On Chorded Cycles

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Oct. 24, 2016

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Theorem (Corradi and Hajnal)

If $\delta(G) \geq 2k$ and $|G| \geq 3k$ then G contains k vertex disjoint cycles.

Over the years there have been many results that find conditions sufficient for cycles (often with various properties like containing a set of vertices, or a set of edges, etc.).

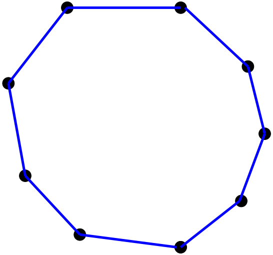
But the one property that was greatly ignored was the following:

An Old Question by Posa, 1960

Question

What conditions imply a graph contains a cycle with a chord?

Here a **chord** is an edge between two vertices on the cycle that is not an edge of the cycle.



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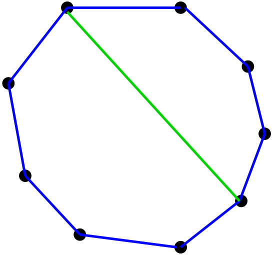

First answer by J. Czipzer 1963 - using min deg $\delta(G)$

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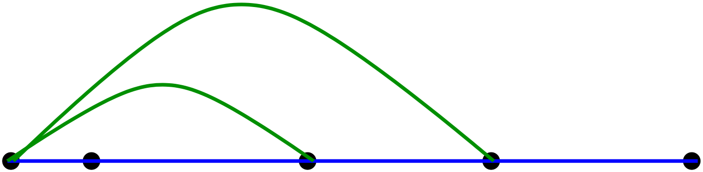
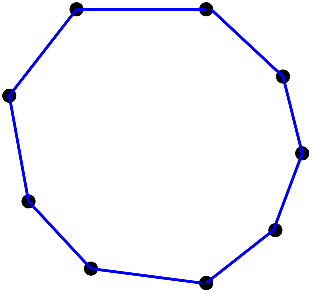


Sharpness Example

Minimum degree 2 is not enough! Simply take any cycle.



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- Other conditions for a chorded cycle.
- Some specified number of disjoint chorded cycles.
- Some specified number of disjoint doubly chorded cycles.
- Cycles with a designated minimum number of chords.

Some History (45 years later) - D. Finkel, 2008

Theorem

If G is a graph on $n \geq 4k$ vertices with minimum degree $\delta(G) \geq 3k$, then G contains at least k vertex disjoint chorded cycles.

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$\delta(G) \geq 3$ **implies chorded cycle.**

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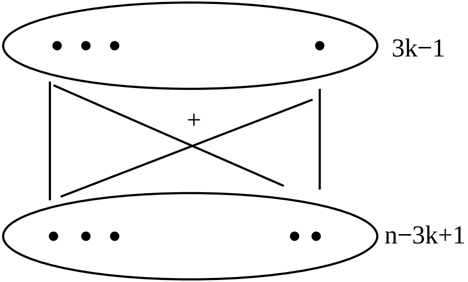
$\delta(G) \geq 2$ **implies cycle.**

Clearly, $n \geq 4k$ is needed as the cycles need at least 4 vertices each.

For $n \geq 6k$, the graph $K_{3k-1,n-3k+1}$ has $\delta = 3k - 1$ and no collection of k vertex disjoint chorded cycles, as chorded cycles here require 3 vertices from each partite set.



$K_{3k-1, n-3k+1}$

Bialostocki, Finkel and Gyarfas, 2008

Conjecture

Let r, s be nonnegative integers and G a graph with order at least $3r + 4s$ and minimum degree $\delta(G) \geq 2r + 3s$.
Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.

They proved this conjecture for $r = 0$, $s = 2$ and for $s = 1$ and every r.

Bialostocki, Finkel and Gyarfas, 2008

Theorem

Let G be a graph with order at least 8 and $\delta(G) \geq 6$, then G contains two vertex disjoint chorded cycles.

They also settled the extremal problem of the minimum number of edges in a graph on n vertices ensuring two vertex disjoint chorded cycles.

Chiba, Fujita and Gao, 2010

Settled the conjecture completely, actually proving more.

Theorem

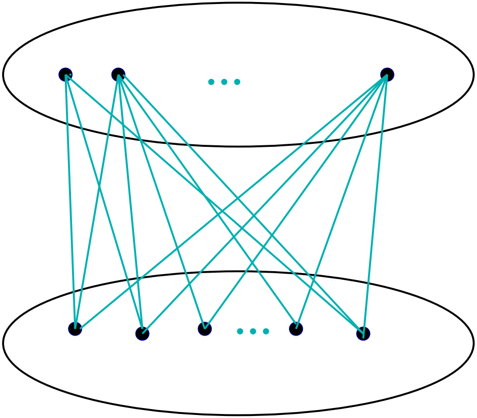
Let r and s be integers with $r + s \geq 1$. Let G be a graph of order at least $3r + 4s$. If

$$\sigma_2(G) \geq 4r + 6s - 1,$$

then G contains a collection of $r + s$ vertex disjoint cycles, such that s of them are chorded.

Sharpness example

$K_{2r+3s-1\,,\,n-2r-3s+1}$



Theorem

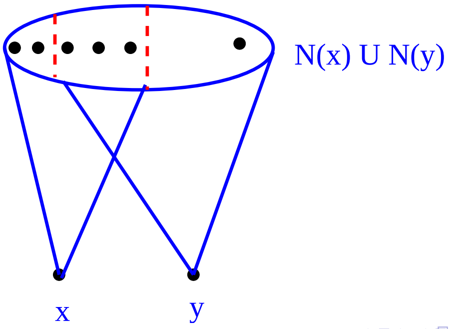
Let k be a positive integer and G a graph of order $n \geq 4k$ with $\sigma_2(G) \geq 6k - 1$. Then G contains k vertex disjoint chorded cycles.

Theorem

If G is a graph on $n \geq 4k$ vertices such that for any pair of non-adjacent vertices x, y,

$$|N(x, y)| \geq 4k + 1,$$

then H contains at least k vertex disjoint chorded cycles.



S. Qiao and S. Zhang, 2010

Theorem

If G is a graph with at least $4k$ vertices and minimum degree at least $\lceil \frac{7k}{2} \rceil$, then G contains k vertex disjoint cycles, each with at least 2 chords.

Call such cycles *doubly chorded cycles (or DCC's).*

RG, K. Hirohata, P. Horn

Theorem

If G is a graph on $n \geq 6k$ vertices with

$$\sigma_2(G) \geq 6k - 1,$$

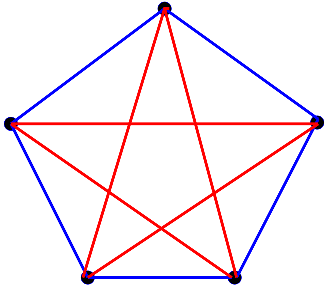
then G contains k vertex disjoint doubly chorded cycles.

Question

How many chords should we expect or hope to find?

Cycles with many chords?

A special case: Cliques



K_5: 4-regular but with 5 chords

In general:

Given a K_{k+1}: It is k-regular with

$$f(k) = \frac{(k-2)(k+1)}{2}$$

chords. We think of $f(k)$ chorded cycles as "loose" K_{k+1} cliques.

Note: There are no single chorded cliques.

Theorem

Ali, Staton - 1999

If $\delta(G) = k$, then G contains a

$$\left\lceil \frac{k(k-2)}{2} \right\rceil - \textit{ chorded cycle.}$$

Note: There are no single chorded cliques.

Theorem

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If $\delta(G) = k$, *then* G *contains a*

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Corollary

If $\delta(G) \geq 3$, *then* G *contains a doubly chorded cycle - that is, a loose* K_4.

Theorem

If $\delta(G) = k$, then G contains an

$$f(k) = \frac{(k+1)(k-2)}{2} - \text{ chorded } \text{ cycle.}$$

RG, P. Horn, C. Magnant: Our first step -

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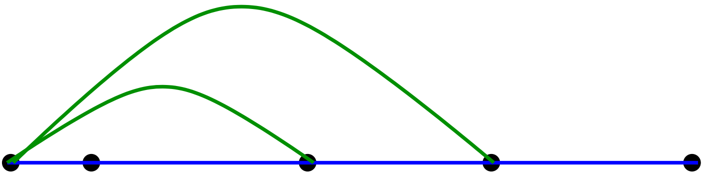

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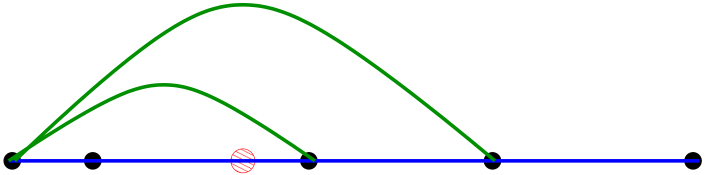

The story so far:

Lower End: $\delta(G) \geq k$ implies an $f(k) = \frac{(k-1)(k+2)}{2}-$chorded cycle.

Upper End:

Theorem

Hajnal and Szemerédi

If $\delta(G) \geq kt$, $|G| = (k+1)t$, then G can be covered by t vertex disjoint K_{k+1}'s.

Conjecture

If $\delta(G) \geq kt$, and $|G| \geq (k+1)t$ then G contains t

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Tight End: Hajnal-Szemerédi Theorem.
If $t = 1$, this is our first Theorem.
We show it is true for some classes of graphs, and for graphs with some extra "room".

RG, P. Horn, C. Magnant 2012

Theorem

There exist k_0, t_0 such that if $\delta(G) \geq kt$, where $k \geq k_0$, $t \geq t_0$ and $n \geq n_0(k,t)$, then G contains t disjoint cycles with at least

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- Bounds for k_0 and t_0 show a tradeoff.
- Bounds for n_0 quite large.

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- minimum degree $\leq k-1$,
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- Problem: Not very useful!

Possibly More Useful:

Theorem

Let $d = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil$.

- If G has average degree at least $2d$, then G contains a $f(k) = \frac{(k+1)(k-2)}{2}$-chorded cycle.
- There exist graphs with average degree $2d - o(1)$ with no $f(k)$-chorded cycle.

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- $k = 2, 3, 4$: Trivial induction removing vertex of lowest degree if $< \delta$.
- $k \geq 5$ much tougher induction.

Bondy's Meta-Conjecture

Conjecture

Almost any non-trivial condition on a graph which implies the graph is hamiltonian (contains a spanning cycle) also implies the graph is pancyclic (contains cycles of all possible lengths). There may be a simple family of exceptional graphs (including small order exceptions).

With M. Cream and K. Hirohata we extended Bondy's meta-conjecture.

Definition

We call a graph chorded pancyclic is it contains a chorded cycle of all possible lengths from 4 to the order of the graph.

Conjecture

Almost any non-trivial condition on a graph which implies the graph is hamiltonian (contains a spanning cycle) also implies the graph is chorded pancyclic (contains cycles of all possible lengths). There may be a simple family of exceptional graphs (including small order exceptions).

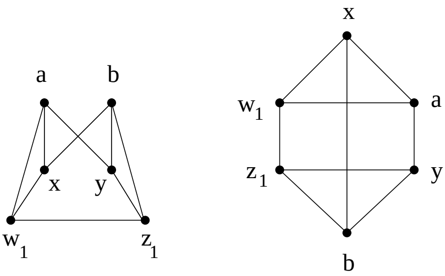
Evidence

Theorem

(Bondy, 1971) If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is pancyclic or $G = K_{n/2,n/2}$.

Theorem

Let G be a graph of order $n \geq 4$. If $\sigma_2(G) \geq n$, then G is chorded pancyclic, $G = K_{n/2,n/2}$, or $G = G_6$.



Two drawing of G_6 which has no chorded 4-cycles.

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- Can we place k-path linear forest on k disjoint chorded cycles?
- Can we control the order of the chorded cycles?
- Can we expand our chorded cycle system to span $V(G)$?

Question

Can we make many short chorded cycles?

with Chen, Hirohata, Ota and Song

Theorem

Let k be a natural number. Then there exists a positive integer n_k such that if G is a graph with $\delta(G) \geq 3k + 8$ and order at least n_k, then G contains k vertex disjoint chorded cycles of the same length.

Theorem

Let G be a multigraph of order n and minimum degree at least 5. Then G contains a chorded cycle of length at most $c_0 \log_2 n$, where $30 \leq c_0 \leq 300$ is a constant.

Specifing certain edges to be chords

Question

Can we make an independent set of k edges the chords of k vertex disjoint cycles?

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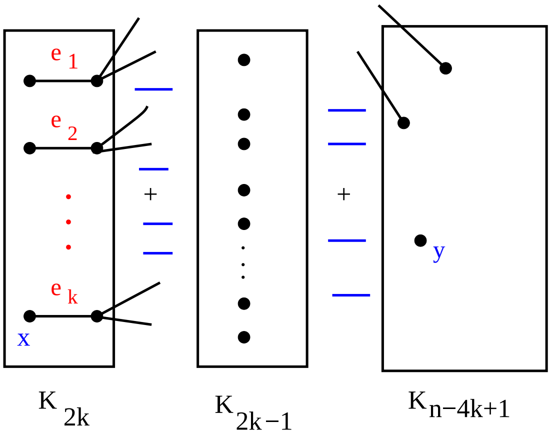
Can we make an independent set of k edges chords of k vertex disjoint cycles?

Theorem

Let $k \geq 1$ be an integer and G be a graph of order $n \geq 14k$. If $\sigma_2(G) \geq n + 3k - 2$, then for any k independent edges $e_1, e_2, \ldots, e_k$ of G, the graph G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that e_i is a chord of C_i for all $1 \leq i \leq k$. Furthermore, $4 \leq |V(C_i)| \leq 5$ for each i.

Sharpness Example



Question- Placing vertices on chorded cycles

Question

When can we distribute k vertices on k disjoint chorded cycles?

Placing vertices on chorded cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 16k - 12$. If $\delta(G) \geq n/2$ then for any set of k vertices $\{v_1, v_2, \ldots, v_k\}$ there exists a collection of k vertex disjoint chorded cycles $\{C_1, \ldots, C_k\}$ such that $v_i \in V(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let G be a graph of order $n \geq 18k - 2$ and let $e_1, e_2, \ldots, e_k$ be a set of k independent edges in G. If

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k chorded cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

[with M. Cream, R. Faudree and K. Hirohata]

As a Corollary to the proof we obtain the fact the edges

$$e_1, e_2, \ldots, e_k$$

can be a mix of either chords or edges of the cycles (again one edge per cycle).

Further, we can show that the cycle system can also be extended to span $V(G)$.

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let G be a graph of order $n \geq 22k - 2$ and let $e_1, \ldots, e_k$ be k independent edges in G. Then if

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k vertex disjoint doubly chorded cycles $C_i, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Corollary

The above system can be extended to span $V(G)$.

Containing Linear Forests

Fact

Given independent path $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ with each $r_i \geq 2$ let $r = \sum r_i$. Then the number of interior vertices in this path system is $r - 2k$.

Theorem

Let $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ be a linear forest in a graph G of order $16k + r - 2$ with

$$\delta(G) \geq n/2 + r - 1 - k.$$

Then there exists a system of k chorded cycles $C_1, \ldots C_k$ such that the path P_{r_i} lies on the cycle C_i and $|V(C_i)| \leq r_i + 4$.