Some Unusual Applications of Math

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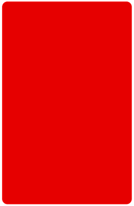
The Tools: A man has three cards, one **red** on both sides, one **blue** on both sides, and one **blue** on one side and **red** on the other side.

Side 1



Side 2

Game 1 - Three Card Game

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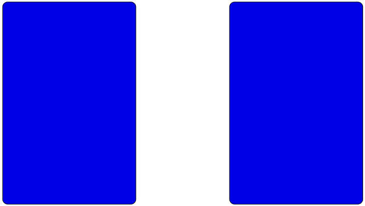

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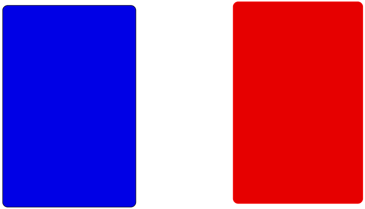

Side 1 Side 2

Play

The Game: He places the three cards into a hat and asks you to blindly select one card and only look at one side of the card. You show him that side of the card and he offers to bet even money he can tell you the color of the other side of the card.

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Is this a fair bet?

Common Thinking

Suppose the side you show is **red**.

Then, most people say this is a fair bet since:

The card is either **red-red**

or the card is **red**-**blue**

and the man has a 50% chance of guessing correctly.

The Scam - I mean the real solution

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- The key is - 2 cards have the same color on both sides.
- The man will always guess the same color as that shown and be correct 2/3 of the time!

Guarini's Problem - Knights on a 3×3 chessboard

K		K
K		K

Problem: Have the knights switch row positions using only legal moves.

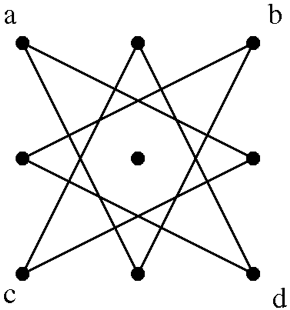
Recall that knights only move in an L shape.

Two questions are now clear:

1. Can this be done?

2. If it can be done, what is the minimum number of moves needed?

A Graph Model



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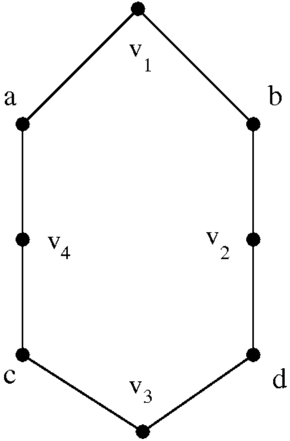
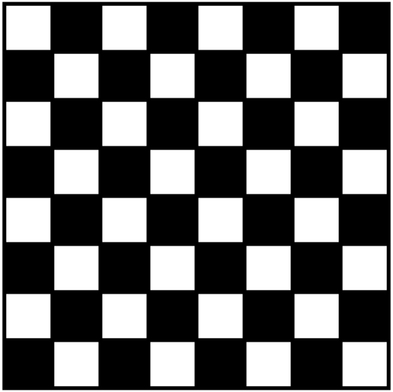


Tiling the chessboard with dominoes



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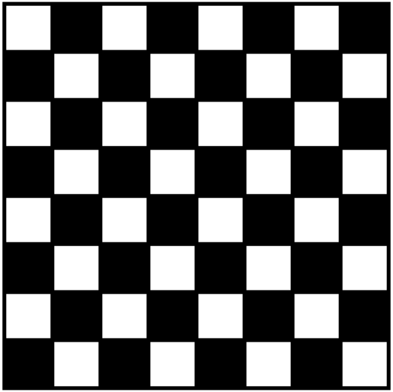
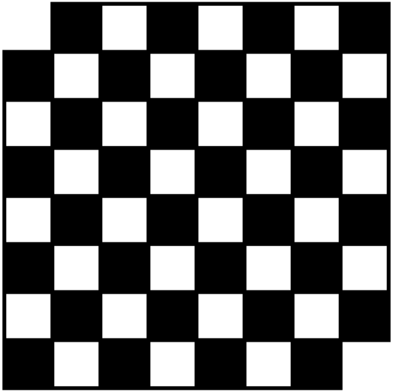


Holes in the Chessboard



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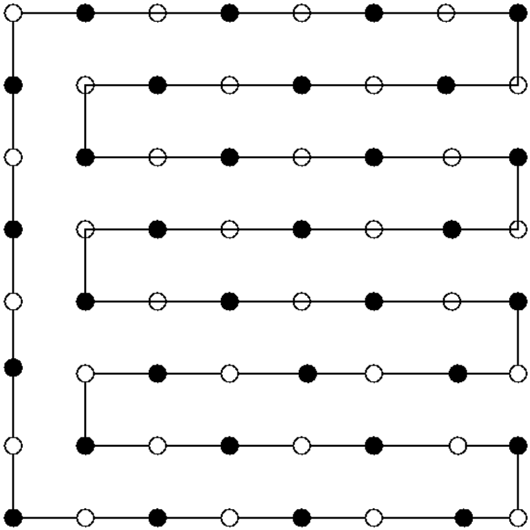
With both the upper left and lower right squares removed, still no tiling is possible, as each dominoe covers one white and one black square, and 30 white and 32 black squares remain!

Theorem

If you remove one black and one white square from anywhere on the 8×8 chessborad, then you can tile the remaining board with 31 dominoes.

Proof of Gomory's Theorem



The coin game - for 2 players

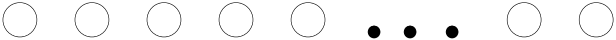
One hundred randomly selected coins are placed in a row. The sum of the values of these coins is T **which is odd**.

GAME: Player 1 selects one coin from either end of the row. Player 2 selects one coin from either end of the row. Play continues, alternating turns, until all coins have been selected.

Problem: Show that there is a strategy so that

Player 1 always wins! (i.e. get most of the money)

Sketch of proof



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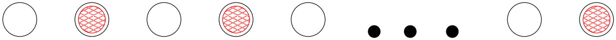

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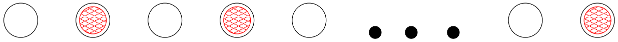

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2. Count the value of the coins in one color to determine which color has the most $(> T/2)$ WLOG say white.

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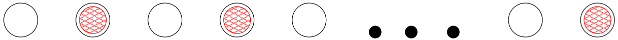

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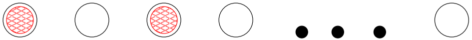

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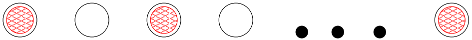

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3. Player 1 selects the end coin that is white.
4. Player 2 now has only red coins on either end to select from.
5. Player 1 now selects the white coin uncovered by the choice Player 2 just made.
6. Player 2 again has only reds to choose from. This continues until the game ends with Player 1 having all the white coins and the win.

The Moral of the Story

Never give a non-mathematician an even break!

The Paint Ball War

The Game: Three friends, **Matt** (the mathematician) **Phil** (the physicist) and **Ed** (the engineer) decide to play paintball.

Matt - limited experience: 50% shooter.
Phil - plays more: 75% shooter.
Ed - plays constantly: 100% shooter.

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- **Ed** will get the third shot (100%).

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Because of the ability differences, they decide to follow some old dueling traditions... so

- **Matt** will get the first shot (50%).
- **Phil** will get the second shot (75%).
- **Ed** will get the third shot (100%).
- Should it be needed, another round then follows among any survivors.

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- Surrender and go to a bar and wait for the winner to join him there later!

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- If he shoots (and hits) Ed, then there is a 75% chance Phil shoots him.
- If he shoots Phil, then Ed immediately will shoot him!
- Surrender and go to a bar and wait for the winner to join him there later!
- Matt's best move is to **purposely miss!**

Why is this a good move?

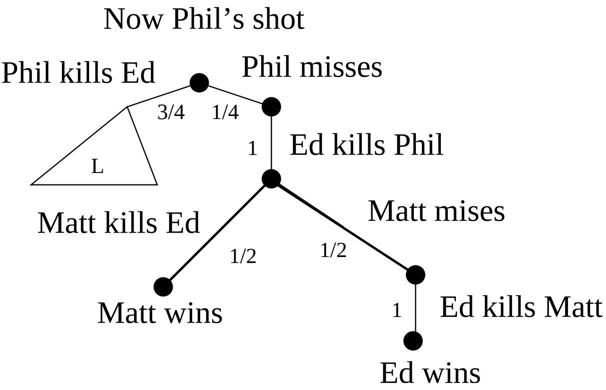

Figure : The strategy tree diagram.

$P(\text{ Ed wins }) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = .125$

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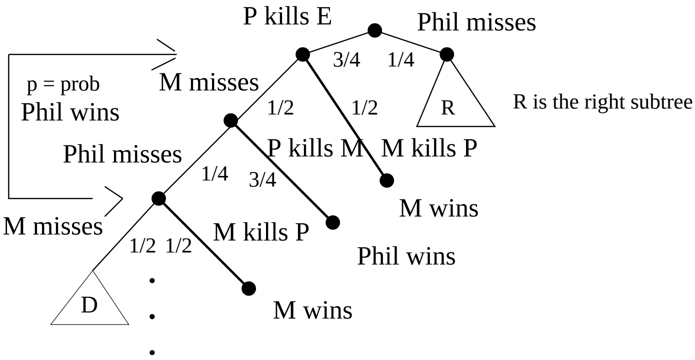

Figure : The strategy tree diagram.

Why is this a good move?

Use the fact we are repeating turns.

The subtree D contains this repeated pattern.

$$(3/4 \times 1/2 \times 3/4) = x$$

then

$$x \times 1/4 \times 1/2 = y$$

then

$$y \times 1/4 \times 1/2 \text{ etc.}$$

So we get a geometric series with first term
$a = 3/4 \times 1/2 \times 3/4 = 9/32$ and ratio $r = 1/4 \times 1/2$
Hence, $P(\text{Phil wins}) = a/(1-r) = 72/224 \approx .312$.
Now

$$P(Matt \text{ wins}) = 1 - P(Phil \text{ wins}) - P(Ed \text{ wins}) = 1 - .321 - .125 = .554.$$

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The 5 card trick - due to Fitch Chaney, 1950

Suppose you and a partner want to perform the following 5 card trick.
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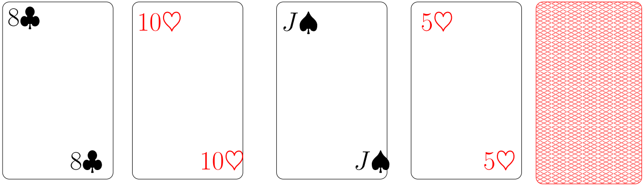

The Trick

Now your partner enters the scene and identifies the face down card exactly!

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2. The **suit** and the **rank** of the down card!

How the suit information is encrypted

Given any 5 cards from a standard deck, the

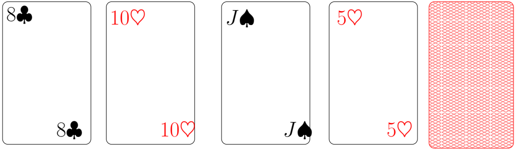
The Pigeon Hole Principle:
There must be **at least 2 cards** of the same suit!

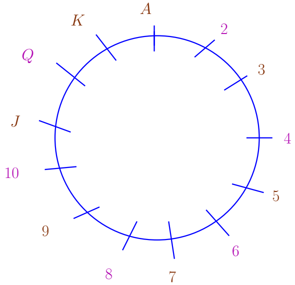
Thus, we can use one card from that suit placed in a predetermined position to signal the suit. Say for now - the left most **position**.
Now how do we handle the rank?

3 "unused cards" - Use the relative ranks to have a **low (L)**, **medium (M)** and **high (H)** card among the 3 remaining cards. The lexicographic orderings of these three produce:

L M H (call this 1)
L H M (call this 2)
M L H (call this 3)
M H L (call this 4)
H L M (call this 5)
H M L (call this 6)

The solution





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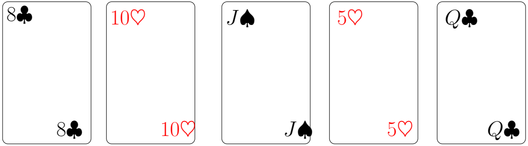

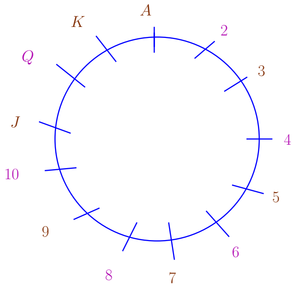

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The Game of Penny Ante

This game was invented by Walter Penny in 1969.

The Game: Ask your opponent to select any pattern of **heads (H)** and **tails (T)** of length three they wish. Then you select a different pattern. Flip pennies repeatedly until one of the two patterns occurs on three consecutive tosses. The person with that pattern wins an even money bet.

Is this a fair game?

Many people think the game is unfair because the second person only gets to choose their pattern after the first person has selected. This means they do not have all the choices the first person had.

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Those are the conditions - but that is not the advantage!

Suppose, for ease of argument, your opponent selected **HHH** .

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Then you will select **THH** .

Case Analysis

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H H H

… ___ ___ ___ ___ …

P = 1/8 that first three flips are heads

Case Analysis

Now suppose this is the first HHH in the sequence.

... ___ H ___ H ___ H ___ ...

Case Analysis

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...	T	H	H	H	...

Then THH had to occur first!

Thus, 7/8 of the time THH wins!

An analysis of all the other possibilities (more involved tree structures and infinite geometric series) shows that the following rule always wins with probability at least 2/3.

Let the opponent select a pattern of three. Now move their first two choices to your last two choices and fill the first position with what ever choice does not form a palindrome!

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2 decks problem

Suppose two standard decks of cards are individually shuffled and placed on a table. Now, the top card of each deck is turned over to determine if there is an **EXACT MATCH.** This process is continued throughout the remaining cards.

Given an even money bet - would you bet there was an exact match before the cards run out?

The quick answer

One view of this problem is that when the card from deck 1 is turned over, there is a $\frac{1}{52}$ chance the card in deck 2 will match it. Thus, the expected number of matches when going through the entire deck would be:

$$np = (52)\frac{1}{52} = 1.$$

Thus, the **actual length of the deck** does not seem to matter.

Can we squeeze more out of this?

That rough answer seems to say it would seem a wise bet to say there would be a match.

BUT HOW WISE???.

What is really going on here?

Another way to look at this problem, and one that brings far more to the table, is to think of the 2nd deck as a permutation of the first deck. Now, the question we are asking is how many permutations of the deck have at least one position unchanged (a fixed point)?

This unchanged position gives us an exact match!

A permutation with no fixed points is called a **derangement.**

Counting derangements - use IEP

At this point, students would be familiar with De Morgan's Laws. So the step to the IEP is a natural one.

Suppose we want to count how many permutations (of our 52 cards) have at least i matches with deck 1.

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Then, applying the IEP we obtain:

Counting derangements via IEP:

$$D_{52} \; = \; 52! \sum_{i=0}^{i=52} \frac{(-1)^i}{i!}$$

.

But we know from Calculus that:

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e} \approx .3679.$$

Hence, $P(derangement) \approx .3679$ and thus $P(match) \approx .63$.
Thus, it is a very good even money bet!

Game: Given $t \geq 2$ piles of chips, two players proceed as follows:

Player 1 removes some number of chips from exactly one pile.

Player 2 then removes some number of chips from exactly one pile.

Players alternate turns until no chips remain. The first player that cannot remove some chips loses.

Is there a strategy for optimal play?

To find the strategy, it really helps to start at the end of the game.

When no chips remain we call this the **ZERO POSITION.**

Your goal as a player is to put your oponent in the zero position.
How can we achieve this?

Base two representations

The key is to think of each pile of chips as a number and to look at the base 2 representation of this number.

Say we have three piles of 8, 5 and 6 chips.

Base 2 view

8: 1 0 0 0
5: 0 1 0 1
6: 0 1 1 0

Base 2 view

8:	1	0	0	0
5:	0	1	0	1
6:	0	1	1	0

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A: 1 0 1 1
(not a zero position)

What to do?

Player 1 would like to put the other player into a zero position.

We think of a zero position as 0 0 0 0

So to get our parity counts above to a zero position we must remove chips from one pile so that all the colums have an even number of ones:

This can always be done!

One such move

Looking at our example:

8: 1 0 0 0
5: 0 1 0 1
6: 0 1 1 0

A: 1 0 1 1
(not a zero position)

Removing 5 chips from the pile of 8 will produce what we need!

3: 0 0 1 1
5: 0 1 0 1
6: 0 1 1 0

A: 0 0 0 0
(zero position)

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Player 1 will then repeat the strategy of moving to a zero position! Eventually player 1 will reduce to the ultimate zero position - no chips, and win the game