R-equivalence in adjoint classical groups over fields of virtual cohomological dimension 2

Amit Kulshrestha and R. Parimala

Abstract

Let F be a field of characteristic not 2 whose virtual cohomological dimension is at most 2. Let G be a semisimple group of adjoint type defined over F. Let $RG(F)$ denote the normal subgroup of $G(F)$ consisting of elements R-equivalent to identity. We show that if G is of classical type not containing a factor of type D_n, $G(F)/RG(F) = 0$. If G is a simple classical adjoint group of type D_n, we show that if F and its multi-quadratic extensions satisfy strong approximation property, then $G(F)/RG(F) = 0$. This leads to a new proof of the R-triviality of F-rational points of adjoint classical groups defined over number fields.

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Introduction

In [Ma, Chapter II, §14] Manin introduced the notion of R-equivalence on a variety X over a field F as follows : two points $x, y \in X(F)$ are *R-equivalent* if there exist $x = x_0, x_1, x_2, \cdots, x_n = y \in X(F)$ and F-rational maps $f_i : \mathbb{P}^1 \dashrightarrow X; 1 \leq i \leq n$, regular at 0 and 1 such that $f_i(0) = x_{i-1}$ and $f_i(1) = x_i$. If X is the underlying variety of a connected algebraic group G, then the set of elements of $G(F)$ which are

R-equivalent to 1, is a normal subgroup $RG(F)$ of $G(F)$. We denote the quotient $G(F)/RG(F)$ by $G(F)/R$. A connected algebraic group is called *R-trivial*, if for all field extensions E of F, we have $G(E)/R = 0$. Colliot-Thélène and Sansuc [CTS] proved that if the variety of a connected algebraic group G is stably rational then G is R-trivial. For example, if G is an adjoint classical group of type ${}^1A_n$, ${}^2A_{2n}$ [VK, pp. 240] or B_n, then G is rational, and hence R-trivial.

Let G be a classical group of adjoint type defined over a number field. The group $G(F)/R$ is then trivial. If $\tilde{G}$ is a simply connected cover of G, the triviality of $G(F)/R$ can be deduced from the following results:

(i) [Y] If $\tilde{G}$ is of type ${}^2A_n$ then $\tilde{G}(F)/R = 0$.

(ii) [PR, Theorem 9.5] The group $\tilde{G}(F)$ is projectively simple provided $\tilde{G}$ does not contain a factor of type A_n. In particular the non-central normal subgroup $R\tilde{G}(F)$ coincides with $\tilde{G}(F)$.

(iii) [G, pp. 222] and [CTGP, proof of Cor. 4.11] The natural map $\tilde{G}(F)/R \to G(F)/R$ is surjective.

Number fields are examples of fields of virtual cohomological dimension two. The aim of this paper is to study the group $G(F)/R$ where G is a classical group of adjoint type defined over a field of virtual cohomological dimension two.

Let Γ_F be the Galois group $\mathrm{Gal}(F_s/F)$, where F_s is the separable closure of F. The *cohomological dimension* of F is the least positive integer n such that for all discrete torsion Γ_F-modules M, the Galois cohomology groups $H^i(\Gamma_F, M)$ are zero for all $i \geq n+1$. A field F is said to have *virtual cohomological dimension* n if the cohomological dimension of $F(\sqrt{-1})$ is n. We write $\mathrm{cd}(F)$ to denote the cohomological dimension and $\mathrm{vcd}(F)$ to denote the virtual cohomological dimension of F. We prove that $G(F)/R = 0$ for adjoint groups G of type ${}^2A_n$ and C_n over a field F of virtual cohomological dimension at most 2. For classical groups of type D_n, we prove that if the cohomological dimension of F is at most 2 then $G(F)/R = 0$.

Further, if the virtual cohomological dimension of F is at most 2, then we show that $G(F)/R = 0$, provided F satisfies *certain* approximation properties. These results, in particular, lead to a new proof of the triviality of $G(F)/R$ for adjoint classical groups over number fields.

The main ingredients in the proof of the our results are Merkurjev's computation of $G(F)/R$ for all adjoint groups of classical type [Me2, Th. 1], as well as results on the classification of hermitian forms over division algebras with involution over fields of virtual cohomological dimension two [BP2].

1 Some known results

In this section, we record some known results which are used in the paper. Let F be a field with $\mathrm{char}(F) \neq 2$. Let $Z = F$, or a quadratic extension of F. Let A be a central simple algebra over Z and σ be an involution on A of either kind. If σ is of second kind, then let $Z^\sigma = F$. An element $a \in A^*$ is said to be a *similitude* of (A, σ) if $\sigma(a)a \in F^*$. The similitudes of (A, σ) form a group which we denote by $Sim(A, \sigma)$. The map $\mu(a) = \sigma(a)a$ is a homomorphism $\mu : Sim(A, \sigma) \to F^*$ whose image is denoted by $G(A, \sigma)$. Elements of $G(A, \sigma)$ are called *multipliers*. Let σ be adjoint to a hermitian form h. Then $\lambda \in G(A, \sigma)$ if and only if $\lambda h \simeq h$ [KMRT, Prop. 12.20]. Let $\mathbf{Sim}(A, \sigma)$ denote the algebraic group whose F rational points are given by $Sim(A, \sigma)$. Let $\mathbf{Sim}_+(A, \sigma)$ be the connected component of identity of $\mathbf{Sim}(A, \sigma)$. Let $Sim_+(A, \sigma)$ denote the F-rational points of $\mathbf{Sim}_+(A, \sigma)$. Elements of $Sim_+(A, \sigma)$ are called *proper similitudes*. We denote the group $\mu(Sim_+(A, \sigma))$ by $G_+(A, \sigma)$. Let $R_{Z/F}$ denote the Weil restriction to F. The group of *projective similitudes* is the quotient group

$$\mathbf{Sim}(A, \sigma)/R_{Z/F}(\mathbf{G}_m)$$

which we denote by $\mathbf{PSim}(A, \sigma)$. The group of F-rational points of $\mathbf{PSim}(A, \sigma)$ is $Sim(A, \sigma)/Z^*$. The connected component of the identity of the group $\mathbf{PSim}(A, \sigma)$ is denoted by $\mathbf{PSim}_+(A, \sigma)$

Let $N(Z) = F^{*2}$ or $N_{Z/F}(Z^*)$ according as σ is of first kind or second kind, respectively. Let $\mathrm{Hyp}(A,\sigma)$ be the subgroup of F^* generated by the norms from all those finite extensions of F, where the involution σ becomes hyperbolic. If A is split, the involution σ is adjoint to a quadratic form q over F. The group $G_+(A,\sigma)$ is then denoted by $G_+(q)$ and the group $G(A,\sigma)$ is denoted by $G(q)$. In fact $G_+(q) = G(q)$, because of the existence of hyperplane reflections in the orthogonal group.

Theorem 1.1 (**[Me2, Th. 1]**) *With the notation as above, $N(Z).\mathrm{Hyp}(A,\sigma)$ is a subgroup of $G_+(A,\sigma)$ and further,*

$$\mathbf{PSim}_+(A,\sigma)(F)/R \simeq G_+(A,\sigma)/N(Z).\mathrm{Hyp}(A,\sigma).$$

□

We now record a lemma due to Dieudonné.

Lemma 1.2 (**Dieudonné, [KMRT, Lemma 13.22]**) *Let q be a quadratic form of even rank and $d = \mathrm{disc}(q)$. Let $L = F(\sqrt{d})$. Then $G(q) \subseteq N_{L/F}(L^*)$.* □

The following result of Merkurjev-Tignol extends Dieudonné's lemma.

Lemma 1.3 (**[MT, Th. A]**) *Let A be a central simple algebra of even degree with an orthogonal involution σ. Let $d = \mathrm{disc}(\sigma)$ and let $L = F(\sqrt{d})$. Then $G_+(A,\sigma) \subseteq N_{L/F}(L^*)$.* □

Let q be a non-degenerate quadratic form of rank r over F. Let τ_q be the adjoint involution on $M_r(F)$. Then $\mathrm{Hyp}(M_r(F),\tau_q) = \mathrm{Hyp}(q)$, the subgroup of F^* generated by $N_{L/F}(L^*)$; L varying over finite extensions of F where q becomes hyperbolic. If r is odd, then $\mathrm{Hyp}(q) = 1$.

Theorem 1.4 (**[Me2, pp. 200]**) *Let A be a central simple algebra of odd degree with an orthogonal involution σ. Let q be a quadratic form over F such that σ is adjoint to q. Then $G_+(A,\sigma) = G(q) = \mathrm{Hyp}(q).F^{*2} = \mathrm{Hyp}(A,\sigma).F^{*2} = F^{*2}$.* □

We now record a result due to Knebusch which describes the group of spinor norms of a quadratic form. Let q be a quadratic form over F and $\mathrm{sn}(q)$ denote the subgroup of F^* generated by F^{*2} and representatives of the square classes in the image of the spinor norm map $\mathrm{sn} : \mathrm{SO}(q) \to F^*/F^{*2}$. For a central simple algebra A over F, let $\mathrm{Nrd} : A \to F$ denote the reduced norm map. For $S \subseteq F^*$, we denote by $\langle S \rangle$, the subgroup generated by S in F^*.

Theorem 1.5 **(Knebusch's norm principle, [L, Theorem VII.5.1])** *For a quadratic form q over F we have:*

$$\mathrm{sn}(q) = \langle \{N_{L/F}(L^*) : L/F \text{ is a quadratic extension over } F \text{ and } q_L \text{ is isotropic}\} \rangle.$$

$\square$

The two results recorded below describe the group $G(A, \sigma)$ in the case when σ is unitary or symplectic under further assumptions on the degree of A.

Theorem 1.6 **([Me2, §2])** *Let F be a field with $\mathrm{char}(F) \neq 2$. Let A be a central simple algebra over Z of odd degree with an involution σ of second kind with $Z^\sigma = F$. Then $G_+(A, \sigma) = G(A, \sigma) = \mathrm{Hyp}(A, \sigma) = N(Z)$.* $\square$

Theorem 1.7 **([Me2, §2, Lemma 3])** *Let F be a field with $\mathrm{char}(F) \neq 2$. Let A be a central simple algebra over F of degree $2n$ with n odd. Let σ be a symplectic involution on A. Then $G_+(A, \sigma) = G(A, \sigma) = \mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(A)$.* $\square$

Next results we record are local criteria for elements to be reduced norms or spinor norms for formally real fields F with $\mathrm{vcd}(F) \leq 2$.

Theorem 1.8 **([BP2, Theorem 2.1])** *Let F be a formally real field with $\mathrm{vcd}(F) \leq 2$. Let Ω denote the set of orderings on F. Let A be a central simple algebra over F and $A_v = A \otimes_F F_v$, F_v denoting the real closure of F at v. Let $\lambda \in F^*$ be such that $\lambda >_v 0$ at those orderings $v \in \Omega$ where A_v is non-split. Then $\lambda \in \mathrm{Nrd}(A^*)$.* $\square$

Theorem 1.9 **([BP2, Cor. 7.10])** *Let F be a formally real field with $\mathrm{vcd}(F) \leq 2$. Let q be a quadratic form over F. Then $sn(q)$ consists of elements of F^* which are positive at each $v \in \Omega$ such that q is definite at F_v.* $\square$

We say that a quadratic form q over F is *locally isotropic* if over each real closure F_v, $v \in \Omega$, the form q is isotropic.

Corollary 1.10 *With the notation as in 1.9, if q locally isotropic, then $sn(q) = F^*$.* $\square$

Let Γ_F denote the Galois group $\mathrm{Gal}(F_s/F)$. For a discrete Γ_F-module M, let $H^n(F, M)$ denote the Galois cohomology group $H^n(\mathrm{Gal}(F_s/F), M)$. We now record some results of Arason which we shall use in the paper.

Theorem 1.11 **(Corollar 4.6, [A1])** *Let $Z = F(\sqrt{\delta})$ be a quadratic extension of F. Then we have a long exact sequence of abelian groups*

$$\cdots \to H^n(F, \mu_2) \xrightarrow{\mathrm{res}} H^n(Z, \mu_2) \xrightarrow{\mathrm{cores}} H^n(F, \mu_2) \xrightarrow{\cup_{n,1}(\delta)} H^{n+1}(F, \mu_2) \to \cdots$$

where res *and* cores *denote the restriction and corestriction maps respectively.* $\square$

In view of 1.11 and the isomorphism $H^2(F, \mu_2) \simeq {}_2\mathrm{Br}(F)$, we have the following

Proposition 1.12 *Let $Z = F(\sqrt{\delta})$ be a quadratic extension of F and let A be a central simple algebra over Z with $\exp(A) = 2$ and $\mathrm{cores}_{Z/F}([A]) = 0 \in H^2(F, \mu_2)$. Then there exists a central simple algebra A_0 over F such that $A_0 \otimes_F Z$ is Brauer equivalent to A.* $\square$

We say that a field extension L/F is a *quadratic tower over F* if there exist fields F_i such that $F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r = L$ and each F_i/F_{i-1} is a quadratic extension for $1 \leq i \leq r$. We denote by $\mathcal{F}_2(F)$ the set of quadratic towers of F in an algebraic closure of F. Let $I(F)$ denote the fundamental ideal of the Witt ring $W(F)$ of F. For each $n \geq 1$, we denote by $I^n(F)$ to denote the ideal $I(F)^n$.

Lemma 1.13 **([A1, Satz 3.6])** *Let $I^3(F) = 0$ and L/F be a quadratic tower. Then $I^3(L) = 0$.* $\square$

Theorem 1.14 **([A2, Prop. 2])** *Let F be a field with $\mathrm{cd}(F) \leq 2$. Then $I^3(F) = 0$.* $\square$

A non-trivial element $\chi \in H^r(F, \mu_2)$ is called (-1)-torsion-free if for every $s \geq 1$, the element $\chi \cup (-1) \cup (-1) \cup \cdots \cup (-1) \in H^{r+s}(F, \mu_2)$ is non-trivial. The following is a consequence of 1.11

Proposition 1.15 *Let F be a field with $\mathrm{vcd}(F) \leq n$. Then $H^{n+1}(F, \mu_2)$ is (-1)-torsion-free.* □

The following lemma relates the conditions $\mathrm{vcd}(F) \leq 2$ and $I^3(F)$ being torsion-free.

Lemma 1.16 (**[BP2, Lemma 2.4]**) *Let F be a field with virtual cohomological dimension at most two. Then $I^3(F)$ is torsion-free.*

Proof Since $\mathrm{vcd}(F) \leq 2$, by [AEJ] the invariants $e_r : I^r(F) \to H^r(F, \mu_2)$ have kernel $I^{r+1}(F)$ for each $r \geq 0$ and $H^r(F(\sqrt{-1}), \mu_2) = 0$ for $r \geq 3$. Then it is evident from Arason exact sequence 1.11 for the quadratic extension $F(\sqrt{-1})/F$ that $H^r(F, \mu_2) \stackrel{\cup(-1)}{\longrightarrow} H^{r+1}(F, \mu_2)$ is an isomorphism for $r \geq 3$. Let $q \in I^3(F)$ be a torsion-element. Then $2^s.q = 0 \in W(F)$ for some integer $s \geq 0$. As a consequence $e_3(q) \cup (-1) \cup (-1) \cup \cdots \cup (-1) = 0 \in H^{3+s}(F, \mu_2)$. Since $H^r(F, \mu_2) \stackrel{\cup(-1)}{\longrightarrow} H^{r+1}(F, \mu_2)$, $r \geq 3$, are isomorphisms, we conclude that $e_3(q) = 0$; i.e. $q \in \ker(e_3) = I^4(F)$. By a similar argument $q \in I^r(F)$ for each $r \geq 3$ and hence $q \in \bigcap_r I^r(F)$. By a theorem of Arason-Pfister [L, Cor. X.3.2], $q = 0 \in W(F)$ and hence $I^3(F)$ is torsion-free. □

The following result is a weaker form of [Se, Prop. 10, §II.4.1].

Theorem 1.17 *Let F be a field and $\mathrm{cd}(F) \neq \mathrm{vcd}(F)$. Then F has orderings.* □

2 Some norm principles

Let F be a field with $\mathrm{char}(F) \neq 2$ and $I^3(F) = 0$. Let A be a central simple algebra with $\exp(A) = 2$. Then by [Me1], there are quaternion algebras H_i; $1 \leq i \leq r$, such that $A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$. We define an integer $r(A)$ associated to A as follows:

$$r(A) := \min\{r : A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r\}.$$

If A is split, then we define $r(A) = 0$. Given a central simple algebra B over a field Z with $[Z : F] \leq 2$ and a field extension L of F, we set: $B_L = B \otimes_F L$.

Proposition 2.1 *Let $I^3(F) = 0$ and A be a central simple algebra over F. If $\exp(A)$ is a power of 2 then*

$$F^* = \langle \{N_{L/F}(L^*) : L \text{ is a quadratic tower of } F \text{ with } A_L \text{ split}\} \rangle = \mathrm{Nrd}(A^*)$$

In fact, for each $\lambda \in F^*$ there is a quadratic tower L/F and $\alpha \in L^*$ such that $\lambda = N_{L/F}(\alpha)$.

Proof By the classical norm principle for reduced norms, over any field we have the inclusion

$$\langle \{N_{L/F}(L^*) : L \text{ is a quadratic tower of } F \text{ with } A_L \text{ split}\} \rangle \subseteq \mathrm{Nrd}(A^*)$$

Thus to complete the proof, it suffices to show that under the assumption $I^3(F) = 0$,

$$F^* \subseteq \langle \{N_{L/F}(L^*) : L \text{ is a quadratic tower of } F \text{ with } A_L \text{ split}\} \rangle. \qquad (1)$$

Let $\exp(A) = 2^m$. We prove the lemma by induction on m. Suppose $m = 1$. Then $\exp(A) = 2$ and hence by Merkurjev's Theorem [Me1], we write $A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$, where $r = r(A)$ and each H_i is a quaternion algebra over F. We proceed further by induction on r. If $r = 1$ the result holds by [BP2, Prop. 2.7]. Let $r \geq 2$ and $\lambda \in F^*$. By [BP2, Prop. 2.7] there exists a quadratic extension L of F which splits H_1 and $\lambda \in N_{L/F}(L^*)$. Then $r(A_L) < r$ and by 1.13 we have $I^3(L) = 0$. Induction on r leads to (1).

Suppose that $m \geq 2$. Then $\exp(A \otimes_F A) = 2^{m-1}$. Let $\lambda \in F^*$. By induction, there exists a quadratic tower L over F and $\alpha \in L^*$ such that $\lambda = N_{L/F}(\alpha)$ and $(A \otimes_F A)_L$ is split. Then $\exp(A_L) = 2$ and by 1.13, $I^3(L) = 0$. By the previous case, there exists a quadratic tower M of L with $\alpha \in N_{M/L}(M^*)$ and A_M is split. Thus M is a quadratic tower of F such that $\lambda \in N_{M/F}(M^*)$ and A_M is split. This completes the proof. □

Proposition 2.2 *Let* $I^3(F) = 0$ *and* Z *be a quadratic extension of* F. *Let* A *be a central simple algebra over* Z *such that* $\mathrm{cores}_{Z/F}(A) = 0$ *and* $\exp(A) = 2^m$. *Then for each* $\lambda \in F^*$, *there exists a quadratic tower* L/F *such that* $\lambda \in N_{L/F}(L^*)$ *and* A_L *is split.*

Proof We prove this by induction on m. Suppose m = 1. Since $\exp(A) = 2$ and $\mathrm{cores}_{Z/F}(A) = 0$, by 1.12 there exists a central simple algebra A_0 of exponent 2 over F such that $A \sim A_0 \otimes_F Z$. Let $\lambda \in F^*$. Since $I^3(F) = 0$, by 2.1, there exists a quadratic tower L/F such that $(A_0)_L$ is split and $\lambda \in N_{L/F}(L^*)$. Clearly the extension L splits A and the proposition follows.

Suppose $m \geq 2$. Let $\lambda \in F^*$. Since $\exp(A \otimes_Z A) = 2^{m-1}$, by induction there exists a quadratic tower L/F such that $\lambda = N_{L/F}(\alpha)$ for some $\alpha \in L^*$, and $(A \otimes_Z A)_L$ splits. Clearly $\exp(A_L) = 2$, and by the previous case we have a quadratic tower M/L such that A_M splits and $\alpha \in N_{M/L}(M^*)$. Then M/F is a quadratic tower such that $\lambda \in N_{M/F}(M^*)$ and A_M is split. This completes the proof. □

We shall now describe norm principles for fields F with $\mathrm{vcd}(F) \leq 2$. If F has no orderings by 1.17, $\mathrm{cd}(F) \leq 2$, and the results follow from the previous discussion. We shall assume in the rest of the section that F has orderings. We denote by Ω, the set of orderings on F. If A is a central simple algebra over F then A is said to be *locally split* if $A \otimes_F F_v = A_v$ is split for each $v \in \Omega$.

Proposition 2.3 *Let* $\mathrm{vcd}(F) \leq 2$ *and* A *be a central simple algebra over* F *with* $\exp(A) = 2^m$. *Then*

$$F^* = \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } \mathrm{index}(A_M) \leq 2\} \big\rangle.$$

Proof We prove the proposition by induction on m. Let $m = 1$. Then $\exp(A) = 2$ and we proceed by further induction on $r(A)$. The statement is obvious if $r(A) \leq 1$. Let $r(A) \geq 2$ and $A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$ with $r = r(A)$ and $H_i, 1 \leq i \leq r$, quaternion algebras over F. Let $H_1 = (a, b)$ and $H_2 = (c, d)$. Then to the algebra $H_1 \otimes H_2$ is associated the Albert form (*cf.* [KMRT, §16.A])

$$q = \langle -a, -b, ab, c, d, -cd \rangle.$$

Since $\operatorname{disc}(q) = 1$ and $\dim(q) = 6$, the form q is isotropic at F_v for each $v \in \Omega$. Thus by 1.10, $\operatorname{sn}(q) = F^*$ and by 1.5 we have

$$F^* = \operatorname{sn}(q) = \big\langle \{N_{L/F}(L^*) : L \text{ is a quadratic extension of } F \text{ and } q_L \text{ is isotropic}\} \big\rangle$$

Let L be a quadratic extension of F with q_L isotropic. By Albert's Theorem [KMRT, Th. 16.5], we have $r((H_1 \otimes H_2)_L) \leq 1$. Thus $r(A_L) < r(A)$ and by induction we have,

$$L^* = \big\langle \{N_{M/L}(M^*) : M \in \mathcal{F}_2(L) \text{ and } \operatorname{index}(A_M) \leq 2\} \big\rangle.$$

and therefore taking norms from L to F we have,

$$\begin{aligned} F^* = \operatorname{sn}(q) &= \big\langle \{N_{L/F}(L^*) : L \text{ is a quadratic extension of } F \text{ and } q_L \text{ is isotropic}\} \big\rangle \\ &\subseteq \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } \operatorname{index}(A_M) \leq 2\} \big\rangle. \end{aligned}$$

This completes the case $m = 1$. Now let $m \geq 2$. Then $\exp(A \otimes_F A) = 2^{m-1}$ and by induction

$$F^* = \big\langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } \operatorname{index}((A \otimes_F A)_L) \leq 2\} \big\rangle. \tag{2}$$

Let $L \in \mathcal{F}_2(F)$ be such that $\operatorname{index}((A \otimes_F A)_L) \leq 2$. Since the Brauer group of a real-closed field is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, it follows that $(A \otimes_F A)_L$ is locally split. Thus by 1.8, $\operatorname{Nrd}((A \otimes_F A)_L) = L^*$. Since $\operatorname{index}((A \otimes_F A)_L) \leq 2$, we have

$$\begin{aligned} &\operatorname{Nrd}((A \otimes_F A)_L) \qquad (3) \\ &\subseteq \big\langle \{N_{N/L}(N^*) : N \text{ is a quadratic extension of } L \text{ and } (A \otimes_F A)_N \text{ is split}\} \big\rangle. \end{aligned}$$

Let N be a quadratic extension of L such that $(A \otimes_F A)_N$ is split. Then $\exp(A_N) = 2$ and by the case $m = 1$

$$N^* = \big\langle \{N_{M/N}(M^*) : M \in \mathcal{F}_2(N) \text{ and } \operatorname{index}((A_M) \leq 2\} \big\rangle. \tag{4}$$

Now it is clear from (2), (3) and (4) that

$$F^* = \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } \operatorname{index}(A_M) \leq 2\} \big\rangle.$$

□

We refine 2.3 to the following:

Proposition 2.4 *Let F be a field with $\mathrm{vcd}(F) \leq 2$. Let A be a central simple algebra over F with $\exp(A) = 2^m$ for some $m \geq 1$. Then,*

$$F^* = \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\}\big\rangle.$$

Proof Let L be a quadratic tower over F such that $\mathrm{index}(A_L) \leq 2$. Then $A_L \sim (a, b)$, $a, b \in L^*$. Let Ω_L denote the set of orderings on L. For each $w \in \Omega_L$, the quadratic form $q' = \big\langle -1, -a, -b, ab\big\rangle$ is isotropic over L_w, where L_w denotes the real closure of L at w. Therefore by [BP2, Prop. 7.7] we have $\mathrm{sn}(q') = L^*$. Thus, in view of 1.5 we have:

$$L^* = \big\langle \{N_{M/L}(M^*) : M \text{ is a quadratic extension of } L \text{ and } q'_M \text{ is isotropic }\}\big\rangle.$$

Let M be a quadratic extension of L such that q'_M is isotropic. Then the form $\big\langle -a, -b, ab\big\rangle_M$ represents 1, and we can write: $\big\langle -a, -b, ab\big\rangle_M \simeq \big\langle 1, x, y\big\rangle_M$; with $x, y \in M^*$. Comparing the discriminants, we have $\big\langle -a, -b, ab\big\rangle_M \simeq \big\langle 1, x, x\big\rangle_M$. Thus $\big\langle 1, -a, -b, ab\big\rangle_M \simeq \big\langle 1, 1, x, x\big\rangle_M$ and $(a, b)_M \simeq (-1, -x)$. Thus,

$$\begin{aligned} L^* &= \big\langle \{N_{M/L}(M^*) : M \text{ is a quadratic extension of } F \text{ and } q'_M \text{ is isotropic }\}\big\rangle \\ &\subseteq \big\langle \{N_{M/L}(M^*) : M \in \mathcal{F}_2(L) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\}\big\rangle \end{aligned}$$

and

$$N_{L/F}(L^*) \subseteq \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\}\big\rangle.$$

This together with 2.3 gives

$$F^* \subseteq \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\}\big\rangle.$$

This completes the proof. □

Corollary 2.5 *Let $\mathrm{vcd}(F) \leq 2$. Let A_1 and A_2 be central simple algebras over F with $\exp(A_i)$ a power of 2 for $i = 1, 2$. Then we have:*

$$\begin{aligned} F^* = \big\langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) &\text{ and } A_{1M} \sim (-1, -x), \\ &A_{2M} \sim (-1, -y) \text{ for some } x, y \in M^*\}\big\rangle \end{aligned}$$

□

The following is a refinement of the surjectivity of the reduced norm (Theorem 1.8) for locally split algebras with centre a quadratic extension of F.

Proposition 2.6 *Let* $\mathrm{vcd}(F) \leq 2$ *and* F *has orderings. Let* Ω *denote the set of orderings on* F*. Let* $Z = F(\sqrt{\delta})$ *be a quadratic extension of fields. Let* A *be a central simple* Z*-algebra which is split at each* $v \in \Omega$*. Further assume that* $\exp(A) = 2^m$ *for some integer* m *and* $\mathrm{cores}_{Z/F}(A) = 0$*. Then for each* $\lambda \in F^*$*, there exist extensions* E_i *over* F *and* $\lambda_i \in E_i^*$ *such that each* $A \otimes_F E_i$ *is split and* $\lambda = \prod_i N_{E_i/F}(\lambda_i)$*.*

Proof We proceed by induction on m. Let $m = 1$. Since $\mathrm{cores}_{Z/F}(A) = 0$, by 1.12 there is a central simple algebra A_0 over F with $\exp(A_0) = 2$ and $A \sim A_0 \otimes_F Z$. By [Me1], there are quaternion algebras $H_i; 1 \leq i \leq r = r(A_0)$ over F such that $A_0 \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$. Suppose $r = 1$. Then $A_0 \sim H_1 = (a, b)$ for some $a, b \in F^*$. Let q denote the quadratic form $\langle 1, -a, -b, ab\delta \rangle$ over F. Then by [CTSk, Prop. 2.3] we have $\mathrm{sn}(q) = \mathrm{Nrd}((H_1 \otimes_F F(\sqrt{\delta}))^*) \cap F^*$. Since A is locally split, by 1.8 $\mathrm{Nrd}((H_1 \otimes_F F(\sqrt{\delta}))^*) = \mathrm{Nrd}(A^*) = Z^*$. Therefore $\mathrm{sn}(q) = F^*$. Thus by 1.5, for each $\lambda \in F^*$, there exist quadratic extensions E_i/F and $\lambda_i \in E_i^*$ such that each q_{E_i} is isotropic and $\lambda = \prod_i N_{E_i/F}(\lambda_i)$. Further $A \otimes_F E_i \sim (a, b) \otimes_{E_i} E_i(\sqrt{\delta})$ and the norm form of $(a, b) \otimes_{E_i} E_i(\sqrt{\delta})$ is isometric to $q_{E_i(\sqrt{\delta})}$, which is isotropic. It follows therefore that each $A \otimes_F E_i$ is split. Thus F^* is generated by the norms from those extensions of F where the algebra A is split.

Now suppose $r \geq 2$. Then by 2.3 we have

$$F^* = \left\langle \{N_{L/F}(L^*) : \mathrm{index}((A_0)_L) \leq 2\} \right\rangle.$$

The proposition follows immediately from the case $r = 1$.

Let $m \geq 2$. Then $\exp(A \otimes_Z A) = 2^{m-1}$ and at each $v \in \Omega$ the algebra $(A \otimes_Z A) \otimes_F F_v$ is split since $\mathrm{Br}(F_v) = \mathbb{Z}/2\mathbb{Z}$. Thus by induction, F^* is generated by norms from extensions M_i over F such that the algebra $(A \otimes_Z A) \otimes_F M_i$ splits. It is clear that $\exp(A \otimes_F M_i) = 2$. Thus by the exponent 2 case, it follows that each M_i^* is generated by norms from extensions E_i of M_i such that $A \otimes_F E_i$ is split. We conclude therefore, that F^* is generated by norms from those extensions of F where A splits. $\square$

3 Fields with $\mathrm{cd}(F) \leq 2$

In this section, we prove that if $\mathrm{cd}(F) \leq 2$, then for adjoint classical groups G of type ${}^2A_n$, C_n and D_n, $G(F)/R = 0$. We begin with the result leading to the triviality of $G(F)/R$ in the C_n case.

Theorem 3.1 *Let F be a field with $\mathrm{char}(F) \neq 2$ and $I^3(F) = 0$. Let A be a central simple algebra of degree $2n$ over F and σ be a symplectic involution on A. Then* $\mathrm{Hyp}(A, \sigma) = F^*$.

Proof Let $\lambda \in F^*$. Since exponent of A is 2 and $I^3(F) = 0$, by 2.1, there exists a quadratic tower L/F such that L splits A and $\lambda \in N_{L/F}(L^*)$. The involution σ_L is adjoint to a skew-symmetric form h_L over L which is hyperbolic. Therefore $\lambda \in \mathrm{Hyp}(A, \sigma)$. □

Let q be a quadratic form over F of rank $2n$. Let σ be the involution on $M_{2n}(F)$ which is adjoint to q. We denote by $C(q)$ the Clifford invariant of q.

Proposition 3.2 *If $I^3(F) = 0$, then $G(q) \subseteq \mathrm{Hyp}(q)$.*

Proof We first assume that the discriminant of q is trivial. Let $\lambda \in F^*$. The algebra $C(q)$ has exponent 2 and by 2.1, there exists a quadratic tower M of F such that $C(q) \otimes_F M$ is split and $\lambda \in N_{M/F}(M^*)$. By 1.13, $I^3(M) = 0$. Since q_M is an even dimensional quadratic form with trivial discriminant and trivial Clifford invariant, in view of [EL, Th. 3] q_M is hyperbolic and hence $\lambda \in \mathrm{Hyp}(q)$. Thus $\mathrm{Hyp}(q) = F^*$.

Now suppose that $\mathrm{disc}(q)$ is non-trivial, $d \in F^*$ a representative of the square class of $\mathrm{disc}(q)$ in F^*/F^{*2} and $L = F(\sqrt{d})$. Let $\lambda \in G(q)$. By 1.2, $\lambda \in N_{L/F}(L^*)$. Since $\mathrm{disc}(q_L) = 1$, by the previous case $L^* = \mathrm{Hyp}(q_L)$. Taking norms we get $N_{L/F}(L^*) \subseteq \mathrm{Hyp}(q)$. Thus $G(q) \subseteq \mathrm{Hyp}(q)$. □

We prove a similar result when A is not split.

Theorem 3.3 *Let $I^3(F) = 0$. Let A be a central simple algebra with an involution σ of orthogonal type. Let d be the discriminant of σ and $L = F[X]/(X^2 - d)$. Then we have $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma) = N_{L/F}(L^*)$.*

Proof Since A supports an involution of first kind, $\exp(A) \leq 2$. Suppose first that $\mathrm{disc}(\sigma)$ is trivial. Let M be a quadratic tower of F which splits A. By the proof of 3.2 we have $M^* = \mathrm{Hyp}(A_M, \sigma_M)$. Thus $N_{M/F}(M^*) \subseteq \mathrm{Hyp}(A, \sigma)$. This, together with 2.1 implies that $F^* = \mathrm{Nrd}(A^*) \subseteq \mathrm{Hyp}(A, \sigma)$. Hence $F^* = \mathrm{Hyp}(A, \sigma) = G_+(A, \sigma)$. Since $L = F \times F$, we have $N_{L/F}(L^*) = F^*$. Thus $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma) = N_{L/F}(L^*)$.

Suppose that $\mathrm{disc}(\sigma)$ is not trivial. Let $d \in F^*$ represent the class of $\mathrm{disc}(\sigma)$ in F^*/F^{*2}. Let $\lambda \in G_+(A, \sigma)$. Then by 1.3, we have $\lambda \in N_{L/F}(L^*)$ where $L = F(\sqrt{d})$. Clearly $\mathrm{disc}(\sigma_L) = 1$ and by previous case $L^* = \mathrm{Hyp}(A_L, \sigma_L)$. Thus $\lambda \in N_{L/F}(L^*) \subseteq \mathrm{Hyp}(A, \sigma)$. Thus $G_+(A, \sigma) \subseteq N_{L/F}(L^*) \subseteq \mathrm{Hyp}(A, \sigma)$. By 1.1, $\mathrm{Hyp}(A, \sigma).F^{*2} \subseteq G_+(A, \sigma)$. Hence $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma) = N_{L/F}(L^*)$. □

Let Z be a quadratic extension of F and A be a central simple algebra over Z with an involution σ of second kind such that $Z^\sigma = F$. In the next lemma, we consider the case where A splits and the involution σ is adjoint to a Z/F-hermitian form h. In view of 1.6, we further assume that h has even rank; i.e. $\deg(A)$ is even.

Lemma 3.4 *Let $I^3(F) = 0$, let A be split and σ be an involution of second kind on A such that $Z^\sigma = F$. Then $\mathrm{Hyp}(A, \sigma) = F^*$.*

Proof Let $Z = F(\sqrt{\delta})$. Let q_h be the quadratic form over F defined by $q_h(x) = h(x, x)$. Then $q_h \simeq \langle 1, -\delta \rangle \otimes q$ [S, pp. 349, Remark 1.3] for some quadratic form q over F having the same rank as h, which is even. Therefore $q_h \in I^2(F)$ and by a theorem of Jacobson [MH, pp. 114], the form h is hyperbolic over an extension M of F if and only if the quadratic form q_h is hyperbolic over M. Let C denote the Clifford invariant of q_h. Let $\lambda \in F^*$. By 2.1, there exists a quadratic tower M over F such that such that C_M is split and $\lambda \in N_{M/L}(M^*)$. Since $I^3(M) = 0$, by [EL, Th. 3], $(q_h)_M$ is hyperbolic and hence the hermitian form h_M is hyperbolic. Therefore $N_{M/F}(M^*) \subseteq \mathrm{Hyp}(A, \sigma)$. Thus $\mathrm{Hyp}(A, \sigma) = F^*$. □

Theorem 3.5 *If* $I^3(F) = 0$ *and* $\exp(A) = 2^m$ *then* $\mathrm{Hyp}(A, \sigma) = F^*$.

Proof Since A supports an involution σ of second kind, by [S, Th. 9.5] we have $\mathrm{cores}_{Z/F}(A) = 0$. Therefore by 2.2, given $\lambda \in F^*$ there exists a quadratic tower L/F such that A_L splits and $\lambda \in N_{L/F}(L^*)$. Since A_L is split, by 3.4, $L^* = \mathrm{Hyp}(A_L, \sigma_L)$. Taking norms we conclude that $\lambda \in \mathrm{Hyp}(A, \sigma)$. Therefore $\mathrm{Hyp}(A, \sigma) = F^*$. □

Theorem 3.6 *Let* $\mathrm{cd}(F) \leq 2$ *and* Z *be a quadratic extension of* F. *Let* A *be a central simple algebra of even degree over* Z *with an involution* σ *of second kind such that* $Z^\sigma = F$. *Then* $\mathrm{Hyp}(A, \sigma).F^{*2} = F^*$.

Proof By [BP1, Lemma 3.3.1], there exists an odd degree extension L over F such that $\exp(A \otimes_F L)$ is a power of 2. Since the condition $\mathrm{cd}(F) \leq 2$ is preserved under finite extensions of fields [Ar, Th 2.1], we have $\mathrm{cd}(L) \leq 2$. By 1.14 $I^3(L) = 0$ and by 3.5, $\mathrm{Hyp}(A_L, \sigma_L) = L^*$. Hence $N_{L/F}(L^*) \subseteq \mathrm{Hyp}(A, \sigma)$. Let $\lambda \in F^*$ and $[L : F] = 2s + 1$. Then $\lambda^{2s+1} = N_{L/F}(\lambda) \in \mathrm{Hyp}(A, \sigma)$ and we have $\lambda \in \mathrm{Hyp}(A, \sigma).F^{*2}$. This implies that $\mathrm{Hyp}(A, \sigma).F^{*2} = F^*$. □

Theorem 3.7 *If* $\mathrm{cd}(F) \leq 2$ *and* G *an adjoint group of classical type defined over* F, *then* $G(F)/R = 0$.

Proof A classical adjoint group G is a direct product of groups $R_{L_i/F}(G_i)$, where L_i/F are finite extensions and G_i are absolutely simple adjoint groups of classical type defined over L_i [T, 3.1.2]. Moreover, $G_i(L_i)/R = R_{L_i/F}(G_i)(F)/R$ and R-equivalence commutes with direct products [CTS, pp. 195]. In view of this, it suffices to prove the theorem for an absolutely simple classical adjoint group G defined over F. By [We] such an algebraic group is isomorphic to $\mathbf{PSim}_+(A, \sigma)$ for a central simple algebra A over a field Z, $[Z : F] \leq 2$, with an involution σ. In view of 1.1 and 1.14, the result follows in ${}^2A_n$ case from 3.6 and 1.6, in B_n case from 1.4 in C_n case from 3.1 and 1.7, and in D_n case from 3.3. □

Remark Theorem 3.7 for groups of type A_n and C_n also follows from [CTGP, Cor. 4.11], using the fact that $G(F)/R = 0$ if G is simply connected of type A_n or C_n, and [G, pp. 222].

4 Fields with $\mathrm{vcd}(F) \leq 2$: Symplectic groups

In this section F denotes a formally real field with $\mathrm{vcd}(F) \leq 2$, and Ω, the set of orderings on F. Let A be a central simple algebra over F of degree $2n$ and σ be an involution of symplectic type on A. In view of 1.7, we assume that n is even. We say that σ is *locally hyperbolic* if for each $v \in \Omega$, the involution σ_v on $A_v = A \otimes_F F_v$ is hyperbolic, F_v denoting the real closure of F at v.

Proposition 4.1 *Let A be a central simple algebra over F of degree $2n$, where n is an even integer. Let σ be a symplectic involution on A. If σ is locally hyperbolic then* $\mathrm{Hyp}(A, \sigma) = F^*$.

Proof First assume that $A = M_n(H)$, where H is a quaternion algebra over F. Let bar denote the canonical involution on H and h a hermitian form of rank n over $(H, -)$ such that σ is adjoint to h. Since σ is locally hyperbolic, so is h and hence $\mathrm{sgn}(h) = 0$. Thus h has even rank and trivial signature and by [BP2, Th. 6.2], the form h itself is hyperbolic. Thus $\mathrm{Hyp}(A, \sigma) = F^*$.

Suppose A is arbitrary. Since A supports an involution, $\exp(A) = 2$ [S, Th. 8.4] and by 2.3, we have

$$F^* = \left\langle \{N_{M/F}(M^*) : \mathrm{index}(A_M) \leq 2\} \right\rangle \qquad (*)$$

Let M be a finite extension of F such that $\mathrm{index}(A_M) \leq 2$. Then $A_M \simeq M_n(H)$ where H is a quaternion algebra over M. Since σ is locally hyperbolic, so is σ_M and by the previous case, $M^* = \mathrm{Hyp}(A_M, \sigma_M)$. Therefore $N_{M/F}(M^*) \subseteq \mathrm{Hyp}(A, \sigma)$ and in view of $(*)$ we get $\mathrm{Hyp}(A, \sigma) = F^*$. □

Theorem 4.2 *Let F be a formally real field with* $\mathrm{vcd}(F) \leq 2$*. Let A be a central simple algebra over F of degree $2n$ and σ be a symplectic involution on A. Then* $G(A, \sigma) \subseteq \mathrm{Hyp}(A, \sigma)$.

Proof In view of 1.7, we assume that n is even. Let $\lambda \in G(A,\sigma)$ and $K = F(\sqrt{-\lambda})$. Let Ω_K denote the set of orderings on K. For each $w \in \Omega_K$, $\lambda \equiv -1$ modulo K_w^{*2} is a similarity factor for σ_K and hence $\mathrm{sgn}(\sigma_K) = 0$. Further $\deg(A)$ is divisible by 4 and hence the involution σ_K is locally hyperbolic. Thus by 4.1, we have $\mathrm{Hyp}(A_K, \sigma_K) = K^*$. Therefore

$$\lambda = N_{K/F}(\sqrt{-\lambda}) \in N_{K/F}(K^*) = N_{K/F}(\mathrm{Hyp}(A_K, \sigma_K)) \subseteq \mathrm{Hyp}(A, \sigma).$$

$\square$

5 Fields with $\mathrm{vcd}(F) \leq 2$: Unitary groups

Let F be an arbitrary field with $\mathrm{char}(F) \neq 2$. Let $Z = F(\sqrt{\delta})$ be a quadratic extension of F. Let A be a central simple algebra over Z and σ be an involution on A such that $Z^\sigma = F$. In view of 1.6, we assume throughout this section that A has even degree.

Let $\deg(A) = 2m$ and $D = D(A,\sigma)$ denote the discriminant algebra of (A,σ) (*cf.* [KMRT, §10.E]). The algebra D is a central simple algebra over F and carries an involution $\overline{\sigma}$ of first kind, which is of symplectic type if m is odd and of orthogonal type if m is even [KMRT, Prop. 10.30]. For $1 \leq i \leq 2m$, let $\wedge^i A$ be the i^{th} *$\wedge$-power* of A (*cf.* [KMRT, §10 (10.4)]). By [KMRT, Prop. 14.3], there is a homogeneous polynomial map $\wedge^i : A \to \wedge^i A$ of degree i, $1 \leq i \leq 2m$. If $A = \mathrm{End}_F(V)$ then $\wedge^i A = \mathrm{End}_F(\Lambda^i V)$ and $\wedge^i(f) = \Lambda^i(f)$, the i^{th} exterior power of the linear map $f \in \mathrm{End}_F(V)$.

Theorem 5.1 *Let F be a field with $\mathrm{char}(F) \neq 2$. Let A be a central simple algebra of degree $2m$ over a field Z with m odd. Let σ be an involution of second kind on A such that $Z^\sigma = F$. Let $D = D(A,\sigma)$ be the discriminant algebra of (A,σ). Then $G(A,\sigma) \subseteq \mathrm{Nrd}(D^*).N_{Z/F}(Z^*)$.*

Proof Let $x \in G(A,\sigma)$ and $g \in \mathrm{Sim}(A,\sigma)$ be such that $\mu(g) = \sigma(g)g = x$. Then $N_{Z/F}(\mathrm{Nrd}(g)) = \mu(g)^{2m}$ and by Hilbert Theorem-90, there exists $\alpha \in Z^*$ such that

$\mu(g)^{-m}\,\mathrm{Nrd}(g) = \alpha^{-1}\overline{\alpha}$, where bar denotes the non-trivial automorphism of Z over F. By [KMRT, Lemma 14.6], we have

$$\overline{\sigma}(\alpha^{-1}\wedge^m g)\alpha^{-1}\wedge^m g = N_{Z/F}(\alpha)^{-1}\mu(g)^m.$$

Since m is odd, $x = \mu(g) \in G(D,\overline{\sigma}).N_{Z/F}(Z^*)$. Thus

$$G(A,\sigma) \subseteq G(D,\overline{\sigma}).N_{Z/F}(Z^*) \qquad (*)$$

Let $y \in G(D,\overline{\sigma})$ be arbitrary and $h \in \mathrm{Sim}(D,\overline{\sigma})$ be such that $\mu(h) = \overline{\sigma}(h)h = y$. Since m is odd, the involution $\overline{\sigma}$ is of symplectic type and by [KMRT, Prop. 12.23] we have $\mu(h)^m = \mathrm{Nrd}(h)$. Again, since m is odd, we have $y = \mu(h) \in \mathrm{Nrd}(D^*).F^{*2}$. Thus

$$G(D,\overline{\sigma}) \subseteq \mathrm{Nrd}(D^*).F^{*2} \qquad (**)$$

and combining the inclusions $(*)$ and $(**)$ above, we get

$$G(A,\sigma) \subseteq \mathrm{Nrd}(D^*).N_{Z/F}(Z^*).$$

This completes the proof. □

In this section, from now onwards we assume that $\mathrm{vcd}(F) \leq 2$, and F has orderings and denote by Ω, the set of orderings on F. A quadratic form q over F is called *locally hyperbolic* if q is hyperbolic at every real closure F_v, $v \in \Omega$.

Lemma 5.2 *If q is a locally hyperbolic quadratic form of even rank and trivial discriminant over F,* $\mathrm{Hyp}(q) = F^*$.

Proof Since q is locally hyperbolic the Clifford algebra $C(q)$ of q is locally split. Thus by 1.8 we have $\mathrm{Nrd}(C(q)^*) = F^*$. Let $\lambda \in F^*$ and let L/F be a finite extension such that $\lambda \in N_{L/F}(L^*)$ and $C(q)_L$ is split. Then q_L has even dimension, trivial discriminant, trivial Clifford invariant and $\mathrm{sgn}(q_L) = 0$. Therefore by [EL, Th. 3], the form q_L is hyperbolic and $\lambda \in N_{L/F}(L^*) \in \mathrm{Hyp}(q)$. □

Proposition 5.3 *Let $Z = F(\sqrt{\delta})$ be a quadratic extension. Let $A = M_r(Z)$, where r is an even positive integer, support a locally hyperbolic Z/F-involution σ. Then* $\mathrm{Hyp}(A,\sigma) = F^*$.

Proof Let the involution σ be adjoint to a Z/F-hermitian form h. Then the rank of h is r. Let q_h be the quadratic form over F given by $q_h(x) = h(x,x)$. Then $q_h \simeq \langle 1, -\delta \rangle \otimes q$ [S, pp. 349, Remark 1.3], where q is a quadratic form over F of the same rank as that of h, which is even. Therefore $q_h \in I^2(F)$. By Jacobson's theorem [MH, p. 114], the form h_M is hyperbolic if and only if the quadratic form $(q_h)_M$ is hyperbolic. It follows that $\mathrm{Hyp}(A,\sigma) = \mathrm{Hyp}(q_h)$. Since h is locally hyperbolic, the form q_h is locally hyperbolic as well. By 5.2, we have $\mathrm{Hyp}(q_h) = F^*$. Thus $\mathrm{Hyp}(A,\sigma) = \mathrm{Hyp}(q_h) = F^*$. □

The following is a consequence of 5.3 and 2.6.

Proposition 5.4 *Let A be a locally split central simple Z-algebra and σ be a locally hyperbolic Z/F-involution on A. Let* $\exp(A) = 2^m$. *Then* $\mathrm{Hyp}(A,\sigma) = F^*$.

Proof Since A supports an involution σ of second kind with $Z^\sigma = F$, by [S, Th. 9.5], $\mathrm{cores}_{Z/F}(A) = 0$. Thus by 2.6 we have

$$F^* = \big\langle \{N_{L/F}(L^*) : A_L \text{ is split }\}\big\rangle.$$

Let L/F be an extension which splits A. By 5.3, $\mathrm{Hyp}(A_L,\sigma_L) = L^*$ and taking norm from L/F, we conclude that $\mathrm{Hyp}(A,\sigma) = F^*$. □

Proposition 5.5 *Let A be a central simple algebra over Z of degree $2m$, where m is odd. Let σ be a Z/F-involution on A with* $\mathrm{sgn}(\sigma) = 0$. *Let $D = D(A,\sigma)$ be the discriminant algebra of (A,σ). Then* $\mathrm{Nrd}(D^*) \subseteq \mathrm{Hyp}(A,\sigma).F^{*2}$. *Further*

$$G(A,\sigma) = \mathrm{Nrd}(D^*).N_{Z/F}(Z^*) = \mathrm{Hyp}(A,\sigma).N_{Z/F}(Z^*).$$

Proof We first show that $\mathrm{Nrd}(D^*) \subseteq \mathrm{Hyp}(A,\sigma).F^{*2}$. Assume first that $\exp(A)$ is a power of 2. Since $\deg(A) = 2m$ with m odd, $\mathrm{index}(A) = \exp(A) = 2$ and $A = M_m(H)$ for some quaternion algebra H over Z. By [KMRT, §10.4], [KMRT, Prop. 10.30] and the hypothesis that m is odd, it follows that

$$D \otimes_F Z \simeq \wedge^m(M_m(H)) \sim H^{\otimes m} \sim H.$$

Thus if M is a finite extension of F such that D_M is split, then H_M is split and $\mathrm{sgn}(\sigma_M) = 0$. Thus by 5.3, $\mathrm{Hyp}(A_M, \sigma_M) = M^*$ and taking norms, $N_{M/F}(M^*) \subseteq \mathrm{Hyp}(A,\sigma)$. In view of the classical norm principle for reduced norms, $\mathrm{Nrd}(D^*) \subseteq \mathrm{Hyp}(A,\sigma)$.

Now suppose that $\exp(A)$ is arbitrary. By [BP1, Lemma 3.3.1], there exists an odd degree extension L/F such that $\exp(A_L)$ is a power of 2. Let $\lambda \in \mathrm{Nrd}(D^*)$. Then $\lambda \in \mathrm{Nrd}(D_L^*)$. By the previous case, $\lambda \in \mathrm{Hyp}(A_L, \sigma_L)$. Taking norm from L/F and using the hypothesis that m is odd, we conclude that $\lambda \in \mathrm{Hyp}(A,\sigma).F^{*2}$. This proves the first assertion of 5.5. It follows immediately that

$$\mathrm{Nrd}(D^*).N_{Z/F}(Z^*) \subseteq \mathrm{Hyp}(A,\sigma).N_{Z/F}(Z^*) \qquad (*)$$

From 5.1, it is clear that

$$G(A,\sigma) \subseteq \mathrm{Nrd}(D^*).N_{Z/F}(Z^*) \qquad (**)$$

and further by 1.1, $\mathrm{Hyp}(A,\sigma).N_{Z/F}(Z^*) \subseteq G(A,\sigma)$. In view of this and the inclusions $(*)$ and $(**)$ we conclude that

$$G(A,\sigma) = \mathrm{Nrd}(D^*).N_{Z/F}(Z^*) = \mathrm{Hyp}(A,\sigma).N_{Z/F}(Z^*).$$

□

Theorem 5.6 *Let F be a field with $\mathrm{vcd}(F) \leq 2$ and Z be a quadratic extension of F. Let A be a central simple algebra over Z of degree $2m$, where m is odd. Let σ be a Z/F-involution on A. Then $G(A,\sigma) = \mathrm{Hyp}(A,\sigma).N_{Z/F}(Z^*)$.*

Proof Let $\lambda \in G(A,\sigma)$. Let $D = D(A,\sigma)$ be the discriminant algebra of (A,σ). By 5.1, $\lambda \in \mathrm{Nrd}(D^*).N_{Z/F}(Z^*)$. Let $\lambda_1 \in \mathrm{Nrd}(D^*)$ and $\alpha \in N_{Z/F}(Z^*)$ be such that

$\lambda = \lambda_1 \alpha$. Since $N_{Z/F}(Z^*) \subseteq \text{Hyp}(A, \sigma) \subseteq G(A, \sigma)$, it follows that $\lambda_1 \in G(A, \sigma)$. Let $K = F(\sqrt{-\lambda_1})$. Then $\text{sgn}(\sigma_K) = 0$ and by 5.5, $\text{Nrd}(D_K^*) \subseteq \text{Hyp}(A_K, \sigma_K).K^{*2}$. Further, since $\lambda_1 \in \text{Nrd}(D_K^*)$ and $\lambda_1 \equiv -1 \bmod K^{*2}$, D_K is locally split and by 1.8, $\text{Nrd}(D_K^*) = K^*$. Thus $\text{Hyp}(A_K, \sigma_K).K^{*2} = K^*$. Taking norms, we get

$$\lambda_1 \in N_{K/F}(K^*) = N_{K/F}(\text{Hyp}(A_K, \sigma_K)) \subseteq \text{Hyp}(A, \sigma).$$

Thus $\lambda = \lambda_1 \alpha \in \text{Hyp}(A, \sigma).N_{Z/F}(Z^*)$. This completes the proof. □

Let $\Sigma(F)$ denote the set of elements of F which are positive at all orderings of F.

Lemma 5.7 *Let $\alpha, \delta \in F^*$. Then we have:*

$$F^* = \big\langle \{N_{L/F}(L^*) : L/F \text{ is a quadratic extension such that there exists } u_L \in L(\sqrt{\delta}) \text{ with } N_{L(\sqrt{\delta})/L}(u_L) = 1 \text{ and } \alpha u_L \in \Sigma(L(\sqrt{\delta}))\} \big\rangle.$$

Proof Since the quadratic form $\phi = \langle 1, \delta, -\alpha, \delta\alpha \rangle$ is locally isotropic, by 1.5 and 1.10,

$$F^* = \text{sn}(\phi) = \big\langle \{N_{L/F}(L^*)\rangle : L \text{ is a quadratic extension of } F \text{ such that } \phi_L \text{ is isotropic}\} \big\rangle \qquad (*)$$

At an extension L/F where ϕ is isotropic, we choose $a, b, c, d \in L^*$ such that $a^2 + \delta b^2 - \alpha c^2 + \delta\alpha d^2 = 0$. If $c^2 + \delta d^2 = 0$ or $a^2 + \delta b^2 = 0$, clearly $L(\sqrt{\delta})$ has no ordering and thus $\Sigma(L(\sqrt{\delta})) = L(\sqrt{\delta})^*$. In this case, we may take $u_L = 1$. Otherwise, we let $\theta = c + d\sqrt{\delta}$ and $u_L = \theta^{-1}\overline{\theta}$, where $\overline{\theta} = c - d\sqrt{\delta}$. It is immediate that $\text{Tr}_{L(\sqrt{\delta})/L}(u_L) = 2(c^2 + \delta d^2)(c^2 - \delta d^2)^{-1}$ and $N_{L(\sqrt{\delta})/L}(u_L) = 1$. Since $a^2 + \delta b^2 - \alpha c^2 + \delta\alpha d^2 = 0$ and both $c^2 + \delta d^2$ and $c^2 - \delta d^2$ are units, it follows that

$$\alpha = \big((a^2 + \delta b^2)(c^2 + \delta d^2)^{-1}\big)\big((c^2 + \delta d^2)(c^2 - \delta d^2)^{-1}\big)$$

Thus

$$2\alpha \text{Tr}_{L(\sqrt{\delta})/L}(u_L) = \big((a^2 + \delta b^2)(c^2 + \delta d^2)^{-1}\big)\Big(\text{Tr}_{L(\sqrt{\delta})/L}(u_L)\Big)^2 \in N_{L(\sqrt{-\delta})/L}(L(\sqrt{-\delta}))$$

and hence the quaternion algebra $\Big(2\alpha \text{Tr}_{L(\sqrt{\delta})/L}(u_L), -\delta\Big)$ over L is split.

Let v be an ordering on L which extends to an ordering w on $L(\sqrt{\delta})$. Then $\delta >_v 0$ and hence $2\alpha \mathrm{Tr}_{L(\sqrt{\delta})/L}(u_L) >_v 0$. Let bar denote the non-trivial automorphism of $L(\sqrt{\delta})$ over L. Since $\alpha u_L \overline{\alpha u_L} = \alpha^2 >_v 0$, both αu_L and $\overline{\alpha u_L}$ have same sign at w. But $\alpha \mathrm{Tr}_{L(\sqrt{\delta})/L}(u_L) = \alpha(u_L + \overline{u_L}) >_v 0$. Thus $\alpha u_L >_w 0$. This is true for every ordering of $L(\sqrt{\delta})$. Thus $\alpha u_L \in \Sigma(L(\sqrt{\delta}))$ and $N_{L(\sqrt{\delta})/L}(u_L) = 1$. This completes the proof of the lemma. □

Let D be a division algebra with centre Z and τ be an involution on D of the second kind. Let $Z^\tau = F$. Let (V, h) be a non-degenerate hermitian space over (D, τ). Then the integer $\dim_D(V)$ is said to be the *rank* of h and is denoted by $\mathrm{rank}(h)$. Let $\mathrm{rank}(h) = n$. For a choice $\{e_1, e_2, \cdots, e_n\}$ of a D-basis of V, the form h determines a matrix $M_h = (h(e_i, e_j)) \in M_n(D)$. The matrix M_h is τ-hermitian symmetric. Let $r = n \deg(D)$. We define the *discriminant* of h to be $(-1)^{r(r-1)/2}\,\mathrm{Nrd}(M_h) \in F^*/N_{Z/F}(F^*)$ and denote it by $\mathrm{disc}(h)$.

We refine the notion of discriminant to the notion of *Discriminant* as follows:

Let $M_h \in M_n(D)$ be a matrix as above, representing the hermitian form h. Let $M'_h \in M_n(D)$ also represent h. Then there exists an invertible matrix $T \in M_n(D)$ such that

$$\mathrm{Nrd}(M'_h) = \mathrm{Nrd}(M_h)\,\mathrm{Nrd}(T)\tau(\mathrm{Nrd}(T))$$

Thus we have the following well defined notion of Discriminant:

$$\mathrm{Disc}(h) = (-1)^{r(r-1)/2}\,\mathrm{Nrd}(M_h) \in F^*/N_{Z/F}(\mathrm{Nrd}(D^*))$$

where $r = n \deg(D)$.

We now quote a classification result for hermitian forms over division algebras with an involution of second kind over fields with $\mathrm{vcd}(F) \leq 2$.

Theorem 5.8 (**[BP2, Theorem 4.8]**) *Let F be a field with $\mathrm{vcd}(F) \leq 2$ and D be a division algebra with an involution τ of second kind such that $(\mathrm{centre}(D))^\tau = F$.*

Let h be a hermitian form over (D,τ), Then h is hyperbolic if and only if $\mathrm{rank}(h)$ *is even,* $\mathrm{Disc}(h)$ *is trivial and h has trivial signature.* □

Lemma 5.9 *Let D be a central division algebra over Z, τ be a Z/F-involution over D and h be a hermitian of rank $2s$ over (D,τ). Let* $\mathrm{disc}(h)=1$. *Then*

$$F^* = \big\langle \{N_{M/F}(M^*) : \mathrm{Disc}(h_M)=1\}\big\rangle.$$

Proof Let $M_h \in M_{2s}(D)$ be a matrix representing h. Since $\mathrm{disc}(h) = 1 \in F^*/N_{Z/F}(Z^*)$, we have $\mathrm{Nrd}(M_h) = d \in N_{Z/F}(Z^*)$. Let $z \in Z$ be such that $d = N_{Z/F}(z)$. Let $\beta = \mathrm{Tr}_{Z/F}(z)$ and $\gamma = z\beta^{-1}$. Let w be an ordering on Z which extends an ordering v of F such that D_w is not split. Then $\mathrm{Nrd}(M_h) = d = N_{Z/F}(z) >_w 0$. Thus $\mathrm{Tr}_{Z/F}(z) = \beta >_w 0$ if and only if $z >_w 0$. This implies that $\gamma = z\beta^{-1} >_w 0$ and thus by 1.8, $\gamma \in \mathrm{Nrd}(D^*)$. Let $x \in D^*$ be such that $\mathrm{Nrd}(x) = \gamma$. Let

$$M_h' = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & . & \\ & & & x \end{pmatrix} M_h \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & . & \\ & & & \tau(x) \end{pmatrix}^t$$

Then $\mathrm{Nrd}(M_h') = (d\beta^{-1})^2$ and we conclude that for a suitable choice of a matrix M_h representing the hermitian form h, $\mathrm{Nrd}(M_h) = \alpha^2$, $\alpha \in F^*$. Let $\lambda \in F^*$. By 5.7, there exist quadratic extensions L_i/F, $\lambda_i \in L_i^*$ and $u_i \in L_i(\sqrt{\delta})$, $1 \le i \le r$, such that $\lambda = \prod_i N_{L_i/F}(\lambda_i)$, $\alpha u_i \in \Sigma(L(\sqrt{\delta}))$ and $N_{L_i(\sqrt{\delta})/L}(u_i) = 1$. Then

$$\alpha^2 = N_{L_i(\sqrt{\delta})/L_i}(\alpha u_i) \in N_{L_i(\sqrt{\delta})/L_i}(\mathrm{Nrd}(D_{L_i(\sqrt{\delta})}))$$

and hence $\mathrm{Disc}(h_{L_i}) = 1$ for $1 \le i \le r$. Thus $\lambda \in \big\langle \{N_{M/F}(M^*) : \mathrm{Disc}(h_M) = 1\}\big\rangle$ and we conclude that $F^* = \big\langle \{N_{M/F}(M^*) : \mathrm{Disc}(h_M) = 1\}\big\rangle$. □

The following propositions are used in the proof of 5.13, which is the main result of this section.

Proposition 5.10 *Let $A \simeq M_r(D)$ where D is a division algebra over Z and r is even. Let σ be a locally hyperbolic Z/F-involution on A. Then* $\mathrm{Hyp}(A,\sigma) = F^*$.

Proof Let σ be adjoint to a hermitian form h of rank r. Let $d \in F^*/N_{Z/F}(Z^*)$ denote the discriminant of h. Since σ is locally hyperbolic, for each $v \in \Omega$, the quaternion algebra (δ, d) splits at F_v. Thus by 1.8, $\mathrm{Nrd}((\delta, d)) = F^*$. Let $\lambda \in F^*$. There exists a finite extension E/F such that $\lambda \in N_{E/F}(E^*)$ and (δ, d) splits over E. Then $\mathrm{disc}(h_E)$ is trivial. By 5.9 we have

$$E^* = \big\langle \{ N_{M/E}(M^*) : \mathrm{Disc}(h_M) = 1 \} \big\rangle \qquad (*)$$

Let M/E be an extension such that and $\mathrm{Disc}(h_M) = 1$. Since σ is locally hyperbolic, $\mathrm{sgn}(h_M) = 0$. Thus by 5.8, the form h_M is hyperbolic and $\mathrm{Hyp}(h_M) = M^*$. Hence by $(*)$, $\mathrm{Hyp}(h_E) = E^*$ and

$$\lambda \in N_{E/F}(E^*) = N_{E/F}(\mathrm{Hyp}(h_E)) \subseteq \mathrm{Hyp}(h) = \mathrm{Hyp}(A, \sigma)$$

which implies that $\mathrm{Hyp}(A, \sigma) = F^*$. This completes the proof. □

Proposition 5.11 *Let A be a central simple algebra over Z with $\deg(A) \equiv 0(4)$. Let $\exp(A) = 2^m$ for some positive integer m. Let σ be a locally hyperbolic Z/F-involution on A. Then $\mathrm{Hyp}(A, \sigma) = F^*$.*

Proof Suppose $m = 1$. Since $\mathrm{cores}_{Z/F}(A) = 0$, by 1.12 $A \sim A_0 \otimes_F Z$ for some central simple F-algebra A_0 with $\exp(A_0) = 2$. Let M be a finite extension of F such that $A_{0M} \sim H$ for some quaternion algebra H over M. Then $A_M = M_r(H \otimes_F Z)$. Since $\deg(A) \equiv 0(4)$, the integer r is even. Thus by 5.10, $\mathrm{Hyp}(A_M, \sigma_M) = M^*$. In view of 2.4 we have

$$F^* = \big\langle \{ N_{M/F}(M^*) : A_{0M} \sim H \text{ for some quaternion algebra } H \text{ over } M \} \big\rangle.$$

It follows that $\mathrm{Hyp}(A, \sigma) = F^*$.

Suppose $m \geq 2$. Since $\mathrm{Br}(Z_w) = \mathbb{Z}/2\mathbb{Z}$ for each ordering $w \in \Omega_Z$, the algebra $A \otimes_Z A$ splits locally. Clearly $\exp(A \otimes_Z A) = 2^{m-1}$ and $\mathrm{cores}_{Z/F}(A \otimes_Z A) = 0$. Let $\lambda \in F^*$. By 2.6, there exist extensions L_i/F, $1 \leq i \leq s$ and $\lambda_i \in L_i^*$ such that each $(A \otimes_Z A) \otimes_F L_i$ is split and $\lambda = \prod_i N_{L_i/F}(\lambda_i)$. Then $\exp(A \otimes_F L_i) = 2$ for each i and by the case $m = 1$, $\lambda_i \in \mathrm{Hyp}(A_{L_i}, \sigma_{L_i})$. Hence

$$\lambda = \prod_i N_{L_i/F}(\lambda_i) \in \prod_i N_{L_i/F}(\mathrm{Hyp}(A_{L_i}, \sigma_{L_i})) \subseteq \mathrm{Hyp}(A, \sigma)$$

and it follows that $\text{Hyp}(A, \sigma) = F^*$. $\square$

Proposition 5.12 *Let A be a central simple algebra over Z with $\deg(A) \equiv 0(4)$. Let σ be a locally hyperbolic Z/F-involution on A. Then we have $\text{Hyp}(A, \sigma).F^{*2} = F^*$.*

Proof By [BP1, Lemma 3.3.1], there exists an odd degree extension M of F such that $\exp(A_M)$ is a power of 2 and by 5.11, $\text{Hyp}(A_M, \sigma_M) = M^*$. Taking norm from M/F and using that $[M : F]$ is odd, we conclude that $\text{Hyp}(A, \sigma).F^{*2} = F^*$. $\square$

Theorem 5.13 *Let F be a field with $\text{vcd}(F) \leq 2$ and let Z be a quadratic extension over F. Let A be a central simple algebra over Z and σ be a Z/F-involution on A. Then, $G(A, \sigma) \subseteq \text{Hyp}(A, \sigma).N_{Z/F}(Z^*)$.*

Proof The cases where $\deg(A)$ is odd or $\deg(A) \equiv 2(4)$ are covered by 1.6 and 5.6 respectively. We assume that $\deg(A) \equiv 0(4)$. Let $\lambda \in G(A, \sigma)$. At each $v \in \Omega$, the involution σ_v is adjoint to an even rank hermitian form which is hyperbolic if and only if $\text{sgn}(\sigma_v) = 0$. Therefore $\lambda >_v 0$ at those $v \in \Omega$, where σ_v is not hyperbolic. Let $K = F(\sqrt{-\lambda})$. Then $\deg(A_K) \equiv 0(4)$ and σ_K is locally hyperbolic. Thus by 5.12, we have $\text{Hyp}(A_K, \sigma_K).K^{*2} = K^*$. Let $\sqrt{-\lambda} = \alpha\beta^2$, where $\alpha \in \text{Hyp}(A_K, \sigma_K)$ and $\beta \in K^*$. Then $\lambda = N_{K/F}(\sqrt{-\lambda}) = N_{K/F}(\alpha)(N_{K/F}(\beta))^2$. But $N_{K/F}(\alpha) \in N_{K/F}(\text{Hyp}(A_K, \sigma_K)) \subseteq \text{Hyp}(A, \sigma)$. Thus $\lambda \in \text{Hyp}(A, \sigma).F^{*2}$. This completes the proof. $\square$

6 Fields with $\text{vcd}(F) \leq 2$: Orthogonal groups

Let F be an arbitrary field with $\text{char}(F) \neq 2$. Let D be a central division algebra over F with an orthogonal involution τ. We first recall from [BP2], certain invariants associated to hermitian forms over (D, τ).

Discriminant: Let D and τ be as above and h be a hermitian form of even rank over (D, τ). Let $\text{rank}(h) = 2m$ and let $M_h \in M_{2m}(D)$ represent the hermitian form h. Let

$$\text{Disc}(h) = (-1)^{r(r-1)/2} \, \text{Nrd}(M_h) \in F^*/(\text{Nrd}(D^*))^2,$$

where $r = 2m\deg(D)$. If $M'_h \in M_{2m}(D)$ is another matrix representing h then there exists an invertible matrix $T \in M_{2m}(D)$ such that $M_h = TM_h(\tau(T)^t)$. Thus $\mathrm{Nrd}(M'_h) = \mathrm{Nrd}(M_h)\,\mathrm{Nrd}(T)^2$ and $\mathrm{Disc}(h)$ is well defined. We call $\mathrm{Disc}(h)$ the *Discriminant* of h.

Clifford invariant: We recall from [KMRT, §8.B], the notion of the *Clifford algebra* $\mathrm{C}(A,\sigma)$ associated to a central simple algebra A over a field F with an involution σ of orthogonal type. If A is split and σ is adjoint to a quadratic form q then $\mathrm{C}(A,\sigma)$ is the even Clifford algebra $\mathrm{C}_0(q)$ of the quadratic form q. If $\mathrm{disc}(\sigma)$ is trivial, $\mathrm{C}(A,\sigma)$ decomposes into a product $\mathrm{C}_+(A,\sigma)\times\mathrm{C}_-(A,\sigma)$, each of the factors being a central simple algebra over F such that

$$[\mathrm{C}_+(A,\sigma)] + [\mathrm{C}_-(A,\sigma)] = [A] \in \mathrm{Br}(F).$$

Let D, τ and h be as above. Let $\mathrm{disc}(h)$ be trivial and $A = M_{2m}(D)$. Let τ_h be the orthogonal involution on A which is adjoint to h. We define the *Clifford invariant* of h as follows:

$$\mathcal{C}\ell(h) = [C_+(M_{2m}(D),\tau_h)] \in \mathrm{Br}(F)/[D]$$

Let H_{2m} denote the matrix $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \in M_{2m}(D)$ where I_m is the identity matrix of size m. The matrix H_{2m} represents the hyperbolic form of rank $2m$ over (D,τ). Let $\mathrm{U}_{2m}(D,\tau)$, $\mathrm{SU}_{2m}(D,\tau)$ and $\mathrm{Spin}_{2m}(D,\tau)$ denote respectively, the unitary, special unitary group and spin group with respect to the hyperbolic form H_{2m} over (D,τ). We have an exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}_{2m}(D,\tau) \to \mathrm{SU}_{2m}(D,\tau) \to 1$$

from which one gets the exact sequence of pointed sets

$$\to H^1(F,\mathrm{Spin}_{2m}(D,\tau)) \to H^1(F,\mathrm{SU}_{2m}(D,\tau)) \xrightarrow{\delta} H^2(F,\mu_2)$$

Let $\mathfrak{S}$ denote the set of ordered pairs (X,a), where $X \in \mathrm{GL}_{2m}(D)$ and $a \in F^*$ satisfy $\tau(X) = X^t$ and $\mathrm{Nrd}(X) = \mathrm{Nrd}(H_{2m})a^2$. The elements of $H^1(F,\mathrm{SU}_{2m}(D,\tau))$

are equivalence classes of $\mathfrak{S}$ under the following equivalence relation: $(X, a) \sim (X', a')$ if and only if there exists $Y \in \mathrm{GL}_{2m}(D)$ with $X' = YX\overline{Y}^t$ and $a' = \mathrm{Nrd}(Y)a$.

Let h be a hermitian form over (D, τ) with $\mathrm{rank}(h) = 2m$ and $\mathrm{disc}(h) = 1$. Let M_h be a matrix which represents h and $\mathrm{Nrd}(M_h) = a^2$, $a \in F^*$. The two elements $\xi_a = (M_h, a)$ and $\xi_{-a} = (M_h, -a)$ in $H^1(F, \mathrm{SU}_{2m}(D, \tau))$ map to $[h]$ under $H^1(F, \mathrm{SU}_{2m}(D, \tau)) \to H^1(F, \mathrm{U}_{2m}(D, \tau))$. Let $C_+(h) = \delta(\xi_a)$ and $C_-(h) = \delta(\xi_{-a})$. We recall the following lemma from [BMPS, Lemma 3.1].

Lemma 6.1 *If F is a formally real field and v is an ordering on F such that D_v is not split, then the algebra $C_+(h)$ is split at v if and only if $a >_v 0$.* □

Rost invariant: Let h be a hermitian form over (D, τ) with $\mathrm{rank}(h) = 2m$, trivial discriminant and trivial Clifford invariant. Consider the exact sequence

$$1 \to \mathrm{SU}_{2m}(D, \tau) \to \mathrm{U}_{2m}(D, \tau) \to \mu_2 \to 1.$$

This gives rise to the following exact sequence of pointed sets

$$\to \mathrm{U}_{2m}(D, \tau)(F) \to \{\pm 1\} \to H^1(F, \mathrm{SU}_{2m}(D, \tau)) \to H^1(F, \mathrm{U}_{2m}(D, \tau)) \to$$

Since $\mathcal{C}\ell(h) = 0$, there exists $\xi \in H^1(F, \mathrm{SU}_{2m}(D, \tau))$ mapping to $[h] \in H^1(F, \mathrm{U}_{2m}(D, \tau))$ such that $\delta(\xi) = 0$. Let $\tilde{\xi} \in H^1(F, \mathrm{Spin}_{2m}(D, \tau))$ be a preimage of $\xi \in H^1(F, \mathrm{SU}_{2m}(D, \tau))$. Let $G = \mathrm{Spin}_{2m}(D, \tau)$ and $\mathcal{R}_G : H^1(F, G) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ denote the Rost invariant of G [Me3]. The *Rost invariant of h* is defined as follows: [BP2, pp. 664]

$$R(h) = \mathcal{R}_G(\tilde{\xi}) \in \frac{H^3(F, \mathbb{Q}/\mathbb{Z}(2))}{F^* \cup [D]}$$

The element $\mathcal{R}_G(\tilde{\xi})$ takes values in $H^3(F, \mathbb{Z}/4)$ [BP2, Remark 1], where $\mathbb{Z}/4$ has the trivial Galois module structure. We now recall a proposition which we shall use often.

Proposition 6.2 (**[BP2, Cor. 2.6]**) *Let F be a formally real field and let $I^3(F)$ be torsion-free. Let Ω be the set of orderings on F. Then the natural map*

$$H^3(F, \mathbb{Z}/4) \to \prod_{v \in \Omega} H^3(F_v, \mathbb{Z}/4)$$

is injective. □

We now record a classification result for hermitian forms over central division algebras with orthogonal involutions over fields with $\mathrm{vcd}(F) \leq 2$.

Theorem 6.3 **([BP2, Th. 7.3])** *Let F be a field with $\mathrm{vcd}(F) \leq 2$ and D be a central division algebra over D with an orthogonal involution τ. Let h be a hermitian form over (D, τ). Then h is hyperbolic if and only if h has even rank, trivial Discriminant, trivial Clifford and Rost invariant and trivial signature.* □

Let F be a field with $\mathrm{vcd}(F) \leq 2$ and (A, σ) be a central simple algebra over F with orthogonal involution. If A is split, $\deg(A)$ is even, σ is locally hyperbolic and $\mathrm{disc}(\sigma) = 1$, so that by 5.2 we have $\mathrm{Hyp}(A, \sigma) = F^*$. We now consider the case where A is locally split.

Lemma 6.4 *Let $\mathrm{vcd}(F) \leq 2$ and A be a central simple algebra of even degree over F. Let σ be an orthogonal involution on A. If A is locally split then*

$$G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma).F^{*2}.$$

Proof Let $\lambda \in G_+(A, \sigma)$ and $K = F(\sqrt{-\lambda})$. Clearly $\lambda \in N_{K/F}(K^*)$. Let $\mathrm{disc}(\sigma) = d$ and $L = F(\sqrt{d})$. By 1.3, $\lambda \in N_{L/F}(L^*)$. Let $M = F(\sqrt{-\lambda}, \sqrt{d})$. Using [W, Lemma 2.14] for the biquadratic extension M/F, there exist $x \in M^*$ and $y \in F^*$ such that $\lambda = N_{M/F}(x)y^2$. Further A_M is locally split and by 1.8, $\mathrm{Nrd}(A_M) = M^*$. Let E/M be an extension such that $x = N_{E/M}(\alpha)$ for some $\alpha \in E^*$ and A_E be split. Clearly $\mathrm{disc}(\sigma_E) = 1$, σ_E is locally hyperbolic and A_E is split. Thus by 5.2, $\mathrm{Hyp}(A_E, \sigma_E) = E^*$ and hence $x = N_{E/M}(\alpha) \in \mathrm{Hyp}(A_M, \sigma_M)$. Thus $\lambda = N_{M/F}(x)y^2 \subseteq \mathrm{Hyp}(A, \sigma).F^{*2}$. We conclude that $G_+(A, \sigma) \subseteq \mathrm{Hyp}(A, \sigma).F^{*2}$. In view of 1.1 we have $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma).F^{*2}$. □

We continue with some lemmas which will be used in the proofs of the main results of this section.

Lemma 6.5 *Let $\mathrm{vcd}(F) \leq 2$ and $\chi \in H^3(F, \mu_2)$. Then*

$$F^* = \big\langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } \chi_L = (-1) \cup (-1) \cup (-x) \text{ for some } x \in L^*\}\big\rangle.$$

Proof Since $\mathrm{vcd}(F) \leq 2$, $H^3(F(\sqrt{-1}), \mu_2) = 0$ and in view of the Arason exact sequence 1.11, the map $H^2(F, \mu_2) \overset{\cup(-1)}{\longrightarrow} H^3(F, \mu_2)$ is surjective. Let $\xi \in H^2(F, \mu_2)$ be such that $(-1) \cup \xi = \chi$. Let D_ξ be a central division algebra over F, whose Brauer class is represented by ξ. Then $\exp(D_\xi) = 2$. Let $L \in \mathcal{F}_2(F)$ be such that $(D_\xi)_L \sim (-1) \cup (-x)$ for some $x \in L$. Then

$$\chi_L = (-1) \cup \xi_L = (-1) \cup (D_\xi)_L = (-1) \cup (-1) \cup (-x).$$

In view of this and 2.4, we have that

$$\begin{aligned} F^* &= \left\langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } (D_\xi)_L = (-1) \cup (-x) \text{ for some } x \in L^*\} \right\rangle \\ &\subseteq \left\langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } \chi_L = (-1) \cup (-1) \cup (-x) \text{ for some } x \in L^*\} \right\rangle \end{aligned}$$

□

For $\chi \in H^r(F, \mu_2)$, we set $N(\chi) = \langle \{N_{L/F}(L^*) : \chi_L = 0\} \rangle$.

Lemma 6.6 *Let* $\mathrm{vcd}(F) \leq 2$ *and* $\chi \in H^r(F, \mu_2), r \geq 2$. *Then following three groups coincide:*

(i) $N(\chi)$

(ii) $\{\lambda \in F^* : \lambda >_v 0 \text{ at those } v \in \Omega \text{ where } \chi_v \neq 0\}$.

(iii) $\{\lambda \in F^* : (\lambda) \cup \chi = 0\}$.

Proof Since $\mathrm{vcd}(F) \leq 2$, in view of 1.15 the cohomology groups $H^{r+1}(F, \mu_2)$ are (-1)-torsion-free for $r \geq 2$ and thus the groups (*ii*) and (*iii*) coincide. We show that $N(\chi) \subseteq \{\lambda \in F^* : (\lambda) \cup \chi = 0\}$. Let $\lambda \in N(\chi)$ be such that $\lambda = N_{L/F}(\mu)$ for an extension L/F with $\chi_L = 0$. Then $((\mu) \cup \chi)_L = 0$ and thus we have

$$\mathrm{cores}_{L/F}((\mu) \cup \chi)_L = (\lambda) \cup \chi = 0.$$

Hence $N(\chi) \subseteq \{\lambda \in F^* : (\lambda) \cup \chi = 0\}$. To complete the proof, we show that $\{\lambda \in F^* : \lambda >_v 0 \text{ at those } v \in \Omega \text{ where } \chi_v \neq 0\} \subseteq N(\chi)$. Let $\lambda \in F^*$ be such that $\lambda >_v 0$ at those $v \in \Omega$ where $\chi_v \neq 0$. Let $L = F(\sqrt{-\lambda})$. Then it follows that $\chi_w = 0$ for each ordering w of L. It follows from [Ar, Th. 2.1] that $\mathrm{vcd}(L) \leq 2$ and thus by 1.15, $H^3(L, \mu_2)$ is (-1)-torsion-free. Therefore $\chi_L = 0$. Thus $\lambda = N_{L/F}(\sqrt{-\lambda}) \in N(\chi)$. This completes the proof. □

In 6.7, 6.8 and 6.9 below, the only restriction on F is that $\mathrm{char}(F) \neq 2$. Let D be a central division algebra over F and τ be an orthogonal involution on D. Let h be a hermitian form of rank $2m$ and trivial discriminant over (D, τ). Let $a \in F^*$ be such that $\mathrm{Nrd}(M_h) = a^2$, where M_h is a matrix representing the form h. Since $\mathrm{disc}(h) = 1$, we recall from [MT, Prop. 1.12] that $G_+(h) = G(h)$.

Lemma 6.7 *Let D be a central division algebra over a field F of characteristic different from 2 with an orthogonal involution τ. Let h and h' be two even rank hermitian forms of trivial discriminant over (D, τ). Then we have the following additive property for Clifford invariants:*

$$\mathcal{C}\ell(h \perp h') = \mathcal{C}\ell(h) + \mathcal{C}\ell(h') \in H^2(F, \mu_2)/[D].$$

Proof We extend the scalars to the function field of the Brauer-Severi variety of D. Using the fact that the invariant e_2 of quadratic forms is additive on forms of trivial discriminant and that the kernel of the scalar extension map $H^2(F, \mu_2) \to H^2(F(X_D), \mu_2)$ is generated by the class of D in $H^2(F, \mu_2)$ [MT, Cor. 2.7], the lemma follows. □

From this lemma and the fact that two similar hermitian forms with even rank and trivial discriminant have the same Clifford invariants [BP1, pp. 204], we immediately have

Corollary 6.8 *Let D, τ and h be as in 6.7. Then for each $\lambda \in F^*$, the Clifford invariant $\mathcal{C}\ell(h \perp -\lambda h)$ is trivial.* □

In the following lemma, we compute the Rost invariant of the hermitian form $h \perp -\lambda h$, where h is as in 6.7 and $\lambda \in F^*$ is an arbitrary scalar.

Lemma 6.9 *Let D be a central division algebra of even degree over a field F of characteristic different from 2. Let τ be an orthogonal involution on D and h be a hermitian form over (D,τ) of even rank and trivial discriminant. Let $\lambda \in F^*$. Then,*

$$R(h \perp -\lambda h) = (\lambda) \cup [C_+(h)] \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^* \cup [D].$$

Proof Let $\operatorname{rank}(h) = 2m$ and $A = M_{2m}(D)$. Let τ_h be the involution on A which is adjoint to h. We denote by $\mathbf{PGO}_+(h)$ the group $\mathbf{PSim}_+(A, \tau_h)$ of similitudes. We have an exact sequence

$$1 \to \mu_2 \to \mathrm{SU}(h) \to \mathbf{PGO}_+(h) \to 1$$

which induces a map on the cohomology sets $H^1(F, \mu_2) \to H^1(F, \mathrm{SU}(h))$. We claim that under this map $(\lambda) \in H^1(F, \mu_2)$ is mapped to an element $\xi_\lambda \in H^1(F, \mathrm{SU}(h))$ which corresponds to the class of the hermitian form λh in $H^1(F, \mathrm{U}(h))$. In fact, the cocycle $(\lambda) \in Z^1(F, \mu_2)$ given by $s \mapsto s(\sqrt{\lambda})(\sqrt{\lambda})^{-1}$ for $s \in \mathrm{Gal}(F_s/F)$, when treated as a cocycle with values in $\mathrm{U}(h)$, represents $[\lambda h]$ in $H^1(F, \mathrm{U}(h))$.

Since $\deg(A) \equiv 0(4)$, the centre of $\mathrm{Spin}(h)$ is $\mu_2 \times \mu_2$ and the kernel of the map $\mathrm{Spin}(h) \to \mathrm{SU}(h)$ is (ϵ, ϵ), where $\epsilon = \pm 1$. The quotient of $\mu_2 \times \mu_2$ by μ_2 under the diagonal embedding maps isomorphically onto the centre of $\mathrm{SU}(h)$. By [MPT, Th. 1.14], the Rost invariant of the image $\tilde{\xi}_\lambda$ of $(1, \lambda) \in H^1(F, \mu_2 \times \mu_2)$ in $H^1(F, \mathrm{Spin}(h))$ is $(\lambda) \cup [C_+(h)]$. Thus $\tilde{\xi}_\lambda \in H^1(F, \mathrm{Spin}(h))$ maps to $\xi_\lambda \in H^1(F, \mathrm{SU}(h))$, which in turn maps to the class of λh in $H^1(F, \mathrm{U}(h))$ as is seen above. Thus we conclude that the hermitian form λh admits a lift $\tilde{\xi}_\lambda$ such that $\mathcal{R}(\tilde{\xi}_\lambda) = (\lambda) \cup [C_+(h)]$.

We now compute $R(h \perp -\lambda h)$. Let $i : \mathrm{Spin}(-h) \to \mathrm{Spin}(-h \perp h)$ be the natural map and $\tilde{i} : H^1(F, \mathrm{Spin}(-h)) \to H^1(F, \mathrm{Spin}(-h \perp h))$ the induced map on the cohomology sets. In view of [BP2, Lemma 3.6], $\mathcal{R}(\tilde{i}(\xi)) = \mathcal{R}(\xi)$ for every $\xi \in H^1(F, \mathrm{Spin}(-h))$. The group $\mathrm{Spin}(-h \perp h)$ maps isomorphically onto $\mathrm{Spin}_{4m}(D, \tau)$ preserving the Rost invariant. Further, the image of $(1, \lambda)$ in the cohomology set $H^1(F, \mathrm{Spin}(-h))$ maps to the isometry class of $-\lambda h$ in $H^1(F, \mathrm{U}(h))$ and to the isometry class of $-\lambda h \perp h$ in $H^1(F, \mathrm{U}_{4n}(D, \tau))$. This implies that the Rost invariant $R(h \perp -\lambda h)$ is equal to $(\lambda) \cup [C_+(h)] \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^* \cup [D]$. This completes the proof. $\square$

From now on, we assume that $\mathrm{vcd}(F) \leq 2$. Let L/F be a formally real extension and Ω_L be the set of orderings on L. Let h be a hermitian form over (D, τ) - a central division algebra D over F with an orthogonal involution τ. We define $\mathcal{S}_{\ell,L}(h)$ as follows:

$$\mathcal{S}_{\ell,L}(h) = \{\lambda \in L^* : h_{L_w} \simeq \lambda h_{L_w} \text{ for all } w \in \Omega_L\}.$$

If $L = F$ then we simply write $\mathcal{S}_\ell(h)$ to denote $\mathcal{S}_{\ell,F}(h)$. In 6.10 - 6.14 below, h denotes an even rank hermitian form over (D, τ) with trivial discriminant and $a \in F^*$ denotes a scalar which satisfies $\mathrm{Nrd}(M_h) = a^2$ for a choice M_h of a matrix representing h. Further, for $z \in F^*$ and a central simple algebra B over F with $\exp(B) = 2$, we denote by $z \cup B$ the element $(z) \cup [B] \in H^3(F, \mu_2)$.

Proposition 6.10 *We have* $G(h) = (N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_\ell(h)$.

Proof We first prove that $(N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_\ell(h) \subseteq G(h)$. Let $\lambda \in (N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_\ell(h)$. We show that $h \perp -\lambda h$ is hyperbolic. It is clear that $h \perp -\lambda h$ has even rank. Further $\mathrm{Nrd}(M(h \perp -\lambda h)) = a^4$, and since a^2 is totally positive, by 1.8 it belongs to $\mathrm{Nrd}(D^*)$. Thus it follows that $h \perp -\lambda h$ has trivial Discriminant. Moreover since $\mathrm{disc}(h)$ is trivial, by 6.8, it follows that the Clifford invariant of $h \perp -\lambda h$ is trivial. Since $\lambda \in \mathcal{S}_\ell(h)$, the form $h \perp -\lambda h$ has trivial signature as well.

By 6.9 we see that the Rost invariant $R(h \perp -\lambda h) = [(\lambda) \cup C_+(h)]$. We show that $[(\lambda) \cup C_+(h)]$ is trivial in $H^3(F, \mathbb{Z}/4)/F^* \cup [D]$. Let $x \in F^*$ be such that $x \in N(-a \cup D)$ and $\lambda x^{-1} \in N(a \cup D)$. We claim that $(\lambda) \cup [C_+(h)] = (x) \cup [D]$. In view of 6.2, it suffices to check that at each $v \in \Omega$, we have $(\lambda) \cup [C_+(h)_v] = (x) \cup [D_v]$.

Suppose $v \in \Omega$ is such that $\lambda >_v 0$ and $x >_v 0$. In this case $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are both trivial.

Suppose $\lambda >_v 0$ and $x <_v 0$. Then $\lambda x^{-1} <_v 0$. Since $\lambda x^{-1} \in N(a \cup D)$ and $x \in N(-a \cup D)$, in view of 6.6 both $(a) \cup [D]$ and $(-a) \cup [D]$ are split at v. Thus $-1 \in \mathrm{Nrd}(D_v)$ and hence D is split at v. Thus both $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are trivial in this case as well.

Now suppose that $\lambda <_v 0$ and D_v is split. Since $\lambda \in \mathcal{S}_l(h)$, we conclude that h_v is hyperbolic. Thus the Clifford invariant $\mathcal{C}\ell(h)_v = 0$. Further, since D_v is split and h_v is hyperbolic, we have $C_+(h)_v = C_-(h)_v = 0$. Thus we conclude that $C_+(h)_v$ is split and thus $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are both zero.

Next, suppose that $\lambda <_v 0$, D_v is not split and $x <_v 0$. Since $x \in N(-a \cup D)$, by 6.6 $(-a) \cup [D_v] = 0$; i.e. $-a \in \mathrm{Nrd}(D_v)$. Hence $a <_v 0$. Since D_v is not split and $a <_v 0$, by 6.1 $C_+(h)_v$ is not split. Thus we conclude in this case that both $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are non-zero and hence equal.

Now the only remaining case is when $\lambda <_v 0$, D_v is not split and $x >_v 0$. In that case, $\lambda x^{-1} <_v 0$ and since $\lambda x^{-1} \in N(a \cup D)$, by 6.6 we have that $(a) \cup [D_v] = 0$. Thus $a \in \mathrm{Nrd}(D_v)$ and hence $a >_v 0$. Since D_v is non-split, by 6.1 $C_+(h)_v$ is split. Thus both $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are zero in this case.

We conclude therefore that $(\lambda) \cup [C_+(h)_v] = (x) \cup [D_v]$ for all $v \in \Omega$. Thus by 6.2, we have $(\lambda) \cup [C_+(h)] = (x) \cup [D]$ and

$$R(h \perp -\lambda h) = (\lambda) \cup [C_+(h)] = 0 \in H^3(F, \mathbb{Z}/4)/F^* \cup [D].$$

Since $\mathrm{vcd}(F) \leq 2$, by 6.3 we have that $h \perp -\lambda h$ is hyperbolic. Thus $\lambda \in G(h)$.

We now show the inclusion $G(h) \subseteq (N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_\ell(h)$. It is clear that $G(h) \subseteq \mathcal{S}_\ell(h)$. We thus show that $G(h) \subseteq N(a \cup D)N(-a \cup D)$. Let $\lambda \in G(h)$. Then the the form $h \perp -\lambda h$ is hyperbolic and hence its Rost invariant $(\lambda) \cup [C_+(h)]$ is trivial. Thus there exists $x \in F^*$ such that $(\lambda) \cup [C_+(h)] = (x) \cup [D]$. By reading this equality locally at each $v \in \Omega$ and observing the sign pattern, we conclude that $x \in N(-a \cup D)$ and $\lambda x \in N(a \cup D)$. Therefore, $\lambda \in N(a \cup D)N(-a \cup D)$. This completes the proof. □

The following lemma will be used in the proof of 6.12.

Lemma 6.11 *Let D be a central division algebra over F and let τ be an orthogonal involution on D. Let h be an even rank locally hyperbolic hermitian form over (D, τ) with* $\mathrm{Disc}(h) = 1$ *and* $\mathcal{C}\ell(h) = 0$. *Then* $\mathrm{Hyp}(h) = F^*$.

Proof Since the hermitian form h has even rank, trivial discriminant and trivial Clifford invariant, there exists $\tilde{\xi} \in H^1(F, \mathrm{Spin}_{2m}(D,\tau))$ which maps to $[h] \in H^1(F, \mathrm{U}_{2m}(D,\tau))$. Let $\mathcal{R}(\tilde{\xi}) \in H^3(F, \mu_2)$ be the Rost invariant of $\tilde{\xi}$. Let $L \in \mathcal{F}_2(F)$ be such that $\mathcal{R}(\tilde{\xi}_L) = (-1) \cup (-1) \cup (-x)$ for some $x \in L^*$. We claim that $\mathcal{R}(\tilde{\xi}_L) = (-x) \cup D_L$.

Let Ω_L be the set of orderings on L and $w \in \Omega_L$ be such that D_{Lw} is split. Then $\mathcal{R}(\tilde{\xi}_{Lw}) = e_3(h_{Lw})$, where e_3 is the Arason invariant of quadratic forms. Since h_w is hyperbolic by hypothesis, we have $e_3(h_{Lw}) = 0$. Thus $\mathcal{R}(\tilde{\xi}_L)$ and $(-x) \cup D_L$ are both zero at w.

Now suppose D_{Lw} is not split. Then $D_{Lw} = (-1) \cup (-1)$ and thus $\mathcal{R}(\tilde{\xi}_{Lw}) = (-1) \cup (-1) \cup (-x) = (-x) \cup D_{Lw}$. Thus $\mathcal{R}(\tilde{\xi}_L) = (-x) \cup D_L$ at each $w \in \Omega_L$ and by 6.2, $R(h_L) = 0$. Therefore h_L is a locally hyperbolic form with even rank, trivial Discriminant, trivial Clifford invariant and trivial Rost invariant. By 6.3 the form h_L is hyperbolic. In view of this and 6.5 we conclude that $\mathrm{Hyp}(h) = F^*$. □

The following proposition gives an explicit description of the group $\mathrm{Hyp}(h)$.

Proposition 6.12 *We have* $\mathrm{Hyp}(h) = (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h))$.

Proof We first prove that $\mathrm{Hyp}(h) \subseteq (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h))$. Let L/F be a finite extension such that h_L is hyperbolic. Then $\mathrm{Disc}(h_L)$ is trivial in $L^*/\operatorname{Nrd}(D_L^*)^2$ and hence either $a \in \operatorname{Nrd}(D_L^*)$ or $-a \in \operatorname{Nrd}(D_L^*)$; i.e. either $N(a \cup D_L) = L^*$ or $N(-a \cup D_L) = L^*$. We clearly have $\mathcal{S}_{\ell,L}(h) = L^*$. Thus

$$(N(a \cup D_L) \cap \mathcal{S}_{\ell,L}(h)).(N(-a \cup D_L) \cap \mathcal{S}_{\ell,L}(h)) = L^*.$$

Clearly $N_{L/F}(N(a \cup D_L)) \subseteq N(a \cup D)$ and $N_{L/F}(N(-a \cup D_L)) \subseteq N(-a \cup D)$. Further as in [KMRT, Prop. 12.21], $N_{L/F}(\mathcal{S}_{\ell,L}(h)) \subseteq \mathcal{S}_\ell(h)$. Thus

$$\begin{aligned} N_{L/F}(L^*) &\subseteq N_{L/F}((N(a \cup D_L) \cap \mathcal{S}_{\ell,L}(h)).(N(-a \cup D_L) \cap \mathcal{S}_{\ell,L}(h))) \\ &\subseteq (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h)). \end{aligned}$$

Since $N_{L/F}(L^*)$ generate $\mathrm{Hyp}(h)$ as L runs over extensions where h is hyperbolic, it follows that $\mathrm{Hyp}(h) \subseteq (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h))$.

To complete the proof, we show that $N(a \cup D) \cap \mathcal{S}_\ell(h) \subseteq \mathrm{Hyp}(h)$. The inclusion $N(-a \cup D) \cap \mathcal{S}_\ell(h) \subseteq \mathrm{Hyp}(h)$ follows in the similar manner. Let $\lambda \in N(a \cup D) \cap \mathcal{S}_\ell(h)$. By 6.10, $\lambda \in G(h)$. Let $K = F(\sqrt{-\lambda})$. Since $\lambda \in N(a \cup D)$, by 6.6 $(\lambda) \cup (a) \cup [D] = 0 \in H^4(F, \mu_2)$. Thus $(-1) \cup (a) \cup [D_K] = 0 \in H^4(K, \mu_2)$. By 1.15, $(a) \cup [D_K] = 0 \in H^3(K, \mu_2)$. Hence $a \in \mathrm{Nrd}(D_K)$ and $\mathrm{Disc}(h)_K = 1$.

Let w be an ordering on K. Since $\lambda <_w 0$ and $\lambda \in G(h_K)$, the form h_K is locally hyperbolic. Thus the Clifford invariant $\mathcal{C}\ell(h)_K$ is trivial at w. Therefore, if D_{Kw} is split, then $C_+(h)_{Kw} = C_-(h)_{Kw} = 0$. If D_{Kw} is not split, then in view of 1.8, $a >_w 0$ as $a \in \mathrm{Nrd}(D_K)$. By 6.1, $C_+(h)_{Kw}$ is split. We have thus shown that $C_+(h_K)$ is locally split. By 1.8, it follows that $\mathrm{Nrd}(C_+(h_K)) = K^*$. Let L/K be a finite extension and $\alpha \in L^*$ be such that $\sqrt{-\lambda} = N_{L/K}(\alpha)$ and $C_+(h_L) = 0$. Then h_L is an even rank locally hyperbolic form with $\mathrm{Disc}(h_L) = 1$ and $C_+(h_L) = 0$. By 6.11, $\mathrm{Hyp}(h_L) = L^*$. Thus

$$\sqrt{-\lambda} = N_{L/K}(\alpha) \in N_{L/K}(\mathrm{Hyp}(h_L)) \subseteq \mathrm{Hyp}(h_K).$$

Taking norm from K/F we have $\lambda \in \mathrm{Hyp}(h)$. Thus $N(a \cup D) \cap \mathcal{S}_\ell(h) \subseteq \mathrm{Hyp}(h)$. □

With the notation as above, we have following corollaries.

Corollary 6.13 *If h is locally hyperbolic then*

$$\mathrm{Hyp}(h) = N(a \cup D).N(-a \cup D) = G(h).$$

Proof Since h is locally hyperbolic, $\mathcal{S}_\ell(h) = F^*$. From 6.10 and 6.12, it is clear that $\mathrm{Hyp}(h) = N(a \cup D).N(-a \cup D) = G(h)$. □

Corollary 6.14 *If h has trivial Discriminant, then* $\mathrm{Hyp}(h) = \mathcal{S}_\ell(h) = G(h)$.

Proof Since $\mathrm{Disc}(h) = 1$, it follows that either $N(a \cup D) = F^*$ or $N(-a \cup D) = F^*$. In either case, it is immediate from 6.10 and 6.12, that $\mathrm{Hyp}(h) = \mathcal{S}_\ell(h) = G(h)$. □

Let A be a central simple algebra over F with an orthogonal involution σ. Suppose $\mathrm{disc}(\sigma) = 1$ and $\mathrm{C}(A, \sigma) = \mathrm{C}_+(A, \sigma) \times \mathrm{C}_-(A, \sigma)$. We have the following extension of 6.13

Proposition 6.15 *Let F be a field with* $\mathrm{char}(F) \neq 2$. *Let (A, σ) be a central simple algebra of even degree over F with an orthogonal involution. Let* $\deg(A) \equiv 0(4)$ *and* $\mathrm{disc}(\sigma) = 1$. *Then* $\mathrm{Hyp}(A, \sigma) \subseteq \mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \, \mathrm{Nrd}(\mathrm{C}_-(A, \sigma))$. *Further if* $\mathrm{vcd}(F) \leq 2$ *and σ is locally hyperbolic then*

$$\mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \, \mathrm{Nrd}(\mathrm{C}_-(A, \sigma)) = G(A, \sigma).$$

Proof The first assertion follows from the fact that over any extension L/F where σ is hyperbolic, either $\mathrm{C}_+(A_L, \sigma_L)$ or $\mathrm{C}_-(A_L, \sigma_L)$ is split [KMRT, Prop. 12.21].

Suppose $\mathrm{vcd}(F) \leq 2$ and σ locally hyperbolic. Let $\lambda \in \mathrm{Nrd}(\mathrm{C}_+(A, \sigma))$. Let L/F be a finite extension such that $\lambda \in N_{L/F}(L^*)$ and $\mathrm{C}_+(A_L, \sigma_L)$ is split. We show that $\mathrm{Hyp}(A_L, \sigma_L) = L^*$. In view of 2.3, replacing L by a quadratic tower, we may assume that $A_L \simeq M_{2r}(H)$ for some quaternion algebra H over L. Let τ be an orthogonal involution on H and h be a hermitian form over (H, τ) such that σ_L is adjoint to h. Let $M_h \in M_{2r}(H)$ represent h and $\mathrm{Nrd}(M_h) = a^2$ for some $a \in L^*$. Let w be an ordering on L such that H_w is not split. Since $C_+(h) = C_+(A_L, \sigma_L)$ is split, by 6.1 $a >_w 0$. Thus $(a) \cup [H] = 0 \in H^3(L, \mu_2)$ and $N(a \cup H) = L^*$. In view of 6.13, $\mathrm{Hyp}(A_L, \sigma_L) = \mathrm{Hyp}(h) = N(a \cup H).N(-a \cup H) = L^*$. Thus $\mathrm{Hyp}(A_L, \sigma_L) = L^*$. Taking norms from L/F we have $\lambda \in N_{L/F}(L^*) = N_{L/F}(\mathrm{Hyp}(A_L, \sigma_L)) \subseteq \mathrm{Hyp}(A, \sigma)$. Thus $\mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \subseteq \mathrm{Hyp}(A, \sigma)$.

The inclusion $\mathrm{Nrd}(\mathrm{C}_-(A, \sigma)) \subseteq \mathrm{Hyp}(A, \sigma)$ follows from a similar argument. We therefore conclude that $\mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \, \mathrm{Nrd}(\mathrm{C}_-(A, \sigma))$

To complete the proof we show that $G(A, \sigma) \subseteq \mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \, \mathrm{Nrd}(\mathrm{C}_-(A, \sigma))$. Let $\lambda \in G(A, \sigma)$. Then the hermitian form $\langle 1, -\lambda \rangle$ is hyperbolic. Hence the Rost invariant $R(\langle 1, -\lambda \rangle)$ is trivial. As in the proof of 6.9, $R(\langle 1, -\lambda \rangle) = (\lambda) \cup [\mathrm{C}_+(A, \sigma)]$. Since the Rost invariant is trivial, there exists $x \in F^*$ such that $(\lambda) \cup [\mathrm{C}_+(A, \sigma)] = (x) \cup [A]$. If for an ordering v on F, the algebra A_v is split, then h_v being hyperbolic, $\mathrm{C}_+(A, \sigma)_v$ and $\mathrm{C}_+(A, \sigma)_v$ are both split. If A_v is not split and $x <_v 0$ then $\mathrm{C}_+(A, \sigma)_v$ is not split. Hence $\mathrm{C}_-(A, \sigma)_v$ is split. Thus $x \in \mathrm{Nrd}(\mathrm{C}_-(A, \sigma))$ and a similar argument gives $\lambda x \in \mathrm{Nrd}(\mathrm{C}_+(A, \sigma))$. Hence $\lambda = \lambda x.x^{-1} \in \mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \, \mathrm{Nrd}(\mathrm{C}_+(A, \sigma))$. We have thus shown that

$$G(A, \sigma) \subseteq \mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(\mathrm{C}_+(A, \sigma)) \, \mathrm{Nrd}(\mathrm{C}_-(A, \sigma)).$$

The inclusion $\mathrm{Hyp}(A,\sigma) \subseteq G(A,\sigma)$ follows from 1.1 and this completes the proof. $\square$

Theorem 6.16 *Let* $\mathrm{vcd}(F) \leq 2$ *and let* A *be a central simple algebra over* F *with* $\deg(A)$ *even and an involution* σ *of orthogonal type. If* $\mathrm{disc}(\sigma) = 1$ *and* σ *is locally hyperbolic then* $G(A,\sigma) = \mathrm{Hyp}(A,\sigma).F^{*2}$.

Proof Let $\deg(A) = 2n$. Suppose n is odd. Since σ is locally hyperbolic, the algebra A is locally split and by 6.4 the results holds. We can thus assume that n is even. In this case we are through by 6.15. $\square$

7 Fields with $\mathrm{vcd}(F) \leq 2$ satisfying SAP

Let F be a field with orderings and let Ω denote the set of orderings on F. Given $a \in F^*$, we define the corresponding *Harrison set* Ω_a as follows:

$$\Omega_a := \{v \in \Omega : a >_v 0\}.$$

The set Ω has *Harrison topology* for which $\{\Omega_a : a \in F^*\}$ is a sub-basis. With this topology, Ω is a Hausdorff, compact and totally disconnected space. We say that F has *strong approximation property* (SAP), if every closed and open set of Ω is of the form Ω_a for some $a \in F^*$. A quadratic form q is said to be *weakly isotropic*, if for some positive integer s, the s-fold orthogonal sum $s.q = \perp_s q$ is isotropic. Combining [ELP, Th. C] and [P, Satz. 3.1] we have the following

Theorem 7.1 *A field* F *with orderings has* SAP *if and only if for every* $a, b \in F^*$, *the quadratic form* $\langle 1, a, b, -ab\rangle$ *is weakly isotropic.* $\square$

In what follows, for $a_1, a_2, \cdots, a_r \in F^*$ the notation $\langle\langle a_1, a_2, \cdots, a_r\rangle\rangle$ will denote the r-fold Pfister form $\langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_r\rangle$. For a quadratic form q, we denote by $D(q)$ the set of elements of F^* represented by q. We remark that if $q = \langle 1, -a, -b, ab\rangle$ then $D(q) = \mathrm{Nrd}(H^*)$, where H is the quaternion algebra (a, b) over F. Set $\Omega(H) = \{v \in \Omega : H \otimes_F F_v \text{ is split }\}$. The following lemma is recorded in [Ga].

Lemma 7.2 *Let F be a field with orderings. Let $a, b \in F^*$. Let $q_1 = \langle\langle -1, -a\rangle\rangle$, $q_2 = \langle\langle -1, a\rangle\rangle$ and $H = (a, b)$. Suppose there does not exist $c \in F^*$ such that $\Omega_c = \Omega(H)$, then $-b \notin D(q_1)D(q_2)$.*

Proof Suppose $-b \in D(q_1)D(q_2)$ and let $x_1 \in D(q_1)$ and $x_2 \in D(q_2)$ be such that $-b = x_1x_2$. Then $q_1 \perp bq_2$ is isotropic and hence $2\langle 1, a, b, -ab\rangle \simeq q_0 \perp \mathbb{H}$ for some Albert form q_0 and $\mathbb{H} \simeq \langle 1, -1\rangle$. We have

$$C(q_0) = C(q_1 \perp bq_2) = (-1, -1) = C(4\langle 1\rangle \perp \mathbb{H}) \in \mathrm{Br}(F).$$

Therefore by [KMRT, Prop. 16.3], $q_0 \simeq 4\langle c\rangle \perp \mathbb{H}$ for some $c \in F^*$. It is easy to see that $\Omega_c = \Omega(H)$, which contradicts the hypothesis. Thus $-b \notin D(q_1)D(q_2)$. □

Lemma 7.3 *Let F be a field for which $I^3(F)$ is torsion-free. Let $a, b \in F^*$. Let $q_1 = \langle\langle -1, -a\rangle\rangle$, $q_2 = \langle\langle -1, a\rangle\rangle$ and $H = (a, b)$. If $-b \notin D(q_1)D(q_2)$ then there is no element $c \in F^*$ with $\Omega_c = \Omega(H)$.*

Proof Suppose there is an element $c \in F^*$ such that $\Omega_c = \Omega(H)$. Let $q' = \langle 1, a, b, -ab, -c, -c\rangle$. For $v \in \Omega$ if $c <_v 0$, then by the choice of c we have $a <_v 0$ and $b <_v 0$. This implies that the form q' is hyperbolic at v. If $c >_v 0$, then again by the choice of c, either $a >_v 0$ or $b >_v 0$ and in either case q' is hyperbolic at v. We thus conclude that q' is locally hyperbolic. Clearly $q' \in I^2(F)$, therefore $2q' \in I^3(F)$. Since $2q'$ is an even rank quadratic form with trivial signature, it is hyperbolic at each F_v, $v \in \Omega$. Thus by Pfister's local-global principle [L, Th. VIII.4.1], $2q'$ is a torsion element in the Witt group $W(F)$. By the hypothesis, $I^3(F)$ is torsion-free. Thus $2q' = 2\langle 1, a, b, -ab, -c, -c\rangle$ is hyperbolic. Therefore the form $q_1 \perp bq_2$ is isotropic, which implies that $-b \in D(q_1)D(q_2)$. This is a contradiction to the hypothesis. □

Combining 7.2 and 7.3 above, we get the following

Corollary 7.4 *Let $I^3(F)$ be torsion-free. Let $a, b \in F^*$ and $q_1 = \langle\langle -1, -a\rangle\rangle$, $q_2 = \langle\langle -1, a\rangle\rangle$ and $H = (a, b)$. Then $-b \in D(q_1)D(q_2)$ if and only if there exists $c \in F^*$ such that $\Omega_c = \Omega(H)$.* □

Using the results above, we have thus derived

Corollary 7.5 *Let F be a field with $I^3(F)$ torsion-free. Then the following statements are equivalent:*

(i) For all $a \in F^*$ we have $D\left(\langle\langle -1, -a\rangle\rangle\right) D\left(\langle\langle -1, a\rangle\rangle\right) = F^*$.

(ii) The field F has SAP.

(iii) Given a quaternion algebra $H = (a, b)$ over F, there exists an element $c \in F^*$ such that $\Omega_c = \Omega(H)$. □

Lemma 7.6 *Let $I^3(F)$ be torsion-free, and H be a quaternion algebra over F. Then* $\mathrm{Nrd}(H^*) = \{\lambda \in F^* : \lambda >_v 0 \text{ at each } v \in \Omega \backslash \Omega(H)\}$.

Proof Let n_H denote the norm form of the quaternion algebra H. Since $I^3(F)$ is torsion-free, for $\lambda \in F^*$, $\langle 1, -\lambda\rangle \otimes n_H = 0 \in I^3(F)$ if and only if for all $v \in \Omega$, $\langle 1, -\lambda\rangle \otimes n_H = 0 \in I^3(F_v)$. In other words, $\lambda \in \mathrm{Nrd}(H^*)$ if and only if $\lambda \in \mathrm{Nrd}(H \otimes F_v)^*$ at each $v \in \Omega$. This is equivalent to saying that $\lambda >_v 0$ if $v \in \Omega \backslash \Omega(H)$, and the lemma follows. □

Lemma 7.7 *Let $I^3(F)$ be torsion-free, F has orderings and satisfies* SAP. *Then for every $a, b \in F^*$ we have* $\mathrm{Nrd}(a, b)^* . \mathrm{Nrd}(-a, b)^* = F^*$.

Proof Let $H_1 = (a, b)$ and $H_2 = (-a, b)$. Since F has SAP, the closed amd open set $\Omega \backslash \Omega(H_1) = \Omega_x$ for some $x \in F^*$. Similarly $\Omega \backslash \Omega(H_1) = \Omega_y$ for some $y \in F^*$. By 7.6, $\mathrm{Nrd}(H_1^*) = \mathrm{Nrd}(-1, -x)$ and $\mathrm{Nrd}(H_2^*) = \mathrm{Nrd}(-1, -y)^*$. Since at a given ordering $v \in \Omega$, at least one of H_1 and H_2 is split, $\Omega_x \cap \Omega_y = \phi$. Thus $\Omega_x \subseteq \Omega_{-y}$. Now using 7.6, we conclude that $\mathrm{Nrd}(-1, y)^* \subseteq \mathrm{Nrd}(-1, -x)^*$. Thus in view of 7.5, $F^* = \mathrm{Nrd}(-1, y)^* \, \mathrm{Nrd}(-1, -y)^* \subseteq \mathrm{Nrd}(-1, -x)^* \, \mathrm{Nrd}(-1, -y)^*$. This proves $\mathrm{Nrd}(H_1)^* . \mathrm{Nrd}(H_2)^* = F^*$. □

Detlev Hoffmann has suggested the following more direct proof of 7.7.

Lemma 7.8 *Let $I^3(F)$ be torsion-free, F has orderings and satisfies* SAP. *Then for every $a, b \in F^*$ we have* $\mathrm{Nrd}(a, b)^* . \mathrm{Nrd}(-a, b)^* = F^*$.

Proof Let $H_1 = (a, b)$ and $H_2 = (-a, b)$. Let $X_i = \Omega \backslash \Omega(H_i); i = 1, 2$ and let $\lambda \in F^*$ be arbitrary. Since F has SAP, there exists $x \in F^*$ such that $X_1 \cup (X_2 \cap \Omega_\lambda) = \Omega_x$. Using the hypothesis that $I^3(F)$ torsion-free and the observation that $X_1 \cap X_2 = \phi$, one can conclude that $x \in \mathrm{Nrd}(H_1)^*$ and $\lambda x^{-1} \in \mathrm{Nrd}(H_2)^*$. Thus $\lambda = x\lambda x^{-1} \in \mathrm{Nrd}(H_1)^*.\, \mathrm{Nrd}(H_2)^*$. □

From now on, in this section the field F satisfies $\mathrm{vcd}(F) \leq 2$. We say that a field F satisfies SAP *for quadratic towers* if each quadratic tower $L \in \mathcal{F}_2(F)$ has SAP.

Proposition 7.9 *Let* $\mathrm{vcd}(F) \leq 2$ *and* F *has* SAP *for quadratic towers. Let* h *be a locally hyperbolic hermitian form of even rank over a central-division algebra* D *with an orthogonal involution* τ*. Let* $\mathrm{disc}(h) = 1$*. Then we have* $\mathrm{Hyp}(h) = F^*$*.*

Proof Let $L \in \mathcal{F}_2(F)$ be such that $D_L \sim (-1, -x)$ for some $x \in L^*$. Since h_L is locally hyperbolic, by 6.13 we have $\mathrm{Hyp}(h_L) = N(a \cup D_L).N(-a \cup D_L)$. Clearly $N(a \cup D_L) = \mathrm{Nrd}(a, -x)^*$ and $N(-a \cup D_L) = \mathrm{Nrd}(-a, -x)^*$. Since F has SAP for quadratic towers, so does L. Thus by 7.7, we have

$$\mathrm{Nrd}(a, -x)^*.\, \mathrm{Nrd}(-a, -x)^* = L^*$$

and we conclude that $\mathrm{Hyp}(h_L) = L^*$. In view of this and 2.4 we have $\mathrm{Hyp}(h) = F^*$. □

Proposition 7.10 *Let* $\mathrm{vcd}(F) \leq 2$ *and* F *has* SAP *with respect to quadratic towers. Let* A *be a central simple algebra of even degree over* F *and* σ *be a locally hyperbolic involution on* A *with* $\mathrm{disc}(\sigma) = 1$*. Then* $\mathrm{Hyp}(A, \sigma) = F^*$*.*

Proof Let $L \in \mathcal{F}_2(F)$ be such that $A_L \sim (-1, -x)$ for some $x \in L^*$. Let $H = (-1, -x)$. Then $A_L = M_r(H)$ for some positive integer r. Let τ be an orthogonal involution on H and h be a hermitian form over (H, τ) such that σ_L is adjoint to h. Then $\mathrm{Hyp}(A_L, \sigma_L) = \mathrm{Hyp}(h)$.

First assume that r is even. Then it follows from [KMRT, Prop. 7.3(1)] that $\mathrm{disc}(\sigma_L) = \mathrm{disc}(h) = 1$. Since σ_L is locally hyperbolic, the hermitian form h is locally hyperbolic. Thus h is a locally hyperbolic form of even rank and trivial discriminant over (H, τ). Therefore in view of 7.9 we have $\mathrm{Hyp}(A_L, \sigma_L) = \mathrm{Hyp}(h) = L^*$.

Now suppose r is odd. Since σ_L is locally hyperbolic, the hermitian form h is locally hyperbolic. Thus the quaternion algebra H is locally split and by 1.8 we have $\mathrm{Nrd}(H)^* = L^*$. Let $\lambda \in L^*$. Let $L_i \in \mathcal{F}_2(L)$ be such that each H_{L_i} is split and $\lambda = \prod_i N_{L_i/L}(\lambda_i)$, where $\lambda_i \in L_i^*$. Then h_{L_i} is a locally hyperbolic quadratic form over L_i with even rank and trivial discriminant. Thus by 5.2, we have $\mathrm{Hyp}(h_{L_i}) = L_i^*$. Therefore

$$\lambda \in \prod_i N_{L_i/L}(\mathrm{Hyp}(H_{L_i})) \subseteq \mathrm{Hyp}(h)$$

and hence $\mathrm{Hyp}(h) = L^*$.

Thus it follows that if $L \in \mathcal{F}_2(F)$ is such that $A_L \sim (-1, -x)$ for some $x \in L^*$ then $\mathrm{Hyp}(A_L, \sigma_L) = \mathrm{Hyp}(h) = L^*$. Therefore in view of 2.4, we have $\mathrm{Hyp}(A, \sigma) = F^*$. This completes the proof. □

Theorem 7.11 *Let* $\mathrm{vcd}(F) \leq 2$ *and* F *has* SAP *for quadratic towers. Let* A *be a central simple algebra of degree* $2n$ *over* F *and* σ *be an orthogonal involution on* A *with* $\mathrm{disc}(\sigma) = 1$. *Then* $\mathrm{Hyp}(A, \sigma).F^{*2} = G_+(A, \sigma)$.

Proof Let $\lambda \in G_+(A, \sigma)$. Then $\lambda = \sigma(a)a$ for some $a \in A^*$ with $\mathrm{Nrd}(a) = \lambda^n$. Let $K = F(\sqrt{-\lambda})$. First suppose that n is odd. Then $\lambda \in \mathrm{Nrd}(A^*)$ and thus $\lambda >_v 0$ at those $v \in \Omega$ where A_v is not split. If A_v is split then $\lambda >_v 0$ at those $v \in \Omega$ where $\mathrm{sgn}(\sigma_v) \neq 0$. Then A_K is locally split and σ_K is locally hyperbolic.

Now suppose that n is even let and $v \in \Omega$ is such that σ_v is not hyperbolic. Then by (Theorem 3.7, Chapter 10, [S]), A_v is split and σ_v is adjoint to a non-hyperbolic quadratic form q over F_v such that $q \simeq \lambda q$. Thus we conclude that $\lambda >_v 0$ at those orderings $v \in \Omega$ where σ_v is not hyperbolic. Then σ_K is locally hyperbolic.

Thus in either case, by 7.10 we have that $\mathrm{Hyp}(A_K, \sigma_K) = K^*$. Taking norm of $\sqrt{-\lambda}$ from K/F, we conclude that $\lambda \in \mathrm{Hyp}(A, \sigma)$ and therefore $G_+(A, \sigma) \subseteq \mathrm{Hyp}(A, \sigma)$. By 1.1 we have $\mathrm{Hyp}(A, \sigma).F^{*2} \subseteq G_+(A, \sigma)$, and hence we conclude that $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma).F^{*2}$. □

Theorem 7.12 *Let* $\mathrm{vcd}(F) \leq 2$ *and* (A, σ) *be an algebra of type* ${}^2D_n$ *over* F. *Let* F *have* SAP *for quadratic towers. Then we have* $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma).F^{*2}$.

Proof Let $\mathrm{disc}(\sigma) = d$. Let $\lambda \in G_+(A, \sigma)$. By 1.3, we have $G_+(A, \sigma) \subseteq N_{L/K}(L^*)$, where $L = F(\sqrt{d})$. As in the proof of the 7.11, $\sigma_{F(\sqrt{-\lambda})}$ is locally hyperbolic. Let $M = L(\sqrt{-\lambda})$. By the biquadratic lemma [W, Lemma 2.14], it follows that there exist $x \in M^*$ and $y \in F^*$ such that $\lambda = N_{M/F}(x)y^2$. It is clear that $\mathrm{disc}(\sigma_M) = 1$ and σ_M is locally hyperbolic. Thus by 7.10 we have $\mathrm{Hyp}(A_M, \sigma_M) = M^*$ and we easily see that $\lambda y^{-2} \in \mathrm{Hyp}(A, \sigma)$. Thus $G_+(A, \sigma) \subseteq \mathrm{Hyp}(A, \sigma).F^{*2}$. Since by 1.1 $\mathrm{Hyp}(A, \sigma).F^{*2} \subseteq G_+(A, \sigma)$, we conclude that $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma).F^{*2}$. □

Corollary 7.13 *Let* $\mathrm{vcd}(F) \leq 2$, F *has* SAP *for quadratic towers and let* $\mathbf{PSim}_+(A, \sigma)$ *be of type* ${}^1D_n$ *or* ${}^2D_n$. *Then* $\mathbf{PSim}_+(A, \sigma)(F)/R = 0$. □

Since number fields satisfy the conditions of 7.13, we have

Corollary 7.14 *Let* F *be a number field and let* $\mathbf{PSim}_+(A, \sigma)$ *be of type* ${}^1D_n$ *or* ${}^2D_n$. *Then* $\mathbf{PSim}_+(A, \sigma)(F)/R = 0$. □

Remark It is a well known fact that SAP is not preserved under field extensions. As Detlev Hoffmann has pointed out to us, there are examples of fields F with $\mathrm{vcd}(F) = 2$ and quadratic extensions E/F such that F satisfies SAP but not E. Thus the condition 'SAP with respect to quadratic towers' is not redundant in 7.12.

Hoffmann's example is the following: One can construct a formally real field k with the following properties: (i) k has no extension of odd degree. (ii) There is only one ordering on k. (iii) The u-invariant of k is 2. Let $\alpha \in k^* \backslash k^{*2}$ be a sum of two squares. Let $F = k((X))$ and $E = F(\sqrt{\alpha})$. Then $\mathrm{vcd}(F) = 2$ and F has SAP but not E.

Combining together the results of §4, §5 and §7, we have

Theorem 7.15 *Let* F *be a field with* $\mathrm{vcd}(F) \leq 2$. *Let* G *is a classical group of adjoint type defined over* F. *Then,* *(i) If G does not contain a factor of type D_n then $G(F)/R = 0$.*

(ii) If F satisfies SAP *for quadratic towers then $G(F)/R = 0$.*

Proof If F does not have orderings, by 1.17 we have $\mathrm{cd}(F) \leq 2$ and we are through by 3.7. Thus we assume that F has orderings. As in the proof of 3.7, it suffices the prove the theorem for absolutely simple adjoint groups defined over F. In view of 1.1, the Assertion (i) follows from 1.4, 4.2 and 5.13. The Assertion (ii) of the theorem follows immediately from Assertion (i) and 7.13. $\square$

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amitk@math.tifr.res.in
parimala@math.tifr.res.in
School Of Mathematics
Tata Institute Of Fundamental Research
Homi Bhabha Road, Mumbai 400 005, INDIA