Review of Kneser's work on algebraic groups and the Hasse principle and subsequent developments

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Let k be a number field. Let Ω denote the set of places of k and for $v \in \Omega$, let k_v denote the completion of k at v. A classical theorem of Hasse-Minkowski states that a quadratic form q over k represents zero non-trivially provided it represents zero non-trivially over k_v for all $v \in \Omega$; in particular, two quadratic forms over k are isomorphic if they are isomorphic over k_v for all $v \in \Omega$. Another classical theorem of Hasse-Brauer-Noether states that two central simple algebras over k are isomorphic if they are isomorphic over k_v for all $v \in \Omega$ - a consequence of the injectivity of the map $\mathrm{Br}(k) \to \bigoplus_{v\in\Omega} \mathrm{Br}(k_v)$, $\mathrm{Br}(k)$ denoting the Brauer group of k. These results can be formulated as a Hasse principle for Galois cohomology.

Let k be a field, k_s a separable closure of k and Γ_k the Galois group of k_s over k. Let G be a linear algebraic group defined over k. Let $H^1(k, G) = H^1(\Gamma_k, G(k_s))$ be the first non-abelian Galois cohomology set of equivalence classes of continuous one-cocycles $\Gamma_k \to G(k_s)$ [Se4, I.§5]. This pointed set classifies isomorphism classes of principal homogeneous spaces for G over k; a principal homogeneous space defines the trivial class if and only if the underlying k-variety has a k-rational point. The set $H^1(k, \mathrm{PGL}_n)$ classifies isomorphism classes of central simple algebras of degree n over k; given a non-degenerate quadratic form q of rank n over k, $H^1(k, \mathrm{O}(q))$ classifies isomorphism classes of non-degenerate quadratic forms of rank n over k. The results stated above can then be reformulated as the injectivity of the map

$$H^1(k, G) \to \prod_{v\in\Omega} H^1(k_v, G)$$

for $G = \mathrm{PGL}_n$ or $\mathrm{O}(q)$. The injectivity is not true in general for a connected linear algebraic group defined over k [Se4, III.§4, Th.8]. One of the main contributions of Kneser around 1960, as pointed out by Serre, was his idea that "simply connected" is significant for arithmetic; 'he was surely led to that idea by the study of quadratic forms and spinor genera' [K1]. In [K3], Kneser poses the following conjecture.

Conjecture *Let G be a semisimple simply connected linear algebraic group defined over a number field. Then the map*

$$H^1(k, G) \to \prod_{v \in \Omega_\infty} H^1(k_v, G)$$

is injective, Ω_∞ denoting the set of real places of k.

In [K2], Kneser proves that if k is a p-adic field, for a semisimple simply connected linear algebraic group defined over k, $H^1(k, G) = 0$. The proof for classical groups is related to the classification of quadratic and hermitian forms over p-adic fields. There is a classification-free proof of this theorem due to Bruhat-Tits [BT]. In view of this theorem, the set Ω_∞ in the statement of the conjecture may be replaced by Ω. Kneser [K5] proves that surjectivity is true more generally for any connected semisimple linear algebraic group defined over k.

If $G = \mathrm{SL}_A$, A a central simple algebra over k, then $H^1(k, \mathrm{SL}_A) \simeq k^*/\mathrm{Nrd}(A^*)$, $\mathrm{Nrd} : A \to k$ denoting the reduced norm and A^* denoting the group of units of k. The conjecture in this case is a theorem of Hasse-Maass and Schilling : $\lambda \in k^*$ is a reduced norm from A if it is positive at all real places where D is ramified. A proof due to Eichler is contained in [K5]. For special unitary groups, the conjecture is proved using a theorem of Landherr, which in fact is a Hasse principle for hermitian forms over division algebras with unitary involutions. The simplified proof of Landherr's theorem given in [K5] was also independently obtained by Springer. The proof of the conjecture for classical groups - groups of type A_n, B_n, C_n, D_n (D_4 non-trialitarian) is due to Kneser [K5]. The proof of the conjecture for all exceptional groups other than E_8 is due to Harder [H1]. While the results of Kneser and Harder date to the mid-sixties, the case E_8 was settled some twenty years later by Chernousov [Ch]. If k is a global field of positive characteristic and G a semisimple simply connected linear algebraic group defined over k, Harder [H3] proves that $H^1(k, G) = 0$.

As Kneser points out, the following are consequences of the Hasse principle conjecture for totally imaginary number fields : (i) $H^1(k, G) = 0$ if G is semisimple simply connected; (ii) all anisotropic simple groups over k are of type A_n. Sansuc [Sa, Th. 4.2] studies the Hasse principle for principal homogeneous spaces for connected linear algebraic groups over a number field and shows that the only obstruction to Hasse principle is the Brauer-Manin obstruction attached to the Brauer group of a smooth compactification. The proof ultimately reduces to the Hasse principle for G semisimple simply connected. Kneser [K5] proves that if G is a semisimple connected linear algebraic group of classical type defined over a number field and $p : \tilde{G} \to G$ the simply connected cover with kernel μ, the connecting map $\delta : H^1(k, G) \to H^2(k, \mu)$ defined with respect to the exact sequence

$$1 \to \mu \to \tilde{G} \xrightarrow{p} G \to 1$$

is surjective. Sansuc proves [Sa, Cor. 4.5] the surjectivity of δ for any semisimple connected linear algebraic group defined over k; it is a bijection if G is of adjoint type.

Borovoi [B1, Th. 7.2, 7.3] proves that a homogeneous space of a semisimple simply connected linear algebraic group defined over a number field with character-free connected geometric isotropy group satisfies Hasse principle for existence of a rational point - existence of rational points over each real completion ensures the existence of a global point. This is an extension of a classical result for smooth affine quadrics of dimension at least 3. Borovoi uses an abelianisation of the second non-abelian Galois cohomology set (*cf.* [Sp2], [FSS]) to study homogeneous spaces. The final step is to reduce the problem to Kneser's conjecture. He derives as a consequence a theorem of Harder [H2] which states that the Hasse principle holds for projective homogeneous varieties for a connected linear algebraic group defined over a number field. In [B2] Borovoi proves that the Brauer-Manin obstruction is the only obstruction to Hasse principle for homogeneous spaces under connected linear algebraic groups with a connected stabiliser.

Totally imaginary number fields are examples of fields of cohomological dimension 2. A field k is said to have *cohomological dimension* (cd) *at most* n if $H^i(k, M) = 0$ for $i \geq n + 1$ for all finite discrete Γ_k-modules M [Se4, I.§3.1]. Serre posed in the early sixties ([Se1, §4.2],[Se4, III.§3.1]) the following conjecture, which is a far reaching generalisation of Kneser's conjecture.

Conjecture II *Let k be a perfect field of cohomological dimension at most* 2. *Let G be a semisimple simply connected linear algebraic group defined over k. Then* $H^1(k, G) = 0$.

Conjecture I of Serre concerns $H^1(k, G) = 0$ for any connected linear algebraic group G defined over a perfect field with $\mathrm{cd}(k) \leq 1$; this conjecture was settled by Steinberg [St, Th. 11.12)].

Unlike in the arithmetic case, anisotropic groups of large rank occur in general, in all classical types over fields of cohomological dimension 2; thus the induction techniques from arithmetic case to reduce to lower rank groups cannot be extended to the general case.

The first major breakthrough towards the proof of Conjecture II is due to Merkurjev and Suslin [Su] in the early eighties for $G = \mathrm{SL}_A$; their theorem also provides a converse of Conjecture II.

Theorem *Let k be a perfect field. The following are equivalent*

(i) $cd(k) \leq 2$

(ii) For every finite extension K/k and every central simple algebra A over K, we have $\mathrm{Nrd}(A^*) = K^*$.

More than a decade later, the proof of Conjecture II for other classical groups was given by Eva Bayer and Parimala [BP1]. The proof is via classifying hermitian forms over division algebra with involution by the "classical invariants" - dimension, discriminant and the Clifford invariant. Results on the norm principle for algebraic groups due to Merkurjev [M1] are crucial for handling the image $H^1(k, \mu) \rightarrow H^1(k, G)$, μ denoting the center of G. The conjecture is proved if G is of type G_2 or F_4 ([Se2, §8, §9],[BP1]). If G is the split group of type F_4, $H^1(k, G)$ classifies isomorphism classes of exceptional central simple Jordan algebras of dimension 27 over k. The proof of the conjecture in this case uses certain Galois cohomological invariants associated to these algebras ([R], [Se2]) and Springer's classification of "reduced" central simple Jordan algebras [Sp1]. If G is of trialitarian D_4 type, a theorem of Garibaldi [G] states that the image of $H^1(k, \mu) \rightarrow H^1(k, G)$ is trivial, μ denoting the center of G. Conjecture II in this case is reduced to a classification of trialitarian groups by their Tits algebras.

Conjecture II has been proved in several cases under special assumptions on k or G. Gille proves that if G is a group of exceptional type other than E_8, then $H^1(k, G) = 0$ if G is quasi-split [Gi, Th. 4] or if index and exponent coincide for central simple algebras of 2-primary and 3-primary exponents over all finite field extension of k [Gi, Th. 8, 9, 10]. The proof in the arithmetic case due to Chernousov can be adapted to show that for simple groups of type E_8, $H^1(k, G) = 0$ if $\mathrm{cd}(k^{ab}) \leq 1$, k^{ab} denoting the maximal abelian extension of k [Gi, Th. 11].

Index and exponent coincide for central simple algebras over the following classes of fields:

(I) ([FS], [CTOP, Th. 2.1]) k is a 2-dimensional strict henselian field - quotient field of a 2-dimensional excellent henselian local domain with residue field algebraically closed of characteristic 0.

(II) ([dJ]) k is of transcendence degree 2 over an algebraically closed field of characteristic 0.

It is also proved in [CTOP, §2, Th. 2.3] that if k is a 2-dimensional strict henselian field, then $\mathrm{cd}(k^{ab}) \leq 1$, thus leading to Conjecture II for such fields. Several arithmetic properties are satisfied by this class of fields [CTGP]. If G is a simple group of adjoint type over k, $p : \tilde{G} \to G$ is a simply connected covering and $\mu = \mathrm{kernel}(p)$, the bijectivity of the map [CTGP, Th. 2.1(a)] $\delta : H^1(k, G) \to H^2(k, \mu)$ leads to Hasse principle for projective homogeneous varieties over k [CTGP, Th. 5.5]. The proof is an adaption of Borovoi's proof in the arithmetic case.

Colliot-Thélène posed the following conjecture [BP2, pp. 652] analogous to Kneser's conjecture for perfect fields of virtual cohomological dimension 2. A field k is said to have *virtual cohomological dimension* (vcd) *at most* n if $\mathrm{cd}(k(\sqrt{-1})) \leq n$. Number fields are examples of fields of virtual cohomological dimension 2.

Conjecture HP *Let k be a perfect field with* $\mathrm{vcd}(F) \leq 2$. *Let G be a semisimple simply connected linear algebraic group defined over k. Then the map*

$$H^1(k, G) \to \prod_{v \in \Omega_k} H^1(k_v, G)$$

is injective, Ω_k denoting the space of orderings of k and for $v \in \Omega_k$, k_v denoting the real closure of k.

Conjecture HP is settled in the affirmative for all classical groups and groups of type G_2 and F_4 by Eva Bayer and Parimala [BP2]. A Hasse principle for reduced norms analogous to the theorem of Hasse-Maass and Schilling is the first step towards the proof of this theorem. Galois cohomological invariants in degree 3 associated to $H^1(k, G)$ for simply connected groups G constructed by Rost ([M2]) are used in the construction of invariants to classify hermitian forms over division algebras with involution over fields of virtual cohomological dimension 2.

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