Math and Marriage - Don't Call a Lawyer Yet!

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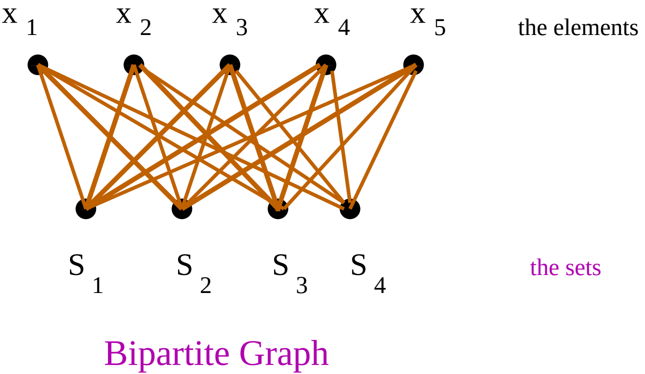
Philip Hall, 1935

Definition
By a **system of distinct representatives** for the sets $S_1, \ldots, S_k$, we mean a sequence $x_1, \ldots, x_k$ such that $x_i \in S_i$, for $i = 1, \ldots, k$ and the x_i are all distinct.

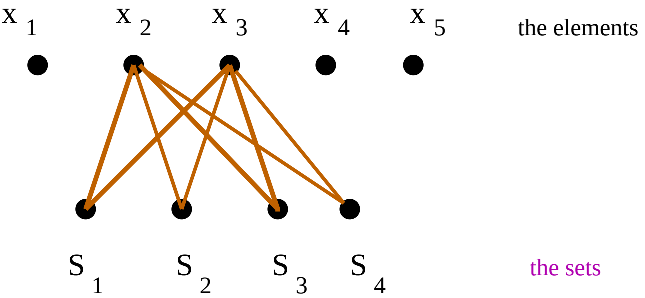
Theorem
The finite sets $S_1, \ldots, S_n$ have a system of distinct representatives if and only if for every $I \subseteq \{1, \ldots, n\}$,

$$|\cup_{i \in I} \; S_i| \geq |I|.$$

Bipartite Graphs - A Natural Model and Condition



Bipartite Graphs - A Natural Model and Condition



Bipartite Graph

Philip Hall - British Group Theorist (1904-1982)

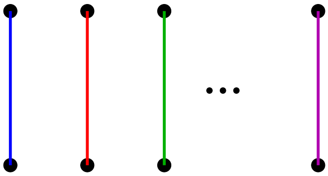
Not really the first - Dénes König, 1916

Theorem

Any k-regular bipartite graph contains a 1-factor, (1-regular spanning subgraph).
In fact, the edge set can be decomposed into k edge disjoint 1-factors.

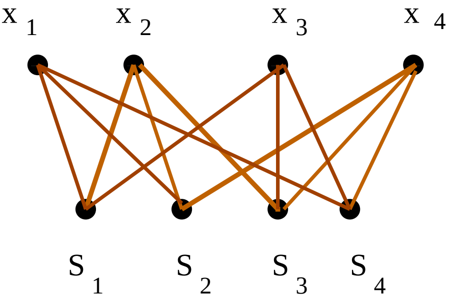
Hence, we can build a set of marriages, remove that set and do it again; a total of k times.

1-factors = 1 regular graphs = marriages



1−factor = set of "marriages"

Example



Bipartite Graph

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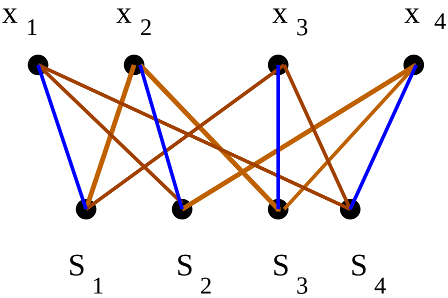

Bipartite Graph

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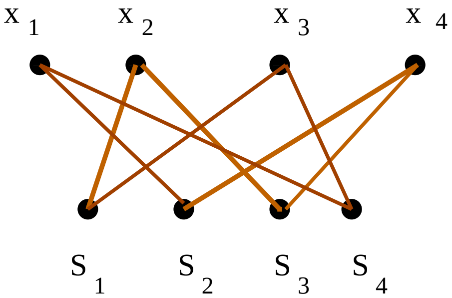
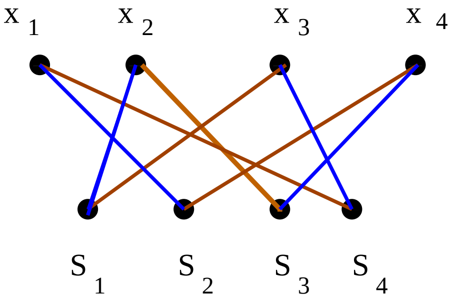


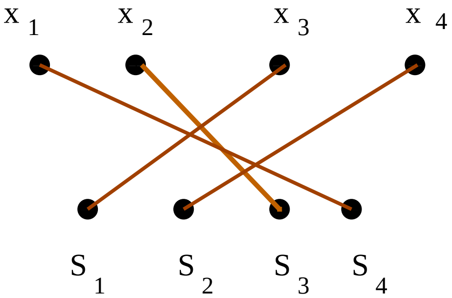
Bipartite Graph

Example



Bipartite Graph

Example



Bipartite Graph

Dénes König - Hungarian Mathematician (1884-1944)

Who coined the term "marriage" theorem?

Paul Halmos and Herbert Vaughn,
The Marriage Problem, American Journal of Mathematics, Vol. 72, No. 1, Jan. 1950, 214-215.

This seems to be the first paper to tie Hall's system of distinct representatives to "marriages".

The sets are the men and the distinct representatives are the wives.

Paul Halmos and Herbert Vaughn

They go on to consider "multi-representatives", that is, sets of representatives associated with each set.
To each mathematician m they assign an integer valued function $g(m)$.

Theorem

A necessary and sufficient condition that each mathematician m may establish a collection of groupies of size $g(m)$ is that for every finite subset I of mathematicians, the number of groupies associated with these mathematicians is at least as large as

$$|\sum_{m \in I} g(m)|.$$

A natural extension of Hall's Theorem.

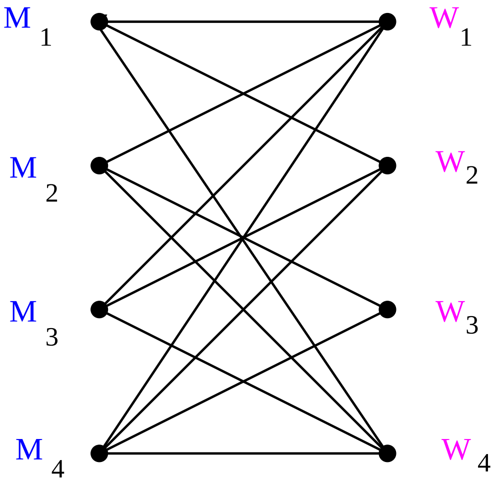
Paul Halmos - Hungarian born American Mathematician (1916-2006)

Algorithms

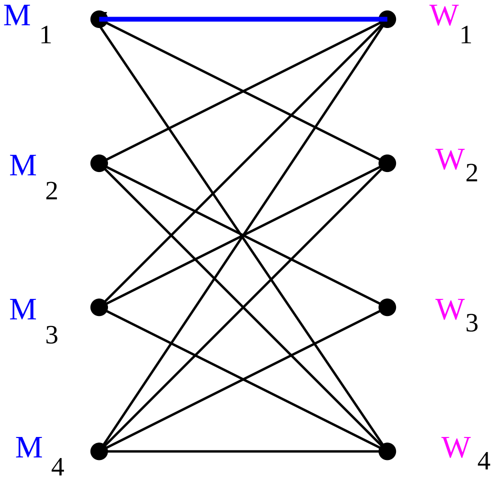
There are algorithms to find matchings in graphs and especially in bipartite graphs.

A simple standard algorithm is based on augmenting paths and the work of Claude Berge.

Augmenting Paths



Augmenting Paths



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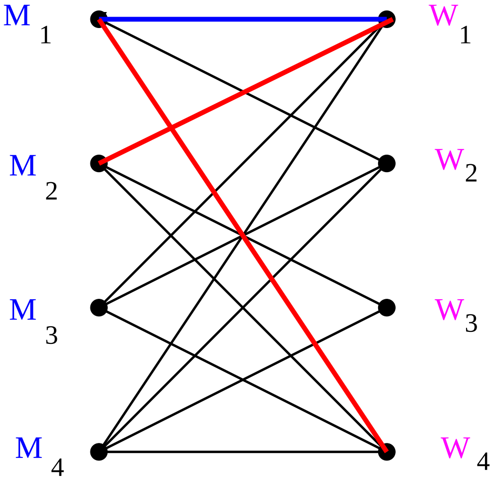

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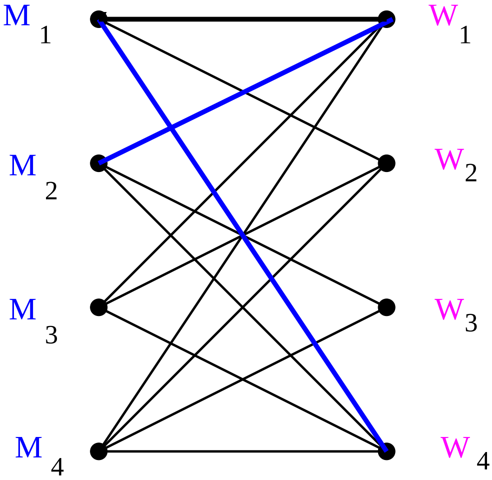

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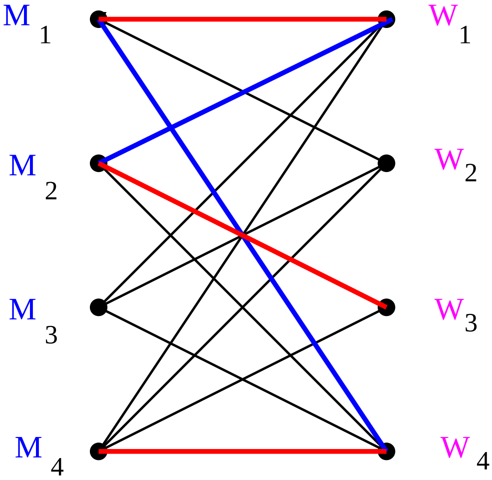

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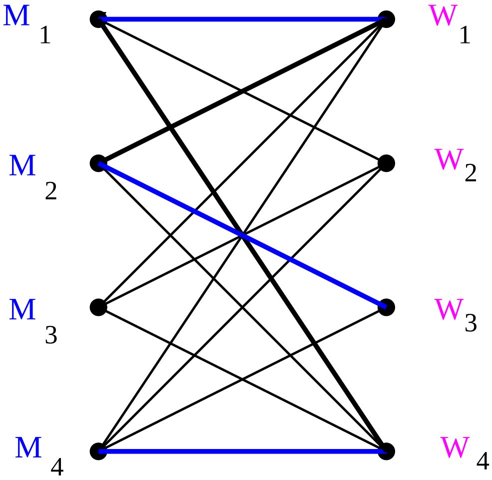

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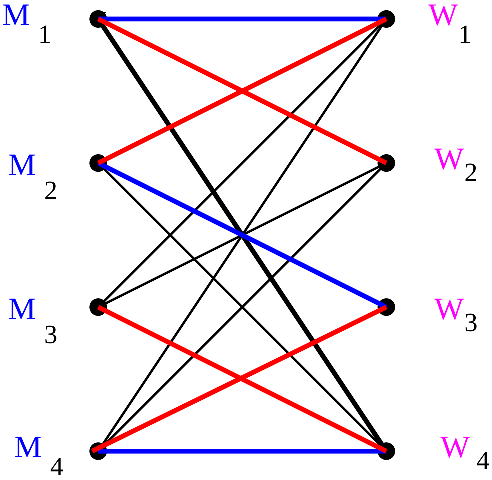

Claude Berge - French Graph Theorist (1926-2002)

Suppose each woman assigns a suitability value $S(m)$ to each man m.

Now suppose we wish to **maximize the minimum suitability**.

This can be done simply by repeatedly finding matchings in this labeled bipartite graph.

Steps

1. Find a set of marriages (perfect matching) in the graph.
2. Find the minimum suitability b in this set of marriages.
3. Remove from the graph all edges with weight b or less.
4. Find (if possible) a new matching in the modified graph. If this is not possible, the last matching was the best you could do. If it is possible, repeat the entire process.

The last problem is called the **bottleneck assignment problem**.

This is the mathematical version of the old proverb that the strength of a chain is only as good as its weakest link!

Stability in Marriage

The classic stability example:

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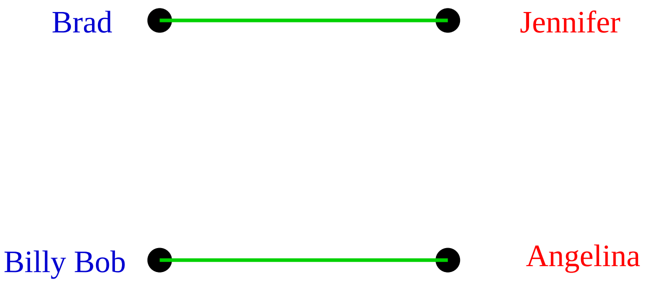

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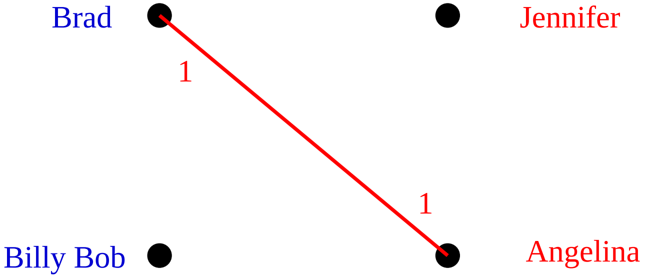

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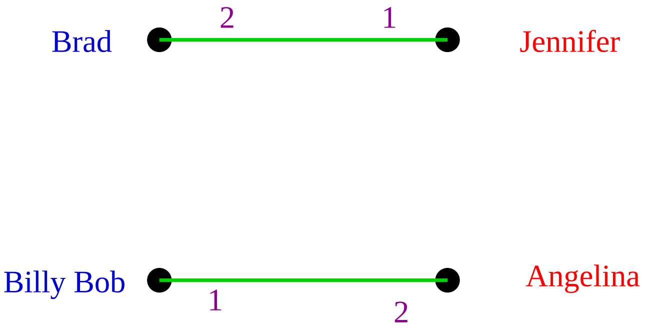

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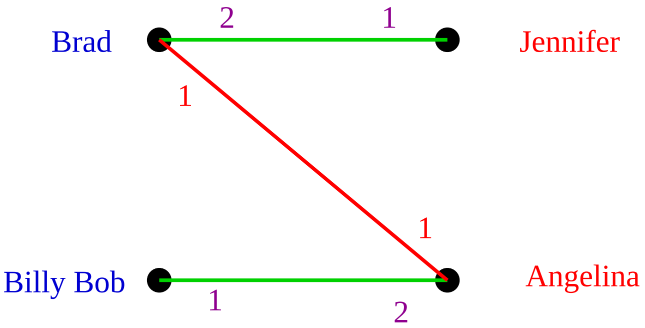


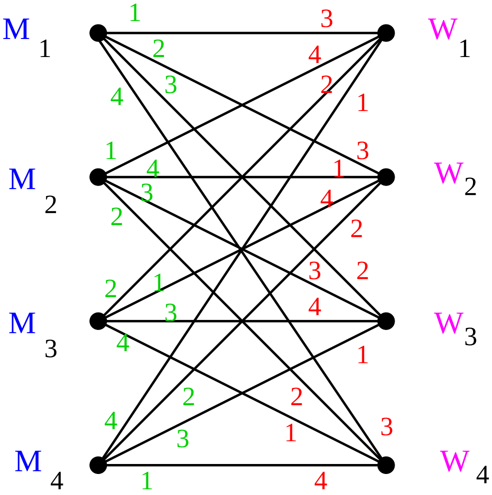
David Gale and Lloyd Shapley - 1962

Problem

The stable marriage problem asks whether it is always possible to find a set of stable marriages when two n-sets each have their own preference lists?

David Gale (1921-2008) and Lloyd Shapley (1923-)

Example - Preference Graph



The Algorithm

* indicates a rejected proposal!

Proposals	P_1	P_2	P_3	P_4	P_5	P_6
m_1	1					
m_2	1*					
m_3	2					
m_4	4					

The Algorithm

* indicates a rejected proposal!

Proposals	P_1	P_2	P_3	P_4	P_5	P_6
m_1	1	1				
m_2	1*	4				
m_3	2	2				
m_4	4	4*				

The Algorithm

* indicates a rejected proposal!

Proposals	P_1	P_2	P_3	P_4	P_5	P_6
m_1	1	1	1			
m_2	1*	4	4			
m_3	2	2	2*			
m_4	4	4*	2			

The Algorithm

* indicates a rejected proposal!

Proposals	P_1	P_2	P_3	P_4	P_5	P_6
m_1	1	1	1	1*		
m_2	1*	4	4	4		
m_3	2	2	2*	1		
m_4	4	4*	2	2		

The Algorithm

* indicates a rejected proposal!

Proposals	P_1	P_2	P_3	P_4	P_5	P_6
m_1	1	1	1	1*	2*	
m_2	1*	4	4	4	4	
m_3	2	2	2*	1	1	
m_4	4	4*	2	2	2	

The Algorithm

* indicates a rejected proposal!

Proposals	P_1	P_2	P_3	P_4	P_5	P_6
m_1	1	1	1	1*	2*	3
m_2	1*	4	4	4	4	4
m_3	2	2	2*	1	1	1
m_4	4	4*	2	2	2	2

Obvious result from algorithm - due to Stephen Stills

Theorem

If you can't be with the one you love,
love the one you're with.

Who benefits most from the algorithm?

Note that in the final set of marriages, each woman is paired with the highest rated man that proposed to her.

Thus, it would seem that the women benefit the most from the algorithm.

But is that really true?

Would men write such an algorithm?

Say person Q is **within the realm of possibility** for person P
if and only if
there is a stable set of marriages where P marries Q.

A person's **optimal mate (OM)** is his or her favorite from within the realm of possibility.

A person's **inferior mate (IM)** is his or her least favorite from the realm of possibility.

The kicker!

Theorem

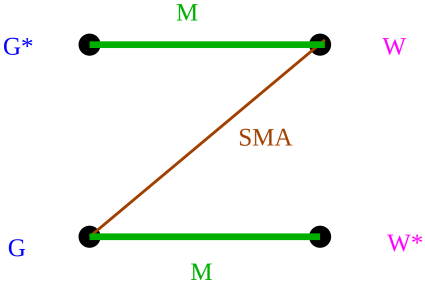
The stable marriage algorithm (SMA)

A. pairs every man with his **OM** *and*

B. pairs every woman with her **IM**.

Proof of B

Suppose A. is true. And assume woman W could actually be married to someone worse than her mate from SMA.



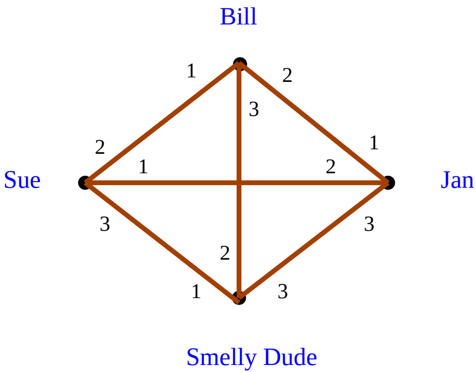
By A., G prefers W to W* and by assumption W prefers G to G*, but this is unstable, a contradiction to the choice of M.

So far we have considered marriages in bipartite graphs.

But what if the graph is not bipartite?
(Not that there is anything wrong with that - Jerry Seinfeld)

This has come to be called **the roomates problem**.

Fact: You cannot ensure stability in the roommate problem.



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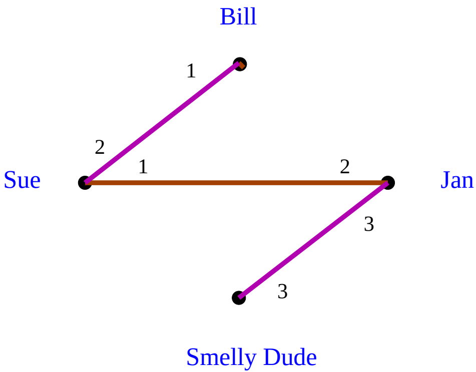

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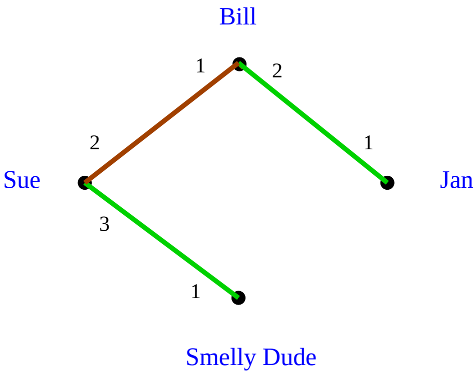

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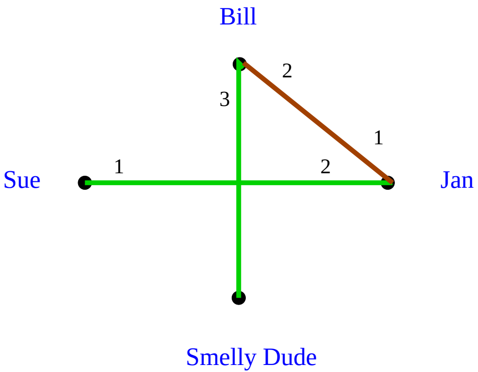


A Different Look

Definition

An $n \times n$ matrix D is **doubly stochastic** if its entries are nonnegative and the sum of any row or column is 1.

Examples:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} .25 & .25 & .25 & .25 \\ 0 & .5 & .5 & 0 \\ .5 & 0 & 0 & .5 \\ .25 & .25 & .25 & .25 \end{bmatrix}$$

Permutation Matrix

General Doubly Stochastic Matrix

Birkoff and von Neumann

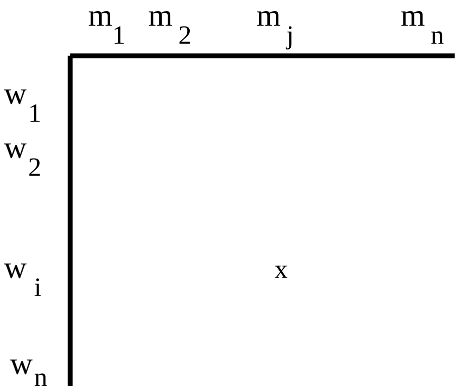
Theorem

Any doubly stochastic matrix $M = (m_{ij})$ can be written as a combination of suitable permutation matrices.

That is, there exist constants $c_1, \ldots, c_n$ and permutation matrices $P_1, \ldots, P_n$ such that

$$M = \sum_{i=1}^{n} c_i \, P_i$$

Interpretation



We can view the entries of a doubly stochastic matrix as the "happiness" measures of individual marriages within a polygamous system.

We can also associate another doubly stochastic matrix

$$T = (t_{ij})$$

which represents the fraction of time each couple spends together.

For the doubly stochastic matrix T, define the happiness function $H(T)$ to be

$$H(T) = \sum_{i,j} m_{ij}\ t_{ij}$$

Our goal of course is to **maximize** $H(T)$.

$$
\begin{aligned}
max_T \ H(T) &= max_{c_1,\ldots,c_n} H(c_1 P_1) + \ldots + H(c_n P_n) \\
&= max_{c_1,\ldots,c_n} c_1 H(P_1) + \ldots + c_n H(P_n) \\
&= max_{c_1,\ldots,c_n} c_1 \sum_{ij} m_{ij} p_{ij}^1 + \ldots + c_n \sum m_{ij} p_{ij}^n \\
&= H(P_i)
\end{aligned}
$$

for some i.

Theorem

The optimal form of marriage is monogamy.

Food for Thought

Essentially the same proof (minimizing) shows that

Theorem

The pessimal form of marriage is monogamy.