong semicomplete subdigraph or a single vertex of D. We call $R[S_1, \ldots, S_r]$ a *round decomposition* of D.

3. Structure of locally semicomplete digraphs

We begin with the structure of non-strong locally semicomplete digraphs.

THEOREM 3.1 ([1]). *Let D be a connected locally semicomplete digraph that is not strong. Then the following holds:*

(a) *If A and B are two strong components of D, then either there is no arc between them or $A \Rightarrow B$ or $B \Rightarrow A$.*
(b) *If A and B are two strong components of D such that A dominates B, then A and B are both semicomplete digraphs.*
(c) *The strong components of D can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.*

The unique sequence $D_1, D_2, \ldots, D_p$ of the strong components of D in Theorem 3.1 (c) is called the *strong decomposition* of D with *initial* component D_1 and *terminal* component D_p.

THEOREM 3.2 ([5]). *Let D be a connected locally semicomplete digraph that is not strong and let $D_1, \ldots, D_p$ be the strong decomposition of D. Then D can be decomposed in $r \geq 2$ subdigraphs $D'_1, D'_2, \ldots, D'_r$ as follows:*

$$D'_1 = D_p, \quad \lambda_1 = p,$$
$$\lambda_{i+1} = \min\{\, j \mid N^+(D_j) \cap V(D'_i) \neq \emptyset\},$$

and

$$D'_{i+1} = D\langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \cdots \cup V(D_{\lambda_i - 1})\rangle.$$

Furthermore, the subdigraphs $D'_1, D'_2, \ldots, D'_r$ satisfy the following:

(a) *D'_i consists of some strong components of D and it is semicomplete for $i = 1, 2, \ldots, r$;*
(b) *D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, \ldots, r-1$;*

(c) *if* $r \geq 3$, *then there is no arc between* D'_i *and* D'_j *for* i, j *satisfying* $|j - i| \geq 2$.

The unique sequence $D'_1, D'_2, \ldots, D'_r$ defined in Theorem 3.2 is called the *semicomplete decomposition* of D.

The following classification of locally semicomplete digraphs was given in [2].

THEOREM 3.3 ([2]). *Let* D *be a connected locally semicomplete digraph. Then exactly one of the following possibilities holds.*

(a) D *is round-decomposable and it hat a unique round decomposition* $R[D_1, D_2, \ldots, D_\alpha]$, *where* R *is a round local tournament on* $\alpha \geq 2$ *vertices and* D_i *is a strong semicomplete digraph for* $i = 1, 2, \ldots, \alpha$;
(b) D *is not round-decomposable and not semicomplete.*
(c) D *is a semicomplete digraph which is not round-decomposable.*

PROPOSITION 3.4 ([2]). *Let* $R[H_1, H_2, \ldots, H_\alpha]$ *be a round decomposition of a strong locally semicomplete digraph* D. *Then, for every minimal separating set* S *of* D, *there are two integers* i *and* $k \geq 0$ *such that* $S = V(H_i) \cup \cdots \cup V(H_{i+k})$.

The following lemma is a partial result of Lemma 3.5 from [2] for locally semicomplete digraphs that are not round-decomposable.

LEMMA 3.5. *If a strong locally semicomplete digraph* D *is not semicomplete and not round-decomposable, then there exists a minimal separating set* $S \subset V(D)$ *such that* $D - S$ *is not semicomplete. Furthermore, if* $D_1, D_2, \ldots, D_p$ *is the strong decomposition and* $D'_1, D'_2, \ldots, D'_r$ *is the semicomplete decomposition of* $D - S$, *then* $r = 3$, $D\langle S\rangle$ *is semicomplete and we have* $D_p \Rightarrow S \Rightarrow D_1$.

DEFINITION 3.6. Let D be a strongly connected locally semicomplete digraph. The *quasi-girth* $g(D)$ (or g if no confusion can arise) of D is defined as follows: If D is round-decomposable and it has a round decomposition $D = R[D_1, D_2, \ldots, D_\alpha]$, then $g(D)$ is the length of a shortest cycle in R; if D is not round-decomposable, then $g(D) = 3$.

We denote the length of a longest induced cycle of D by $\overline{g}(D)$, or $\overline{g}$ if no confusion can arise.

REMARK 3.7. It is not difficult to check that $\overline{g} \leq 2g + 1$ holds and every shortest cycle through a given vertex is of length at most $\overline{g} + 1$ for every strongly connected locally semicomplete digraph.

LEMMA 3.8 ([4]). *Let D be a strongly connected locally semicomplete digraph that is not round-decomposable. Then every induced cycle of D has a length at most 4, i.e., $\overline{g}(D) \leq 4$.*

We end this section with the well-known theorem of Moon.

THEOREM 3.9 ([7]). *Every vertex of a strongly connected semicomplete digraph on $n \geq 3$ vertices is in a t-cycle for $t = 3, 4, \ldots, n$.*

4. Main results

We confirm at first the existence of a k-cycles-factor in locally semicomplete digraphs that are round-decomposable.

THEOREM 4.1. *Let D be a round-decomposable, k-connected locally semicomplete digraph with $n \geq 2(k-1)g$ vertices. Then D contains a g-cycle C such that $D - V(C)$ is $(k-1)$-connected.*

Proof. Let $R[H_1, H_2, \ldots, H_\alpha]$ be a round decomposition of D. We denote by $\mathcal{C}$ the set of all g-cycles in R and for every $C \in \mathcal{C}$, put

$$f(C) := \left| \left\{ S \;\middle|\; \begin{array}{l} S \text{ is a minimum separating set} \\ \text{of } D \text{ with } |S - V(C)| \leq k-2 \end{array} \right\} \right|.$$

We choose an element $C_1 = v_1 v_2 \cdots v_g v_1$ from $\mathcal{C}$ such that $f(C_1) = \min\{f(C) | C \in \mathcal{C}\}$. If $f(C_1) = 0$, then $D - V(C_1)$ is $(k-1)$-connected and we are done. So, we assume that $f(C_1) \geq 1$. Let S_1 be a minimum separating set of D such that $|S_1 - V(C_1)| \leq k-2$. Since D is k-connected, S_1 contains at least 2 vertices of C_1. By Proposition 3.4, we may assume without loss of generality that $S_1 = V(H_1) \cup V(H_2) \cup \cdots \cup V(H_t)$. Of course, $H_\alpha \Rightarrow S_1 \Rightarrow H_{t+1}$ and $D\langle S_1 \rangle$ is semicomplete. Since C_1 is an induced cycle in R, S_1 contains exactly two vertices from C_1 which are adjacent in C_1. This implies that D is not $(k+1)$-connected, and hence, every minimum separating set of D contains exactly k vertices.

Let $v_i \in V(H_{\beta_i})$ for $i = 1, 2, \ldots, g$ and assume without loss of generality that v_1 and v_2 are in S_1, this means that $1 \leq \beta_1 < \beta_2 \leq t$.

It is clear that there is an integer $\beta_2' > \beta_2$ such that

$$\sum_{j=\beta_1+1}^{\beta_2'-1} |V(H_j)| \leq k-1, \quad \text{but} \quad \sum_{j=\beta_1+1}^{\beta_2'} |V(H_j)| \geq k.$$

Since D is k-connected, we have $v_1 \to H_{\beta_2'}$ and $\beta_2' < \beta_3$. Let v_2' be a vertex of $H_{\beta_2'}$. Clearly, $v_2' \to v_3$. So, the new cycle $C_2 = v_1 v_2' v_3 \cdots v_g v_1$ also belongs to $\mathcal{C}$. Since v_1 and v_2' can not belong to a common minimum separating set of D, it is easy to see that if D has no minimum separating set containing the two vertices v_2' and v_3, then $f(C_2) < f(C_1)$, a contradiction to the choice of C_1. Hence, there is a minimum separating set S_2 with $\{v_2', v_3\} \subseteq S_2$. It follows that $|V(H_{\beta_2'})| \leq k-1$.

In general, if we have a g-cycle $C_i = v_1 v_2' \cdots v_i' v_{i+1} \cdots v_g v_1$ from $\mathcal{C}$ such that v_i' and v_{i+1} must belong to a common minimum separating set of D, then we can consider another g-cycle $C_{i+1} = v_1 v_2' \cdots v_i' v_{i+1}' v_{i+2} \cdots v_g v_1$ from $\mathcal{C}$ with $v_{i+1}' \in H_{\beta_{i+1}'}$, where β_{i+1}' is an integer satisfying

$$\sum_{j=\beta_i'+1}^{\beta_{i+1}'-1} |V(H_j)| \leq k-1, \quad |V(H_{\beta_{i+1}'})| \leq k-1 \quad \text{and} \quad \sum_{j=\beta_i'+1}^{\beta_{i+1}'} |V(H_j)| \geq k.$$

Finally, we consider C_g from $\mathcal{C}$. Let $\beta_1' = \beta_1$. It is easy to see that

$$\begin{aligned}
\sum_{\ell=\beta_1'+1}^{\beta_g'} |V(H_\ell)| &= \sum_{i=1}^{g-1} \left[\left(\sum_{j=\beta_i'+1}^{\beta_{i+1}'-1} |V(H_j)| \right) + |V(H_{\beta_{i+1}'})| \right] \\
&\leq \sum_{i=1}^{g-1} [(k-1) + (k-1)] = 2(k-1)(g-1).
\end{aligned}$$

Since D has at least $2(k-1)g$ vertices, we have

$$\begin{aligned}
\sum_{j=\beta_g'+1}^{\alpha} |V(H_j)| + \sum_{i=1}^{\beta_1'} |V(H_i)| &= |V(D)| - \sum_{\ell=\beta_1'+1}^{\beta_g'} |V(H_\ell)| \\
&\geq 2(k-1)g - 2(k-1)(g-1) \\
&= 2(k-1) \geq k
\end{aligned}$$

for $k \geq 2$. It follows that there is no minimum separating set of D which contains both v_g' and v_1. Therefore, $f(C_g) = 0$, a contradiction. □

Note that a 2-regular, round local tournament with $n = 2(m+1) - 1$ vertices is 2-connected, but it has no cycle whose removal leaves a strongly connected digraph, since a shortest cycle in it is of length $m+1$. So, the condition in Theorem 4.1 is best possible in some sense. As an immediate consequence we have the following result:

Corollary 4.2. *Let D be a round-decomposable, k-connected locally semicomplete digraph on $n \geq 2(k-1)\overline{g}$ vertices. Then, for any k*

integers $n_1, n_2, \ldots, n_k \geq \overline{g}$ *with* $n_1 + n_2 + \cdots + n_k = n$, D *has a factor composed of* k *cycles* $C_1, C_2, \ldots, C_k$ *such that* C_i *is of length* n_i *for* $i = 1, 2, \ldots, k$.

Proof. Let $R_1[H_1^1, H_2^1, \ldots, H_{\alpha_1}^1]$ be a round decomposition of $D_1 = D$. By Theorem 4.1, R_1 contains a $g(D_1)$-cycle C_1' such that $D_2 = D_1 - V(C_1')$ is $(k-1)$-connected. Clearly, D_2 is round-decomposable.

Let $R_2[H_1^2, H_2^2, \ldots, H_{\alpha_2}^2]$ be a round decomposition of D_2. It is easy to see that $g(D_2) \leq \overline{g}(D_1)$. By Theorem 4.1 again, we get a $g(D_2)$-cycle C_2' such that $D_2 - V(C_2')$ is $(k-2)$-connected. Note that C_2' is a cycle in R_1.

Successively, we obtain k cycles $C_1', C_2', \ldots, C_k'$, each of which is a cycle in R_1 and of length at most $\overline{g}(D)$. Since every vertex in D has at least one positive and one negative neighbor in C_i' for all $i = 1, 2, \ldots, k$, any $n_i - |V(C_i')|$ vertices in $D - (V(C_1') \cup \cdots \cup V(C_k'))$ can be inserted into C_i' to form an n_i-cycle C_i for all $i \in \{1, 2, \ldots, k\}$. Thus, D contains a required factor. □

In the following, we confirm the existence of a k-cycles-factor in locally semicomplete digraphs that are not round-decomposable.

THEOREM 4.3. *Let* D *be a* k*-connected locally semicomplete digraph with* $n \geq 20(k-1)$ *vertices that is not round-decomposable. If* D *is not semicomplete, then it has a factor composed of* k *cycles, and at least* $(k-2)$ *of them are of length at most* 4.

Proof. Since D is not semicomplete, it has the properties as described in Lemma 3.5. Let S be a minimal separating set of D such that $D - S$ is not semicomplete, and let $D_1, D_2, \ldots, D_p$ be the strong decomposition and D_1', D_2', D_3' be the semicomplete decomposition of $D - S$, respectively. We denote the initial component of D_2' by D_λ.

Since D_2' also contains a minimal separating set S' such that $D - S'$ is not semicomplete, we may assume that S has been chosen such that $|S| \leq |V(D_2')|$. In addition, we assume that $|V(D_1)| \geq |V(D_p)|$ (otherwise, we consider the converse digraph of D, which is obtained by replacing every arc xy of D with yx).

Claim (*) D contains k vertex-disjoint cycles $C_1, C_2, \ldots, C_k$, each of which is of length at least 3, and C_1, C_2 satisfy one of the following conditions:

1) $V(C_1) \cap V(D_i') \neq \emptyset$ for $i = 1, 2, 3$;
2) $V(C_1) \cap V(D_i') \neq \emptyset$ for $i = 2, 3$, $V(C_1) \cap V(D_1') = \emptyset$, and $V(C_2) \cap V(D_j') \neq \emptyset$ for $i = 1, 2$, $V(C_2) \cap V(D_3') = \emptyset$.

Proof. At first, we consider the case when $|V(D_1)| \geq k$. Since D is k-connected, there are k vertex-disjoint paths from D_1 to S. Because of $S \Rightarrow D_1$, such k paths and the arcs from S to D_1 constitute k vertex-disjoint cycles $C'_1, C'_2, \dots, C'_k$, each of which is of length at least 3. If none of them contains a vertex from D'_1, then it is easy to see that we can insert a vertex of D'_1 into C'_1. So, we may assume without loss of generality that C'_1 contains a vertex from D'_1. Clearly, $C'_1, C'_2, \dots, C'_k$ are k required cycles with respect to 1).

Now we consider the case when $|V(D_1)| < k$. Since $D_1 \Rightarrow D_\lambda \Rightarrow D_p \Rightarrow S \Rightarrow D_1$, D contains a cycle $C'_1 = d_3d_2d_1sd_3$ with $s \in S$ and $d_i \in V(D'_i)$ for $i = 1, 2, 3$. Let $C'_2, \dots, C'_\alpha$ be a maximal selection of cycles of length at least 3 such that $C'_1, C'_2, \dots, C'_\alpha$ are vertex-disjoint.

If $\alpha \geq k$, then $C'_1, C'_2, \dots, C'_k$ are the required cycles with respect to 1).

Suppose now that $\alpha < k$. Since D'_2 and D'_3 both are semicomplete, the two subdigraphs

$$\begin{aligned} D''_3 &= D'_3 - (V(C'_1) \cup V(C'_2) \cup V(C'_3) \cup \dots \cup V(C'_\alpha)) \\ \text{and} \quad D''_2 &= D'_2 - (V(C'_1) \cup V(C'_2) \cup V(C'_3) \cup \dots \cup V(C'_\alpha)) \end{aligned}$$

are semicomplete and do not contain any cycle of length more than 2. Since $n \geq 20(k-1)$, $|V(D_1)| < k$, $|V(D_p)| < k$, $|S| \leq |V(D'_2)|$ and by Lemma 3.8, $|V(C'_i)| \leq 4$ for $i = 1, 2, \dots, \alpha$, we have $|V(D''_\ell)| \geq 2k+2$ for $\ell = 2$ or $\ell = 3$. It is easy to see that D''_ℓ contains $2k$ vertices $x_1, x_2, \dots, x_k$ and $y_1, y_2, \dots, y_k$ such that $\{x_1, x_2, \dots, x_k\} \Rightarrow \{y_1, y_2, \dots, y_k\}$. Since D is k-connected, there are k vertex-disjoint paths from $\{y_1, y_2, \dots, y_k\}$ to the set $\{x_1, x_2, \dots, x_k\}$, say P_i from y_i to x_{m_i} for $i = 1, 2, \dots, k$, where $m_1, m_2, \dots, m_k$ is a permutation of $1, 2, \dots, k$. Thus, every path P_i together with the arc $x_{m_i}y_i$ forms a cycle, denoted by C''_i, which is of length at least 3, for $i = 1, 2, \dots, k$.

If there is a cycle, say C''_1, that contains vertices not only from D'_1, but also from D'_3, then we see that $C''_1, C''_2, \dots, C''_k$ are the required cycles with respect to 1).

If at most one of $\{P_1, P_2, \dots, P_k\}$, say P_1 if there is one, contains some vertices of C'_1, then $C'_1, C''_2, C''_3, \dots, C''_k$ are k required cycles with respect to 1).

So, we assume without loss of generality that $V(P_i) \cap V(C'_1) \neq \emptyset$ for $i = 1, 2, \dots, \gamma$, but $V(P_j) \cap V(C'_1) = \emptyset$ for $j > \gamma$, where $\gamma \geq 2$, and furthermore, $V(P_i) \cap V(D'_1) = \emptyset$ or $V(P_i) \cap V(D'_3) = \emptyset$ for every $i \in \{1, 2, \dots, k\}$. Since C'_1 is of length 4, we have $\gamma \leq 4$.

If $\ell = 3$, then, by the assumption above, we have $V(P_i) \cap V(D'_1) = \emptyset$ for all $i \in \{1, 2, \ldots, k\}$. Since P_1 must contain some vertices from D'_2 and S, every vertex of D'_1 can be inserted into C''_1 to form a cycle that satisfies Condition 1).

Let $\ell = 2$. Suppose that there is a path, say P_1, containing d_3. Then P_1 contains at least one vertex from S. If there is another path, say P_2, containing d_1, then C''_1 and C''_2 satisfy Condition 2); if d_1 does not belong to any paths P_i for $i = 1, 2, \ldots, k$, then d_1 can be inserted into C''_1 and we get a cycle satisfying Condition 1). Suppose now that none of $\{P_1, \ldots, P_\gamma\}$ contains d_3, but there is one, say P_1, containing s, then it is easy to see that d_3 can be inserted into P_1, and we are done as above. In the remaining case, we see that there is a path, say P_2, containing d_1. Since P_2 contains at least one vertex from S, it is easy to check that d_3 can be inserted into C''_2 and we get a cycle satisfying Condition 1). □

Let $C_1, C_2, \ldots, C_k$ be k vertex-disjoint cycles, which have the properties as described in Claim (*) above and whose total length is minimal. By Lemma 3.8, we have $3 \le |V(C_i)| \le 4$ for $i = 2, 3, \ldots, k$ with respect to 1) and for $i = 3, 4, \ldots, k$ with respect to 2).

Let $Q = D - (V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k))$ and $Q_i = Q \cap D'_i$ for $i = 1, 2, 3$ and $Q_4 = Q \cap D\langle S \rangle$. Note that if Q_i is not empty, then it is semicomplete, and hence, has a hamiltonian path for $i = 1, 2, 3, 4$.

If C_1 satisfies Condition 1) in Claim (*), i.e. it contains vertices from D'_1 and from D'_3, then it is easy to see that every vertex of Q has a positive and a negative neighbor in C_1. It follows that $D\langle V(C_1) \cup V(Q) \rangle$ is strong, and hence, it has a hamiltonian cycle, say C', by Theorem 3.9. Obviously, $C', C_2, C_3, \ldots, C_k$ constitute the required factor of D.

Suppose now that C_1 and C_2 satisfy Condition 2) in Claim (*). Assume that C_1 and C_2 have been chosen such that no vertex of Q can be inserted into C_1 or into C_2.

Denote $C_1 = a_1 a_2 \cdots a_\mu a_1$ and $C_2 = b_1 b_2 \cdots b_\nu b_1$ and assume that $a_1 \in V(D'_3)$ and $a_2, \ldots, a_i \in V(D'_2)$ and $a_{i+1} \in S$; $b_1 \in V(D'_1)$ and $b_2, \ldots, b_j \in S$ and $b_{j+1} \in V(D'_2)$. Then it is easy to check that $\{a_2, \ldots, a_i\} \to b_1 \to a_{i+1}$ and $b_j \to a_1 \to b_{j+1}$.

It is a simple matter to check that $Q_1 = \emptyset$ and $Q_3 = \emptyset$. We next show that if $Q \neq \emptyset$, then $D\langle V(C_1) \cup V(C_2) \cup V(Q) \rangle$ has two complementary cycles.

Assume that $V(Q_2) \neq \emptyset$ and $V(Q_4) \neq \emptyset$. It is easy to check that $Q_4 \Rightarrow C_1 \Rightarrow Q_2 \Rightarrow C_2 \Rightarrow Q_4$. Hence, $a_1 b_{j+1} \cdots b_\nu b_1 a_{i+1} \cdots a_\mu a_1$ and $a_2 \cdots a_i Q_2 b_2 \cdots b_j Q_4 a_2$ are two complementary cycles of $D\langle V(C_1) \cup V(C_2) \cup V(Q) \rangle$.

Assume now that $Q = Q_4$ (the proof for $Q = Q_2$ is analogous). It is clear that $C_2 \Rightarrow Q \Rightarrow C_1$, and hence, $D\langle V(C_1)\rangle$ and $D\langle V(C_2)\rangle$ are both semicomplete.

If $i \geq 3$, then $a_1 \rightarrow a_3$, and hence $a_1a_3a_4 \cdots a_\mu a_1$ and $a_2b_1b_2 \cdots b_\nu Q_4 a_2$ are two required cycles. So, we assume that $i = 2$.

If $\mu \geq 4$, then in the case when $a_2 \rightarrow a_4$, we have $C_2 \rightarrow a_3$, and hence, $a_1b_{j+1} \cdots b_\nu a_3 \cdots a_\mu a_1$ and $a_2b_1 \cdots b_j Q_4 a_2$ are two required cycles; in the other case when $a_4 \rightarrow a_2$, the 3-cycle $a_2a_3a_4a_2$ and $a_1C_2Q_4a_5 \ldots a_\mu a_1$ are two required cycles.

Therefore, $\mu = 3$. Similarly, it can be shown that $\nu = 3$.

Since Q contains more than two vertices, we see that $a_1b_3qa_3a_1$ and $a_2b_1b_2(Q - q)a_2$ are two required cycles, where q is a vertex of Q.

The proof of the theorem is complete. □

Now we consider semicomplete digraphs.

LEMMA 4.4. *Every k-connected semicomplete digraph D with $n \geq 5k - 2$ vertices contains at least k vertex-disjoint 3-cycles.*

Proof. We prove the statement by induction on k. By Theorem 3.9, every strongly connected semicomplete digraph on at least 3 vertices contains a 3-cycle. Assume that $k \geq 2$. Since D is $(k-1)$-connected, it contains $k-1$ vertex-disjoint 3-cycles $C_1, C_2, \ldots, C_{k-1}$. Let H be the subdigraph induced by the vertices not in the 3-cycles. If H contains a 3-cycle, then we are done. So, we may assume that if H contains some cycles, then they are 2-cycles. Note that $|V(H)| \geq n - 3(k-1) \geq 2k+1$ because $n \geq 5k-2$. Let $P = x_1x_2 \cdots x_m$ be a hamiltonian path of H, $A = \{x_1, x_2, \ldots, x_k\}$ and $B = \{x_{m-k+1}, x_{m-k+2}, \ldots, x_m\}$. Clearly, we have $A \Rightarrow B$. Since D is k-connected, there exist k vertex-disjoint paths from B to A, and each is of length at least 3. Obviously, these paths plus the arcs from A to B form k vertex-disjoint cycles, and hence, D contains k vertex-disjoint 3-cycles. □

THEOREM 4.5. *Let D be a k-connected semicomplete digraph. If D contains $k+1$ vertex-disjoint 3-cycles, then D has a factor composed of k cycles, and at least $k-2$ of them are 3-cycles.*

Proof. Let $C_1, C_2, \ldots, C_k, C$ be $k+1$ vertex-disjoint 3-cycles in D, $F = \{\ C_i \mid 1 \leq i \leq k\}$ and $H = D - \cup_{i=1}^{k} V(C_i)$. Note that C is a 3-cycle in H. Let $H_1, \ldots, H_q$ be the strong decomposition of H. Assume without loss of generality that $N^+(H_q) \cap V(C_i) \neq \emptyset$ for $i = 1, 2, \ldots, \alpha$, and $C_j \Rightarrow H_q$ for $j > \alpha$. If there is an arc from C_j to H_1 for some $j \leq \alpha$, then we see that $D\langle V(C_j) \cup V(H)\rangle$ is strong and we are done by Theorem

3.9. So, we may assume that $H_1 \Rightarrow C_i$ for all $i \leq \alpha$. Let $F_1 = \cup_{i=1}^{\alpha} C_i$. It is clear that for every $i > \alpha$, either $H_1 \Rightarrow C_i$ or C_i has at least one positive neighbor in H_1. Without loss of generality let $F_2 = \cup_{i=\alpha+1}^{\beta} C_i$ and $F_3 = \cup_{j=\beta+1}^{k} C_j$ such that $H_1 \Rightarrow C_i$ for each $i \in \{\alpha+1, \dots, \beta\}$ (if $\beta \geq \alpha+1$) and $N^-(H_1) \cap V(C_j) \neq \emptyset$ for all $j > \beta$. From the connectivity assumption, we conclude that

$$|V(F_1)| \geq k \ \text{ and } \ |V(F_3)| \geq k. \tag{1}$$

Since D is k-connected, there are k vertex-disjoint paths from F_1 to F_3. We denote k such shortest paths by $P_1, P_2, \dots, P_k$, and denote the start vertex (end vertex, respectively) of P_i by x_i (by y_i, respectively) for $i = 1, 2, \dots, k$.

In the following, we assume, to the contrary, that D does not contain k cycles which have the properties as described in the theorem. We consider the following cases:

Case 1. Suppose $q = 1$.

It is clear that $F_3 \Rightarrow H \Rightarrow F_1$, and moreover, either $C_i \Rightarrow C_j$ or $C_j \Rightarrow C_i$ when $i \neq j$. Note that $F_2 = \emptyset$ and P_i is of length 1 for $i = 1, 2, \dots, k$. Assume without loss of generality that P_1 is an arc from C_1 to C_k, i.e., $x_1 \in V(C_1)$ and $y_1 \in V(C_k)$. It follows that $C_1 \Rightarrow C_k$. So, we have $H \Rightarrow C_1 \Rightarrow C_k \Rightarrow H$. Let z be a vertex of H. Then $C' = x_1 y_1 z x_1$ is a 3-cycle. Since $H - z$ has a hamiltonian path, $D\langle V(H) \cup V(C_1) \cup V(C_k)\rangle - V(C')$ is strong, and hence, it has a hamiltonian cycle, say C''. Now we see that $C', C_2, C_3, C_4, \dots, C_{k-1}, C''$ are k cycles, which have the properties as described in the theorem, a contradiction.

Case 2. Suppose $q = 2$.

Assume without loss of generality that C is contained in H_1. If there is an arc from F_1 to F_3, for example, from C_1 to C_k, then we see that $D\langle V(C_1) \cup V(C_k) \cup V(H_2)\rangle$ is strong. Thus, a hamiltonian cycle of H_1 and a hamiltonian cycle of $D\langle V(C_1) \cup V(C_k) \cup V(H_2)\rangle$ together with $C_2, C_3, \dots, C_{k-1}$ yield a contradiction.

Thus, we only need consider the situation that $F_3 \Rightarrow F_1$. It follows that P_i contains at least one vertex of F_2 for each $i \in \{1, 2, \dots, k\}$. This and (1) imply that $|V(F_1)| = |V(F_2)| = |V(F_3)| = k$, and therefore, P_i is of length exactly 2 and P_i plus the arc from y_i to x_i forms a 3-cycle, say C_i', for $i = 1, 2, \dots, k$. Since $V(F_1) = N^+(H_2)$ and $V(F_3) = N^-(H_1)$, the subdigraph $D\langle V(P_1) \cup H\rangle$ is strong, and hence, $C_2', C_3', \dots, C_k'$ and a hamiltonian cycle of $D\langle V(P_1) \cup H\rangle$ yield a contradiction.

Case 3. Suppose $q \geq 3$ and $|V(H_\ell)| \geq 3$ for some ℓ with $1 < \ell < q$.

Assume without loss of generality that C is contained in H_ℓ.

If there is a path (for example, P_1 from C_1 to C_k) that contains neither a vertex from F_2 nor a vertex from C, then $C_2, C_3, \dots, C_{k-1}, C$ and a hamiltonian cycle of $D\langle V(C_1) \cup V(C_k) \cup V(H - V(C))\rangle$ yield a contradiction.

Therefore, P_i contains at least one vertex from $F_2 \cup C$ for every $i \in \{1, 2, \dots, k\}$. This implies that $F_3 \Rightarrow F_1$ and $|V(F_2)| \in \{k,\ k-3\}$. Let C'_i denote the cycle formed by P_i and the arc $y_i x_i$ for $i = 1, 2, \dots, k$, and assume without loss of generality that $|V(C'_1)| \leq |V(C'_2)| \leq \cdots \leq |V(C'_k)|$. Obviously, $|V(C'_1)| \geq 3$ holds. Let $t = \max\{\ i \mid C'_i$ is a 3-cycle$\}$ if $|V(C'_1)| = 3$, otherwise, $t = 0$.

Subcase 3.1. Suppose $|V(F_2)| = k$.

It is clear that $|V(F_1)| = |V(F_2)| = |V(F_3)| = k$, $V(F_1) = N^+(H_q)$ and $V(F_3) = N^-(H_1)$.

Suppose that $t \geq k - 2$. Then, all vertices in $F_2 \cup H$, which do not belong to any cycle C'_i for $i = 1, 2, \dots, n$, can be inserted into C'_k to form a new cycle, and this new cycle with $C'_1, C'_2, \dots, C'_{k-2}, C'_{k-1}$ together yield a contradiction.

Suppose now that $t \leq k-3$. Since the subdigraph $D\langle\{y_{t+1}, y_{t+2}, \dots, y_k\}\rangle$ is semicomplete, it has a hamiltonian path. Assume without loss of generality that $P' = y_{t+1}y_{t+2}\cdots y_k$ is such a path. Let C''_i be the unique 3-cycle containing x_i in C'_i for $i = t+2, t+3, \dots, k$. Since $D\langle V(P_{t+1}) \cup V(P') \cup V(H_1) \cup V(H_q)\rangle$ is strong, it has a hamiltonian cycle, denoted by C''. It is not difficult to see that every vertex in $H_2 \cup H_3 \cup \cdots \cup H_{q-1} \cup F_2$ that does not belong to any cycles of $\{C'_1, \dots, C'_t, C'', C''_{t+2}, \dots, C''_k\}$ can be inserted into the cycle C'' to form a new cycle. This cycle with $C'_1, \dots, C'_t, C''_{t+2}, \dots, C''_k$ yield a contradiction.

Subcase 3.2. Suppose $|V(F_2)| = k - 3$.

Assume without loss of generality that $\alpha = k/3 + 1$, i.e., $|V(F_1)| = k + 3$ and $|V(F_3)| = k$. Note that

$$V(F_3) = N^-(H_1) \quad \text{and} \quad |N^+(H_q) \cap V(F_1)| \geq k. \tag{2}$$

Because of $|V(F_2)| + |V(C)| = k$, every path P_i contains exactly one vertex of $F_2 \cup C$ for $i = 1, 2, \dots, k$.

Claim. There is an integer j such that $D' = D - (\cup_{i \neq j} V(P_i))$ is strong.

Proof. Let x, x', x'' be the three vertices in F_1, which do not belong to any path P_i for $i = 1, 2, \ldots, k$.

Suppose that the subdigraph $D\langle\{x, x', x''\}\rangle$ contains a hamiltonian path, say $x \to x' \to x''$, such that x'' has a positive neighbor in P_j for some j. Clearly, $D\langle\{x, x', x''\} \cup V(P_j)\rangle$ is strong. By (2), we have that $N^+(H_q) \cap \{x, x', x'', x_j\} \neq \emptyset$ and $y_j \in N^-(H_1)$, hence, the subdigraph D' is strong.

Suppose now that $D\langle\{x, x', x''\}\rangle$ does not contain any hamiltonian path whose end-vertex has a positive neighbor in P_i for some $i \in \{1, 2, \ldots, k\}$. Then, it is easy to check that the three vertices x, x', x'' must belong to a common 3-cycle in F_1. Assume without loss of generality that this 3-cycle is C_1. Note that $\{x_1, x_2, \ldots, x_k\} = V(C_2) \cup \cdots \cup V(C_\alpha)$. By the definition of F_1, we may assume that $x_1 \in N^+(H_q)$. Clearly, $D\langle V(D - \cup_{i=2}^k V(P_i))\rangle - \{x, x', x''\}$ is strong, but now, $D\langle V(D - \cup_{i=2}^k V(P_i))\rangle$ must be strong, since C_1 has at least k positive neighbors. □

In the following proof, one can see that we may assume without loss of generality that $j = 1$, i.e. $D\langle V(D - \cup_{i=2}^k V(P_i))\rangle$ is strong. Since $D\langle V(D') \cup W\rangle$ is strong for every subset W of $V(F_2) \cup (\cup_{i=2}^{q-1} V(H_i))$, we only need find $k-1$ vertex-disjoint cycles in $D - V(D') = D\langle \cup_{i=2}^k V(P_i)\rangle$ such that they contain all vertices of $\{x_2, x_3, \ldots, x_k\} \cup \{y_2, \ldots, y_k\}$ and at least $k - 2$ of them are 3-cycles.

If $t \geq k - 1$, then we are done. So, suppose now that $t \leq k - 2$. Let $t' = \max\{1,\ t\}$. Since $D\langle\{y_{t'+1}, y_{t'+2}, \ldots, y_k\}\rangle$ is semicomplete, it has a hamiltonian path, say P' and assume without loss of generality that P' starts at $y_{t'+1}$ and ends at y_k. So, $C''_{t'+1} = x_{t'+1}P_{t'+1}y_{t'+1}P'y_kx_{t'+1}$ is a cycle. Let C''_i be the 3-cycle in C'_i which contains x_i for $i \geq t' + 2$. Now we see that C'_i for i satisfying $2 \leq i \leq t'$ (if $t' \geq 2$) and C''_i for $i = t' + 1, t' + 2, \ldots, k$ are $k - 1$ required cycles.

Case 4. Suppose $q \geq 3$ and $|V(H_i)| \leq 2$ for all i satisfying $1 < i < q$.

Since the 3-cycle C is in H, one of H_1 and H_q contains a 3-cycle. Assume without loss of generality that $|V(H_q)| \geq |V(H_1)|$ and C is contained in H_q.

Subcase 4.1. Suppose $|V(H_q)| \geq 4$.

If $F_2 \neq \emptyset$, then we can change C and C_β and we are done by Case 3.

Suppose now that $F_2 = \emptyset$. Let Q be the terminal component of $H_q - V(C)$. Assume without loss of generality that $x_1 \in V(C_1)$ and $y_1 \in V(C_k)$. If Q has at least one positive neighbor in C_1, then $D\langle V(C_1) \cup$

$V(C_k) \cup V(H)\rangle - V(C)$ is strong and $C, C_2, C_3, \ldots, C_{k-1}$ are $k-1$ cycles of length 3, and hence, we are done. If $C_1 \Rightarrow Q$, then, by changing C and C_1, we are done by Case 3 again.

Subcase 4.2. Suppose $|V(H_q)| = 3$.

Suppose that there is an arc from C_i to H_j for some $i \leq \beta$ and some $j \leq q-1$. Because $H_1 \Rightarrow C_i$, the subdigraph $D\langle V(H) \cup V(C_i)\rangle - V(C)$ is not strong, and furthermore, either its terminal component has at least four vertices or one of its internal components contains C_i. But, in both these cases we are done by Subcase 4.1 or Case 3, respectively.

Suppose now that $H_i \Rightarrow C_j$ for every $i \leq q-1$ and every $j \leq \beta$. If D has an arc xy from F_1 to F_3 (without loss of generality that $x \in V(C_1)$ and $y \in V(C_k)$), then $D\langle V(C_1) \cup V(C_k) \cup V(H)\rangle - V(C)$ is strong and $C, C_2, C_3, \ldots, C_{k-1}$ are $k-1$ cycles of length 3, a contradiction. If $F_3 \Rightarrow F_1$, then it is not difficult to check that $V(P_i) \cap V(F_2) \neq \emptyset$ for $i = 1, 2, \ldots, k$. This and (1) imply that $|V(F_i)| = k$ for $i = 1, 2, 3$. Therefore, $C'_j = x_j P_j y_j x_j$ is a 3-cycle for each $j \in \{1, 2, \ldots, k\}$. Note that $N^+(C) = V(F_1)$ and $N^-(H_1) = V(F_3)$. Since $D\langle V(C'_1) \cup V(H)\rangle$ is strong and $C'_2, C'_3, \ldots, C'_k$ are 3-cycles, we get a contradiction.

The proof of the theorem is complete. □

THEOREM 4.6. *Let D be a k-connected semicomplete digraph on $n \geq 5k+1$ vertices. Then D has a factor composed of k cycles such that one of them is of length at most 5.*

Proof. Since D is k-connected, it contains k vertex-disjoint 3-cycles, say $C_1, C_2, \ldots, C_k$, by Lemma 4.4. Let $F = \cup_{i=1}^k C_i$ and $H = D - V(F)$. If H contains a 3-cycle, then we are done by Theorem 4.5. So, we assume that every strongly connected component of H contains at most two vertices. Let $h_1 h_2 \cdots h_m$ be a hamiltonian path of H. Because of $n \geq 5k+1$, we have $m \geq 2k+1$. Let $A = \{h_1, h_2, \ldots, h_k\}$ and $B = \{h_{m-k+1}, h_{m-k+2}, \ldots, h_m\}$. Then, we have $A \Rightarrow B$.

By the connectivity assumption for D, there are k vertex-disjoint paths from B to A. Let $P_i = v_1^i v_2^i \cdots v_{m_i}^i$, $i = 1, 2, \ldots, k$, be k such shortest paths and assume without loss of generality that $v_2^i, v_{m_i-1}^i \notin V(H)$ for $i = 1, 2, \ldots, k$.

Clearly, $m_i \geq 3$ and $C'_i = v_1^i v_2^i v_3^i v_1^i$ and $C''_i = v_{m_i}^i v_{m_i-2}^i v_{m_i-1}^i v_{m_i}^i$ are 3-cycles in $D\langle V(P_i)\rangle$ for $i = 1, 2, \ldots, k$. If there is a P_i with $m_i \geq 6$, then C'_i and C''_i are vertex-disjoint. It follows that D contains at least $k+1$ vertex-disjoint 3-cycles, and hence, we are done by Theorem 4.5. Therefore, we only need consider the case when $3 \leq m_i \leq 5$ for $i =$

$1, 2, \ldots, k$. By the same argument, we may assume that $P_1, P_2, \ldots, P_k$ go through all 3-cycles in F. Denote $P = \{P_1, P_2, \ldots, P_k\}$.

Suppose that there is a path P_i with $|V(P_i) \cap V(H)| \geq 3$. Then it is easy to check that $v_2^i \in V(C_\mu)$, $v_3^i = h_j$, $v_4^i \in V(C_\nu)$ for some $\mu, \nu \in \{1, 2, \ldots, k\}$ and some j satisfying $k + 1 \leq j \leq m - k$. We call such P_i a W-path.

Assume that $\mu = \nu$. We show that $D\langle V(C_\mu) \cup V(H)\rangle$ contains two vertex-disjoint 3-cycles, i.e., D contains $k + 1$ vertex-disjoint 3-cycles, and hence, we are done by Theorem 4.5. Let v be another vertex of C_μ. Obviously, $C_\mu = v_2^i v v_4^i v_2^i$. If $v \to v_1^i$, then $v_1^i v_2^i v v_1^i$ and C_i'' are two 3-cycles in $D\langle V(C_\mu) \cup V(H)\rangle$; if $v_5^i \to v$, then $v_5^i v v_4^i v_5^i$ and C_i' are two 3-cycles in $D\langle V(C_\mu) \cup V(H)\rangle$; in the remaining case when $v_1^i \to v \to v_5^i$, we see that $v_1^i v v_5^i v_1^i$ and $v_2^i v_3^i v_4^i v_2^i$ are two 3-cycles in $D\langle V(C_\mu) \cup V(H)\rangle$.

Assume now that every W-path goes through two different 3-cycles in F. If there are two W-paths $P_\alpha, P_\beta \in P$ such that v_i^α, v_j^β are in a common cycle C_μ for $i, j \in \{2, 4\}$, then $v_{i-1}^\alpha v_i^\alpha v_{i+1}^\alpha v_{i-1}^\alpha$ and $v_{j-1}^\beta v_j^\beta v_{j+1}^\beta v_{j-1}^\beta$ are two vertex-disjoint cycles in $D\langle V(C_\mu) \cup V(H)\rangle$, and we are done by Theorem 4.5 again. Therefore, there exist at most $\lfloor k/2 \rfloor$ W-paths in P. It follows that $\sum_{i=1}^{k} |V(P_i)| \leq 5k + \lfloor k/2 \rfloor$.

In the following we show that if $\cup_{i=1}^{k} V(P_i)$ does not contain all vertices of F, then we can find k vertex-disjoint cycles $Q_1, Q_2, \ldots, Q_k$ satisfying the following conditions:

a) $|V(Q_i) \cap A| = 1$ and $|V(Q_i) \cap B| = 1$ for $i = 1, 2, \ldots, k$;
b) $\cup_{i=1}^{k} V(Q_i) = \cup_{j=1}^{k} V(P_j) \cup V(F)$.

Suppose that C_ℓ contains a vertex v that is not in $\cup_{i=1}^{k} V(P_i)$ for some $\ell \in \{1, 2, \ldots, k\}$, and there are two paths (say P_α and P_β) going through C_ℓ (assume without loss of generality that $C_\ell = v_i^\alpha v_j^\beta v v_i^\alpha$) such that $v_{j-1}^\beta \Rightarrow v \Rightarrow v_{i+1}^\alpha$ (we call C_ℓ an X-cycle of P). Then $v_1^\alpha v_2^\alpha \cdots v_i^\alpha v_j^\beta v_{j+1}^\beta \cdots v_{m_\beta}^\beta$ and $v_1^\beta \cdots v_{j-1}^\beta v v_{i+1}^\alpha \cdots v_{m_\alpha}^\alpha$ are two paths containing all vertices of $P_\alpha \cup P_\beta \cup C_\ell$. So, it is not difficult to check that there are k vertex-disjoint paths $P_1', P_2', \ldots, P_k'$ which have the following properties:

(1) each of them starts at a vertex of B and ends at a vertex of A;
(2) they do not have any X-cycles;
(3) $\cup_{j=1}^{k} V(P_j) \subseteq \cup_{i=1}^{k} V(P_i') \subseteq \cup_{j=1}^{k} V(P_j) \cup V(F)$.

Let Q_i' be the cycle formed by the path P_i' and the corresponding arc from A to B for $i = 1, 2, \ldots, k$. We show that every vertex in F, but not in $\cup_{i=1}^{k} V(Q_i')$ can be inserted into Q_j' for some $j \in \{1, 2, \ldots, k\}$ if $\cup_{i=1}^{k} V(P_i') \neq \cup_{j=1}^{k} V(P_j) \cup V(F)$.

Let v be a vertex of C_t for some $t \in \{1, 2, \ldots, k\}$ with $v \notin \cup_{i=1}^{k} V(P_i')$. If only one path P_γ' goes through C_t, then $D\langle V(C_t) \cup V(Q_\gamma')\rangle$ is strong, and hence, it has a hamiltonian cycle. Assume now that there are two paths, say P_α and P_β, going through C_t. Since C_t is not an X-path, it is easy to see that one of $D\langle\{v\} \cup V(Q_\alpha')\rangle$ and $D\langle\{v\} \cup V(Q_\alpha')\rangle$ is strong.

Therefore, all vertices in $V(F)$, but not in $\cup_{i=1}^{k} V(P_i')$ can be inserted into some of $Q_1', Q_2', \ldots, Q_k'$. This means that we can find k vertex-disjoint cycles $Q_1, Q_2, \ldots, Q_k$ satisfying the conditions a) and b) above.

Because of $|\cup_{i=1}^{k} V(Q_i)| \leq 5k + \lfloor k/2 \rfloor$, at least one of $Q_1, Q_2, \ldots, Q_k$, say Q_k, is of length at most 5. Since $D\langle V(Q_1) \cup V(H - \cup_{i=1}^{k} V(Q_i))\rangle$ is strong, a hamiltonian cycle of it and $Q_2, Q_3, \ldots, Q_k$ form a factor of D.

The proof of the theorem is complete. □

The next corollary states our main result.

COROLLARY 4.7. *Let D be a k-connected locally semicomplete digraph with $k \geq 3$ and $\overline{g}$ denote the length of a longest induced cycle in D. If D has at least $7(k-1)\overline{g}$ vertices, then it has a factor composed of k cycles, and at least one of them is of length $\overline{g}$ or 5.*

Proof. We only need to consider the three cases (a), (b) and (c) as described in Theorem 3.3. So, this corollary can be confirmed by Corollary 4.2, Theorem 4.3 and 4.6, respectively. □

Another immediate consequence of Theorem 4.6 is the result of [3]:

COROLLARY 4.8 ([3]). *Every k-connected tournament T with at least $8k$ vertices contains k vertex-disjoint cycles that span $V(T)$.*

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