Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Combinatorial Theory, Series B 89 (2003) 159–171
Journal of Combinatorial Theory Series B
http://www.elsevier.com/locate/jctb

Extremal graphs for intersecting cliques

Guantao Chen,[a,1] Ronald J. Gould,[b] Florian Pfender,[b] and Bing Wei[c,2]

[a] *Georgia State University, Atlanta, GA 30303, USA*
[b] *Department of Mathematical and Computational Science, Emory University, Atlanta, GA 30332, USA*
[c] *University of Mississippi, MS 38677, USA*

Received 6 July 2001

Abstract

For any two positive integers $n \geqslant r \geqslant 1$, the well-known Turán Theorem states that there exists a least positive integer $ex(n, K_r)$ such that every graph with n vertices and $ex(n, K_r) + 1$ edges contains a subgraph isomorphic to K_r. We determine the minimum number of edges sufficient for the existence of k cliques with r vertices each intersecting in exactly one common vertex.

MSC: 05C78

Keywords: Extremal graph; Turán graph; Cliques; Matchings

1. Introduction

With integers $n \geqslant r \geqslant 1$, we let $T_{n,r}$ denote the *Turán graph*, i.e., the complete r-partite graph on n vertices where each partite set has either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ vertices and the edge set consists of all pairs joining distinct parts. The number of edges in $T_{n,r}$ is denoted by $ex(n, K_{r+1})$, where K_r represents the complete graph on r vertices.

For a graph G and a vertex $x \in V(G)$, the *neighborhood* of x in G is *denoted* by $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, or when clear, simply $N(x)$, and let

E-mail address: fpfende@emory.edu (F. Pfender).
[1] Supported by NSF Grant DMS-0070059.
[2] Supported by NNSF of China and the innovation funds of Institute of Systems Science, AMSS, CAS.

0095-8956/03/$ - see front matter © 2003 Elsevier Science (USA). All rights reserved.
doi:10.1016/S0095-8956(03)00044-3

$\overline{N_G(x)} = V(G) - N_G(x)$. The *degree* of x in G, denoted by $d_G(x)$, or $d(x)$, is the size of $N_G(x)$. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees, respectively, in G. The order of G is often denoted by $|G| = |V(G)|$. For a subset $X \subset V(G)$, let $G[X]$ denote the subgraph of G induced by X. A *matching* in G is a set of edges from $E(G)$, no two of which share a common vertex, and the *matching number* of G, denoted by $\nu(G)$, is the maximum number of edges in a matching in G.

Suppose that we are given some fixed graph H. What is the maximum number, $ex(n, H)$, of edges in a graph G on n vertices that does not contain a copy of H as a subgraph (often said to *forbid* H)? A graph G on n vertices with $ex(n, H)$ edges and without a copy of H is called an *extremal graph* for H. For $n \geqslant |V(H)|$, adding one more edge to any one of the extremal graphs will produce a copy of H.

A graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex is called a k-fan and denoted by F_k. For each k, the chromatic number of F_k is three, and so by the Erdős–Stone theorem [4], $ex(n, F_k) = (1 + o(1))n^2/4$. The following result is due to Erdős et al. [3].

Theorem 1. *For every $k \geqslant 1$, and for every $n \geqslant 50k^2$, if a graph G on n vertices has more than*

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even}, \end{cases}$$

edges, then G contains a copy of a k-fan. Further, the number of edges is best possible.

A graph on $(r-1)k+1$ vertices consisting of k cliques each with r vertices, which intersect in exactly one common vertex, is called a (k, r)-fan and denoted by $F_{k,r}$. The purpose of this article is to generalize Theorem 1, when k and r are fixed and n is large, as follows.

Theorem 2. *For every $k \geqslant 1$ and $r \geqslant 2$, and for every $n \geqslant 16k^3r^8$, if a graph G on n vertices has more than*

$$ex(n, K_r) + \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even}, \end{cases}$$

edges, then G contains a copy of a (k, r)-fan. Further, the number of edges is best possible.

Note that the number $ex(n, K_r) = |E(T_{n,r-1})|$. To show the lower bound for $ex(n, F_{k,r})$ we present the following graph, $G_{n,k,r}$. For odd k (where $n \geqslant (2k-1)(r-1)+1)G_{n,k,r}$ is constructed by taking a Turan graph $T_{n,r-1}$ and embedding two vertex disjoint copies of K_k in one partite set. For even k (where now $n \geqslant (2k-2)(r-1)+1)G_{n,k,r}$ is constructed by taking a Turán graph $T_{n,r-1}$ and embedding a graph with $2k-1$ vertices, $k^2-(3/2)k$ edges with maximum degree $k-1$ in one partite set.

2. Lemmas

In this section, we give preparatory lemmas for the proof of the main theorem.

Define $f(v, \Delta) = \max\{|E(G)| : v(G) \leqslant v, \Delta(G) \leqslant \Delta\}$. Chvátal and Hanson [2] proved the following theorem.

Theorem 3. *For every $v \geqslant 1$ and $\Delta \geqslant 1$,*

$$f(v, \Delta) = v\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{v}{\lceil \Delta/2 \rceil} \right\rfloor \leqslant v\Delta + v.$$

We will frequently use the following special case proved by Abbott et al. [1]:

$$f(k-1, k-1) = \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even}. \end{cases}$$

The extremal graphs are exactly those we embedded into $T_{n,r-1}$ in the previous section to obtain the extremal $F_{k,r}$-free graph $G_{n,k,r}$.

Let a be a positive integer and let X and Y be two disjoint vertex sets of $V(G)$. We say that X *dominates* Y *with a-deficiency* if $d_Y(x) \geqslant |Y| - a$ for each $x \in X$. Let $V_1, V_2, \ldots, V_m$ be disjoint subsets of $V(G)$. We say that $\{V_1, V_2, \ldots, V_m\}$ *is a-deficiency complete* if V_i dominates V_j with deficiency a for every pair $i \neq j$ with $i, j = 1, 2, \ldots, m$.

The following lemma will be used very heavily in our proof of the main theorem.

Lemma 2.1. *Let a be a positive integer. Let G be a graph and let $\{X_1, X_2, \ldots, X_m\}$ be an a-deficiency complete partition of $V(G)$ with $|X_i| \geqslant ma + 2t$ for each i. Suppose that $C_1, C_2, \ldots, C_t$ are t cliques of G with the properties:*

(1) *$|C_i \cap X_j| \leqslant 2$ for each pair i and j,*
(2) *$|C_i \cap X_j| = 2$ for at most one j for each i.*

Then, there exist t cliques $D_1, D_2, \ldots, D_t$ satisfying:

(1) *$C_i \subseteq D_i$ for each i,*
(2) *$D_1 - C_1, D_2 - C_2, \ldots, D_t - C_t$ are mutually disjoint,*

(3) *for each* i *we have that* $|D_i \cap X_j| = 1$ *for all* j *except possibly one at which* $|D_i \cap X_j| = |C_i \cap X_j| = 2$.

Proof. We need to show that, if $C_i \cap X_j = \emptyset$, there exists a vertex $v_j \in X_j - \bigcup_{\ell=1}^{t} C_\ell$ such that v_j is adjacent to all vertices in C_i. Iteration of this argument will then provide the statement. Without loss of generality, we may assume that $i = j = 1$.

Since $d_{X_1}(v) \geqslant |X_1| - a$ for each $v \in C_1$,

$$\left| \bigcap_{v \in C_1} N_{X_1}(v) \right| \geqslant |X_1| - |C_1|a \geqslant ma + 2t - ma \geqslant 2t.$$

By our assumptions, we have that $|(\bigcup_{i=2}^{t} C_i) \cap X_1| \leqslant 2(t-1)$, thus $\bigcap_{v \in C_1} N_{X_1}(v) - \bigcup_{i=2}^{k} C_i \neq \emptyset$. Lemma 2.1 now follows. □

Lemma 2.2. *Let* G *be a graph and* $Y_1, Y_2, \ldots, Y_m$ *be* m *vertex disjoint subsets of* $V(G)$ *and* $Y_0 \subseteq V(G) - \bigcup_{i=1}^{m} Y_i$ *such that* $|Y_i| \geqslant (i-1)a + k$ *for each* $i = 1, \ldots, m$. *If* Y_i *dominates* Y_j *with a-deficiency for every* $i = 1, 2, \ldots, m, j = 0, 1, \ldots, m$, *and* $i \neq j$, *then, there are* k *vertex disjoint cliques* $C_1, C_2, \ldots, C_k$ *satisfying* $|C_i| = m$ *and* $|C_i \cap Y_j| = 1$ *for each* i *and* $j \geqslant 1$. *Furthermore, if* $|Y_0| \geqslant ma + k$, *then there are* k *vertex disjoint cliques* $D_1, D_2, \ldots, D_k$ *with the property that* $|D_i| = m + 1$ *and* $|D_i \cap Y_j| = 1$ *for each* $i = 1, \ldots, k$ *and* $j = 0, 1, \ldots, m$.

Proof. Let $y_{1,1}, y_{1,2}, \ldots, y_{1,k}$ be k arbitrary vertices in Y_1. Since $|N(y_{1,i}) \cap Y_2| \geqslant |Y_2| - a \geqslant k$, there are k vertices $y_{2,1}, y_{2,2}, \ldots, y_{2,k}$ in Y_2 such that $y_{1,i}y_{2,i} \in E$ for all $i = 1, \ldots, k$. Since $|N(y_{1,i}) \cap N(y_2, i) \cap Y_3| \geqslant |Y_3| - 2a \geqslant k$, there are k vertices $y_{3,1}, y_{3,2}, \ldots, y_{3,k}$ in Y_3 such that $y_{3,i} \in N(y_{1,i}) \cap N(y_{2,i})$ for all $i = 1, 2, \ldots, k$. Continuing in the same fashion, we see that Lemma 2.2 follows. □

The case $k = 1$ of the main theorem is Turan's theorem, the case of $r = 2$ is trivial, and the case of $r = 3$ is Theorem 1. We assume that $k \geqslant 2$ and $r \geqslant 4$. The aim of this section is to prove the following lemma.

Lemma 2.3. *Let* G *be an extremal graph for* $F_{k,r}$ *on* n *vertices with* $n \geqslant 4k^2r^4$, *and with minimum degree* $\delta \geqslant (\frac{r-2}{r-1})n - k$. *Then there exists a partition* $V(G) = V_0 \dot{\cup} V_1 \dot{\cup} \cdots \dot{\cup} V_{r-2}$, *so that* $V_i \neq \emptyset$ *for all* $i = 0, \ldots, r-2$ *and for every* $x \in V_i$, *the following hold*:

$$\sum_{j \neq i} \nu(G[V_j]) \leqslant k - 1 \quad \textit{and} \quad \Delta(G[V_i]) \leqslant k - 1; \tag{1}$$

$$d_{G[V_i]}(x) + \sum_{j \neq i} \nu(G[N(x) \cap V_j]) \leqslant k - 1. \tag{2}$$

Proof. Since G plus any edge contains a copy of $F_{k,r}$, G contains k edge disjoint cliques $D_1, D_2, \dots, D_k$ sharing one vertex v_0 with $|D_1| = r-1$ and $|D_j| = r$ for all $j \geqslant 2$. Let $V(D_1) = \{v_0, v_1, \dots, v_{r-2}\}$. Denote the graph induced by $\bigcup_{i=1}^{k} D_i$ by D. Clearly, $|D| = k(r-1)$. For each $i = 0, \dots, r-2$, we define $X_i = \bigcap_{j \neq i} N(v_j) - V(D)$. Since G does not contain $F_{k,r}$ as a subgraph,

$$X_i \cap X_j = \emptyset \quad \text{for } i \neq j.$$

Since the minimum degree $\delta(G) \geqslant \frac{r-2}{r-1} n - k$,

$$|X_i \cup V(D)| \geqslant \frac{n}{r-1} - (r-2)k.$$

Thus,

$$|X_i| \geqslant \frac{n}{r-1} - (r-2)k - k(r-1) = \frac{n}{r-1} - k(2r-3). \tag{3}$$

For each $i \geqslant 1$, if there is an edge $uv \in E(G[X_i])$, replacing v_i by the edge uv in D we obtain a copy of $F_{k,r}$, a contradiction. Thus,

$$E(G[X_i]) = \emptyset \quad \text{for each } i = 1, 2, \dots, r-2.$$

For every $x_i \in X_i$ and $i \neq 0$, since $d(x_i) \geqslant \frac{r-2}{r-1} n - k$, $d_{X_i}(x_i) = 0$, and $|X_i| \geqslant \frac{n}{r-1} - k(2r-3)$, then

$$\begin{aligned} |\overline{N_{G-X_i}(x_i)}| &= (n - d(x_i)) - |X_i| \\ &\leqslant \left(\frac{n}{r-1} + k\right) - \left(\frac{n}{r-1} - k(2r-3)\right) \\ &= 2k(r-1). \end{aligned}$$

Thus,

$$d_{G-X_i}(x_i) \geqslant |G - X_i| - 2k(r-1),$$

for each $x \in X_i$ where $i = 1, 2, \dots, r-2$. In particular, we have that

$$d_{X_j}(x) \geqslant |X_j| - 2k(r-1) \tag{4}$$

for each $x \in X_i$, i.e., X_i dominates X_j with $2k(r-1)$-deficiency, where $i = 1, 2, \dots, r-2$, $j = 0, 1, \dots, r-2$ and $j \neq i$.

Claim 4. *Let $x_1, x_2, \dots, x_{r-2}$ be $r-2$ vertices such that $x_i \in X_i$ for each $i = 1, \dots, r-2$. Then, for any $Y_0 \subseteq X_0$ with $|Y_0| \geqslant 2k(r-1)^2 \geqslant 2k(r-1)(r-2) + k$, we have the following inequality*:

$$\left| \bigcap_{i=1}^{r-2} N(x_i) \cap Y_0 \right| \geqslant k.$$

Proof. By (4), $d_{X_0}(x_i) \geqslant |X_0| - 2k(r-1)$, and so

$$\left| \bigcap_{i=1}^{r-2} N(x_i) \cap X_0 \right| \geqslant |X_0| - 2k(r-1)(r-2).$$

Claim 4 follows. □

Let X_0^* denote the set of all vertices of X_0 of degree at least $2k(r-1)^2$ in X_0.

Claim 5. $|X_0^*| \leqslant 2k(r-1)(r-2)$.

Proof. Suppose, to the contrary, $|X_0^*| > 2k(r-1)(r-2)$. For each i, let

$$X_0^i = \{x \in X_0^* \mid d_{X_i}(x) \geqslant |X_i|/(2k(r-1)+1)\}.$$

By (4), $d_{X_0}(x_i) \geqslant |X_0| - 2k(r-1)$ for every $x_i \in X_i$, thus $N(S) \supseteq X_i$ for every $S \subseteq X_0^*$ with $|S| = 2k(r-1)+1$, which implies that $|X_0^i| \geqslant |X_0^*| - 2k(r-1)$.

Therefore,

$$\left| \bigcap_{i=1}^{r-2} X_0^i \right| \geqslant |X_0^*| - 2k(r-1)(r-2) > 1.$$

There is an $x_0 \in X_0^*$ such that $|N(x_0) \cap X_i| \geqslant |X_i|/(2k(r-1)+1)$ for each $i = 1, 2, \ldots, r-2$. Recall that by (3) we have $|X_i| \geqslant n/(r-1) - k(2r-3)$ for each $i = 1, \ldots, r-2$. Since $n \geqslant 4k^2r^4$, the following inequality holds:

$$|N_{X_i}(x_0)| \geqslant |X_i|/(2k(r-1)+1) \geqslant 2k(r-1)(r-2)+k.$$

Applying Lemma 2.2 with $Y_0 = N(x_0) \cap X_0$, $Y_1 = N(x_0) \cap X_1, \ldots, Y_{r-2} = N(x_0) \cap X_{r-2}$, and $a = 2k(r-1)$, we obtain k vertex disjoint cliques $C_1, C_2, \ldots, C_k$ of sizes $r-1$ in $N(x_0)$. Then, a copy of $F_{k,r}$ is found, a contradiction. □

Let $Z_0 = X_0 - X_0^*$ and $Z_i = X_i$ for each $i = 1, 2, \ldots, r-2$. By Claim 5 and (3), we have that

$$\left| V - \bigcup_{i=0}^{r-2} X_i \right| \leqslant k(2r-3)(r-1).$$

Thus,

$$\left| V - \bigcup_{i=0}^{r-2} Z_i \right| \leqslant k(2r-3)(r-1) + 2k(r-1)(r-2) < 4k(r-1)^2.$$

Further, the following inequality holds.

$$|Z_0| \geqslant n/(r-1) - k(2r-3) - 2k(r-1)(r-2) = n/(r-1) - k(2r^2-4r+1).$$

Since $\delta(G) \geqslant \frac{r-2}{r-1}n - k$, the following inequalities hold for every $z_0 \in Z_0$ (recall that $Z_0 = X_0 - X_0^*$ and thus by the definition of X_0^* we have $\Delta(G[Z_0]) \leqslant 2k(r-1)^2$):

$$\begin{aligned}|\overline{N_{G-Z_0}(z_0)}| &\leqslant (n - d(z_0)) - (|Z_0| - \Delta(G[Z_0])) \\ &\leqslant \left(\frac{n}{r-1} + k\right) - \left(\frac{n}{r-1} - k(2r^2 - 4r + 1) - 2k(r-1)^2\right) \\ &\leqslant 4kr(r-1).\end{aligned}$$

In particular, for each $z_0 \in Z_0$, we have that for $i > 0$

$$d_{Z_i}(z_0) \geqslant |Z_i| - 4kr(r-1).$$

That is, Z_0 dominates Z_i with $4kr(r-1)$ deficiency.

Claim 6. *For every $v \in V - \bigcup_{i=0}^{r-2} Z_i$, there exists a $j = j(v)$ such that $d_{Z_j}(v) < 2k(r-1)^2 + k < 2kr(r-1)$. Further, such a $j(v)$ is unique.*

Proof. Suppose, to the contrary, there is a $v \in V - \bigcup_{i=0}^{r-2} Z_i$ such that $d_{Z_j}(v) \geqslant 2k(r-1)^2 + k$ for every $j = 0, 1, \ldots, r-2$. Set $a = 2k(r-1)$ and $m = r-1$, then for all $0 \leqslant j \leqslant r-2$,

$$|N_{Z_j}(v)| = d_{Z_j}(v) \geqslant ma + k$$

and

$$d_{Z_j}(z_i) \geqslant |Z_j| - a \quad \text{for } z_i \in Z_i,\ i > 0,\ i \neq j.$$

Applying Lemma 2.2, we see that there are k vertex disjoint cliques of order $r-1$ whose vertex sets are in $N(v)$, a contradiction.

To show the uniqueness of $j(v)$, suppose there are two distinct j_1 and j_2 such that $d_{Z_{j_i}}(v) < 2k(r-1)^2 + k$ for both $i = 1$ and 2. Since $n \geqslant 4k^2r^4 \geqslant 4kr^2(r-1)^2$, we have that

$$\begin{aligned}d(v) &\leqslant n - |Z_{j_1} \cup Z_{j_2}| + 4k(r-1)^2 + 2k \\ &\leqslant n - \left[\left(\frac{n}{r-1} - 2k(r-1)^2\right) + \left(\frac{n}{r-1} - k(2r-3)\right)\right] + 4k(r-1)^2 + 2k \\ &= \frac{r-2}{r-1}n - \frac{n}{r-1} + 2k(r-1)^2 + k(2r-3) + 4k(r-1)^2 + 2k \\ &< \frac{r-2}{r-1}n - k,\end{aligned}$$

a contradiction. $\square$

Adding each $v \in V - \bigcup_{i=0}^{r-2} Z_i$ to $Z_{j(v)}$, we obtain a partition of $V = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_{r-2}$.

Clearly, for each $i = 0, \dots, r-2$,

$$|V_i| \geqslant |Z_i| \geqslant \frac{n}{r-1} - 2k(r-1)^2. \tag{5}$$

For each i and each $v_i \in V_i$, since

$$\Delta(G[V_i]) \leqslant \Delta(G[Z_i]) + \left| V - \bigcup_{i=0}^{r-2} Z_i \right| \leqslant 2k(r-1)^2 + 4k(r-1)^2,$$

we have that

$$\begin{aligned} |\overline{N_{G-V_i}(v_i)}| &\leqslant (n - d(v_i)) - (|V_i| - \Delta(G[V_i])) \\ &\leqslant \left(\frac{n}{r-1} + k\right) - \left(\frac{n}{r-1} - 2k(r-1)^2 - 6k(r-1)^2\right) \\ &= k + 2k(r-1)^2 + 6k(r-1)^2 \\ &< 8kr^2. \end{aligned}$$

In particular, we have that

$$d_{V_j}(v_i) \geqslant |V_j| - 8kr^2. \tag{6}$$

We will show that $V_0, V_1, \dots, V_{r-2}$ satisfy (1) and (2). Let $a = 8kr^2$. Since $n \geqslant 4k^2r^4 \geqslant 8kr^4$, for any j, we have that

$$|V_j| \geqslant \frac{n}{r-1} - 2k(r-1)^2 \geqslant (r-1)a + 2k.$$

Proof of (1). Suppose for some $y \in V_i, |N(y) \cap V_i| \geqslant k$, say the neighbors are $y_1, y_2, \dots, y_k$ in V_i. By Lemma 2.1, there are k cliques $D_1, D_2, \dots, D_k$ such that $y, y_j \in D_j$ and $|D_j| = r$ for each j. Further, $D_j \cap D_\ell = \{y\}$ for all $j \neq \ell$. Thus, a copy of $F_{k,r}$ is found, a contradiction.

Next suppose that $\sum_{j \neq i} \nu(V_j) \geqslant k$. Let $y_1z_1, y_2z_2, \dots, y_kz_k$ be a k-matching with the property that y_j and z_j are in the same V_ℓ for some $\ell \neq i$. Now, since $n \geqslant 4k^2r^4 \geqslant 16k^2r^3$,

$$\left| \bigcap_{j=1}^{k} (N_{V_i}(y_j) \cap N_{V_i}(z_j)) \right| > \left| V_i \right| - 2k(8kr^2) \geqslant \left(\frac{n}{r-1} - 2k(r-1)^2\right) - 16k^2r^2 \geqslant 1.$$

Therefore, there exists a vertex $y \in V_i$, such that $\bigcup_{j=1}^{k} \{y_j, z_j\} \subseteq N(y)$. By Lemma 2.1, there are k cliques $D_1, D_2, \dots, D_k$ such that $y, y_j, z_j \in D_j$ and $|D_j| = r$ for each j. Further, $D_j \cap D_\ell = \{y\}$ for all $j \neq \ell$. Thus, a copy of $F_{k,r}$ is found, a contradiction. □

Proof of (2). Let $v \in V_i$ have neighbors $x_1, x_2, \dots, x_s$ in V_i and neighbors $y_1, z_1, y_2, z_2, \dots, y_t$, and z_t in $V - V_i$ where, for each $j = 1, \dots, t$, y_j and z_j in the same V_ℓ for some $\ell \neq i$ and $y_jz_j \in E(G)$. By (1), both s and t are less than k. Suppose for the moment that $s + t \geqslant k$. Consider k of the cliques $\{v, x_1\}, \dots, \{v, x_s\}$, $\{v, y_1, z_1\}, \dots, \{v, y_t, z_t\}$. Applying Lemma 2.1 again, we obtain k cliques

$D_1, D_2, \ldots, D_k$ which induce a copy of $F_{k,r}$, a contradiction, which completes the proof of Lemma 2.3. $\square$

3. Proof of the main lemma

The following lemma was obtained in [3].

Lemma 3.1. *Let H be a graph and b a nonnegative integer such that $b \leqslant \Delta(H) - 2$, and let $v = v(H)$, $\Delta = \Delta(H)$. Then*

$$\sum_{x \in V(H)} \min\{d_H(x), b\} \leqslant v(b + \Delta). \tag{7}$$

Let G be a graph with a partition of the vertices into $r-1$ non-empty parts

$$V(G) = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_{r-2}.$$

Let $G_i = G[V_i]$ for each $i = 0, 1, \ldots, r-2$, and define

$$G_{cr} = (V(G), \{v_i v_j : v_i \in V_i, v_j \in V_j, i \neq j\}),$$

where "cr" denotes "crossing". For each $i \in \{0, 1, \ldots, r-2, cr\}$ let $d_i(x) = d_{G_i}(x)$ and $v_i = v(G_i)$. We generalized Lemma 6.2 in [3] to the following lemma.

Lemma 3.2. *Suppose G is partitioned as above so that* (1) *and* (2) *are satisfied. If G is $F_{k,r}$-free, then*

$$\sum_{i=0}^{r-2} |E(G_i)| - \left(\sum_{0 \leqslant i < j \leqslant r-2} |V_i||V_j| - |E(G_{cr})| \right) \leqslant f(k-1, k-1). \tag{8}$$

Proof. Observe that G_{cr} is an $(r-1)$-partite graph, and $\sum_{0 \leqslant i < j \leqslant r-2} |V_i||V_j| - |E(G_{cr})|$ is the number of edges missing from the complete $(r-1)$-partite graph. By (1) and the definition of f, we see that $|E(G_i)| \leqslant f(k-1, k-1)$, so the left-hand side of (8) is bounded above by $(r-1)f(k-1, k-1)$. Delete vertices of G so that the left-hand side of (8) is maximal, let G be minimal in this case.

We now claim that for each $i = 0, \ldots, r-2$ and every $x \in V_i$,

$$d_i(x) - (|V - V_i| - d_{cr}(x)) > 0. \tag{9}$$

In fact, if for some $x \in V_i$, $d_i(x) - (|V - V_i| - d_{cr}(x)) \leqslant 0$ holds, then

$$\begin{aligned}
&|E(G_i - x)| + \sum_{j \neq i} |E(G_j)| - \left(\sum_{j \neq i} |V_i - x||V_j| + \sum_{i \neq j < \ell \neq i} |V_j||V_\ell| - |E(G_{cr} - x)| \right) \\
&\quad = \sum_{j=0}^{r-2} |E(G_j)| - \left(\sum_{0 \leqslant j < \ell \leqslant r-2} |V_j||V_\ell| - |E(G_{cr})| \right) \\
&\qquad - (d_i(x) - |V - V_i| + d_{cr}(x)) \\
&\quad \geqslant \sum_{j=0}^{r-2} |E(G_j)| - \left(\sum_{0 \leqslant j < \ell \leqslant r-2} |V_j||V_\ell| - |E(G_{cr})| \right),
\end{aligned}$$

contradicting the minimality of G. Hence (9) holds.

We also claim that for each $i = 0, \ldots, r-2$,

$$d_i(x) - (|V - V_i| - d_{cr}(x)) \leqslant k - 1 - \sum_{j \neq i} v_j. \tag{10}$$

To see (10), we need only observe that,

$$\begin{aligned}
&d_i(x) - (|V - V_i| - d_{cr}(x)) \\
&\quad \leqslant k - 1 - \sum_{j \neq i} [v(G_j[N(x) \cap V_j]) + |V_j| - d_j(x)] \quad \text{by (2)} \\
&\quad \leqslant k - 1 - \sum_{j \neq i} v_j,
\end{aligned}$$

where the last inequality holds since any matching in G_j has at most $|V_j| - d_j(x)$ edges with one or both endpoints outside $N(x) \cap V_j$. This proves (10).

We can also assume that for each $i = 0, \ldots, r-2$,

$$1 \leqslant \sum_{j \neq i} v_j \leqslant k - 2, \tag{11}$$

by the following arguments. If $\sum_{j \neq i} v_j = 0$, then G_j is empty for every $j \neq i$, and in this case by (1),

$$|E(G_i)| - \left(\sum_{j < \ell} |V_j||V_\ell| - E(G_{cr})| \right) \leqslant |E(G_i)| \leqslant f(k-1, k-1);$$

thus (8) holds trivially, verifying the lemma. If $\sum_{j \neq i} v_j = k - 1$, then by (9) and (10), we would have

$$0 < d_i(x) - (|V - V_i| - d_{cr}(x)) \leqslant 0,$$

a contradiction.

We may further suppose that

$$2 \leqslant v_i \quad \text{for each } i = 0, \ldots, r-2. \tag{12}$$

To the contrary, without loss of generality, assume that $v_0 \leqslant 1$, then (11) implies that $\sum_{i=0}^{r-2} v_i \leqslant k-1$. As

$$\sum_{i=0}^{r-2} f(v_i, \Delta) \leqslant f\left(\sum_{i=0}^{r-2} v_i, \Delta\right)$$

always holds, we get that $\sum_{i=0}^{r-2} |E(G_i)| \leqslant f(k-1, k-1)$ and (8) follows.

Now apply Lemma 3.1 for the graph G_i $(i = 0, \dots, r-1)$ with $\Delta = k-1$ and $b = k - 1 - \sum_{j \neq i} v_j \leqslant \Delta - 2$ (by (12)). Using (10) and (7) we get

$$\begin{aligned}
&\sum_{x \in V_i}\left[d_i(x) - \left(\sum_{j \neq i} |V_j| - d_{cr}(x)\right)\right] \\
&\quad \leqslant \sum_{x \in V_i} \min\left\{d_i(x), k - 1 - \sum_{j \neq i} v_j\right\} \\
&\quad \leqslant v_i\left(2(k-1) - \sum_{j \neq i} v_j\right).
\end{aligned} \tag{13}$$

The left side in (13) equals

$$2|E(G_i)| + \sum_{j \neq i} |E(V_i, V_j)| - \sum_{j \neq i} |V_i||V_j|,$$

so adding these $r-1$ sums (for $i = 0, \dots, r-2$) gives

$$\begin{aligned}
2|E(G)| &= 2\sum_{i=0}^{r-2} |E(G_i) + 2|E(G_{cr})| \\
&= \sum_{i=0}^{r-2}\left(2|E(G_i)| + \sum_{i \neq j} |E(V_i, V_j)| - \sum_{j \neq i} |V_i||V_j|\right) + 2\sum_{i<j} |V_i||V_j| \\
&\leqslant \sum_{i=0}^{r-2} v_i\left(2(k-1) - \sum_{j \neq i} v_j\right) + 2\sum_{i<j} |V_i||V_j| \\
&= 2\left[k^2 - 2k + 1 - (k - 1 - v_0)\left(k - 1 - \sum_{j>0} v_j\right) - \sum_{0 \neq j \neq \ell \neq 0} v_j v_\ell\right] \\
&\quad + 2\sum_{i<j} |V_i||V_j|.
\end{aligned}$$

This yields $|E(G)| \leqslant k_2 - 2k + \sum_{i<j} |V_i||V_j|$ (by (11), $k - 1 - v_0 \geqslant 1$ and $k - 1 - \sum_{i \neq 0} v_i \geqslant 1$), and since $f(k-1, k-1) > k^2 - 2k$, this implies (8), finishing the proof of Lemma 3.2. $\square$

4. Proof of the theorem

We can summarize Lemmas 3.2 and 2.3 as follows:

Lemma 4.1. *Suppose that G is an $F_{k,r}$-free graph on n vertices with $n \geqslant 4k^2r^4$, and with minimum degree $\delta \geqslant \frac{r-2}{r-1}n - k$, then $|E(G)| \leqslant ex(n, K_r) + f(k-1, k-1)$.*

Proof. We can assume that G has the maximum number of edges under the conditions of Lemma 4.1 and apply Lemma 2.3 to get a decomposition of G into $G_0, G_1, \ldots, G_{r-2}, G_{cr}$. The graph G_{cr} consists of the edges between V_i and V_j for all distinct pairs i and j. Lemma 3.2 implies that

$$\begin{aligned}
|E(G)| &= \sum_{i=0}^{r-2} |E(G_i)| + |E(G_{cr})| \\
&\leqslant \sum_{i<j} |V_i||V_j| + f(k-1, k-1) \\
&\leqslant ex(n, K_r) + f(k-1, k-1),
\end{aligned}$$

and we are done. □

Since $ex(n, K_r) - ex(n-1, K_r) = \lfloor \frac{r-2}{r-1} n \rfloor$, we see that the following lemma holds.

Lemma 4.2. *Let G be a graph of order n, let k be an integer and c some constant independent from n. If $|E(G)| \geqslant ex(n, K_r) + c$ and $d(x) \leqslant \frac{r-2}{r-1}n - k$, then $|E(G - x)| \geqslant ex(n-1, K_r) + c + k$.*

Proof of Theorem 2. Suppose that $n \geqslant 16k^3r^8$, and that G is an $F_{k,r}$-free graph on n vertices. We need to show that G has at most $ex(n, K_r) + f(k-1, k-1)$ edges. Suppose, to the contrary, that $|E(G)| > ex(n, K_r) + f(k-1, k-1)$. By Lemma 4.1, there exists a vertex $x = x_n$ with degree $d_G(x_n) < \frac{r-2}{r-1}n - k$.

Denote G by G^n, and let $G^{n-1} = G^n - x_n$. By Lemma 4.2,

$$|E(G^{n-1})| \geqslant ex(n-1, K_r) + f(k-1, k-1) + k.$$

If there exists a vertex $x_{n-1} \in V(G^{n-1})$ with degree $d_{G^{n-1}}(x_{n-1}) < \frac{r-2}{r-1}(n-1) - k$, then delete it to obtain $G^{n-2} = G^{n-1} - x_{n-1}$. Continue this process as long as $\delta(G^i) < \frac{r-2}{r-1}i - k$, and after $n - \ell$ steps we get a subgraph G^ℓ with $\delta(G^\ell) \geqslant \frac{r-2}{r-1}\ell - k$. Note that

$$\ell(\ell-1)/2 \geqslant |E(G_\ell)| \geqslant ex(\ell, K_r) + k(n-\ell) + f(k-1, k-1) \geqslant k(n-\ell).$$

We have that $\ell > \sqrt{kn} \geqslant 4k^2r^4$, a contradiction to Lemma 4.1. □

5. Remark

To avoid tedious calculations, we did not attempt to lower the bound $n \geqslant 16k^3r^8$ in the proof, although we strongly believe the bound can be lowered substantially.

References

[1] H.L. Abbott, D. Hanson, H. Sauer, Intersection theorems for systems of sets, J. Combin. Theory Ser. A 12 (1972) 381–389.
[2] V. Chvátal, D. Hanson, Degrees and matchings, J. Combin. Theory Ser. B 20 (1976) 128–138.
[3] P. Erdős, Z. Füredi, R.J. Gould, D.S. Gunderson, Extremal graphs for intersecting triangles, J. Combin. Theory Ser. B 64 (1) (1995) 89–100.
[4] P. Erdős, A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. 52 (1946) 1089–1091.