Iterative Restoration of Wavefront Coded Imagery for Focus Invariance

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Abstract: An important approach for extending depth-of-focus is to insert a phase mask directly into the imaging system to increase the focus invariance of the point spread function (PSF). The process results in an intermediate image that us blurred in a spatially invariant way. In this paper we consider some linear and nonlinear iterative techniques for restoring imagery blurred by such wavefront coding. Tests are performed on data formed by artificially blurring images with an actual cubic phase mask PSF. From these tests we infer that all the methods produce adequate restorations, while a nonlinear, nonnegative constrained iterative algorithm due to Nagy and Strakos, called modified reduced norm steepest descent (MRNSD), appears to produce especially accurate restorations for this problem, in an efficient and stable way.

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1 Problem and Approach

Conventional optical imaging systems are constrained by a fundamental tradeoff between depth-of-field and light gathering. Low numerical aperture systems provide large depth-of-field, but suffer from poor low light performance and often exhibit severe blurring from diffraction by the aperture. A high numerical aperture system provides efficient use of the available light and high resolution at the plane of best focus, but produces blurred imagery away from the plane of best focus. Since each region in the image of a three-dimensional scene has a different amount of blur, computationally intensive space-variant algorithms would be needed to deblur such an image. Significant computational resources would also be needed to estimate the unknown blur in each region. Furthermore, the nature of the blur function for highly defocused regions often leads to an unstable restoration problem in the presence of noise [4].

Dowski and Cathey [3] proposed the introduction of a cubic phase optical element at the pupil of an optical imaging system to present a more tractable problem for image restoration. The cubic phase approach allows a light efficient wide aperture implementation while creating a blur function that is highly invariant with misfocus. Images captured with such a system can be digitally restored in a space-invariant fashion since regions corresponding to severe misfocus have effectively the same blur response as well-focused regions. Dowski and Cathey refer to the general idea of modifying the optical system followed by subsequent restoration as *wavefront coding*.

To date, researchers have emphasized direct linear methods to restore wavefront coded images [1, 3, 5] because of good computational efficiency [2] that allows for real-time restoration. In many applications, real-time results are essential and direct methods are used. We suggest that in some applications, direct linear methods could be used to identify regions of interest in real-time and more computationally intensive iterative algorithms could be subsequently applied in an attempt to improve restoration fidelity [4, 10].

A well-designed incoherent imaging system can be treated as a linear shift-invariant system. The image plane intensity can be expressed in terms of a convolution given by

$$I(x,y) = O(x,y) \star h(x,y) + \eta(x,y), \tag{1}$$

where $I(x,y)$ is the observed object intensity, $O(x,y)$ is the unknown true object intensity, $h(x,y)$ is the incoherent point spread function (PSF), and $\eta(x,y)$ represents noise in the system. The PSF is governed by aberrations and the nature of the pupil aperture. The extended depth-of-field imaging system contains a phase mask that imparts a cubic phase deviation to the pupil plane wavefront as described in reference [3]. The corresponding spatial frequency domain optical transfer function (OTF) is the spatial autocorrelation of the pupil transmission function and $h(x,y)$, the system PSF, is determined by a spatial Fourier transformation of the OTF.

The basic problem is to solve for $O(x,y)$ in equation (1). This deconvolution step is an ill-posed inverse problem that generally requires some form of regularization in order to compute an acceptable solution [4]. Direct linear methods such as the Wiener filter can sometimes be effective. However, exploring the tradeoff between blur and noise and the use of effective regularization is often difficult. Iterative methods allow for a more systematic process, with adaptive computation and expanded regularization options such as truncated iterations and regularized preconditioning [4]. Moreover, nonlinear iterative algorithms offer the potential for more faithful results subject to the appropriateness of the constraints [7].

In this paper we present preliminary results in using linear and nonlinear iterative methods for the particular problem of cubic phase blur restoration. The linear method we consider is the conjugate gradient method for least squares (CGLS) which is often the method of choice for solving large linear systems. CGLS produces a regularized solution with early termination of the iterations [4]. The nonlinear iterative method we consider is a modified residual norm steepest descent algorithm (MRNSD) [7, 9]. This scheme enforces nonnegativity at each iteration, and has been demonstrated to converge much more quickly than other similar methods [9]. We also consider preconditioning for both the linear and the nonlinear approach to reduce the required number of iterations [9, 10]. Such preconditioning is shown to be highly effective in reducing the number of iteration steps, and thus makes the computational complexity of these iterative methods reasonable.

2 Iterative Restoration Methods

The CGLS and MRNSD methods mentioned in the introduction are reviewed first for completeness.

2.1 *Conjugate Gradient Least Squares*

The method of conjugate gradient least squares (CGLS) is an iterative method for solving symmetric positive definite linear systems [8] resulting from the least squares problem

$$\min_{f} \|g - \mathbf{H}f\|_2, \tag{2}$$

where $\mathbf{H}$ is a matrix which has full column rank, so that $\mathbf{H}^T\mathbf{H}$ is symmetric positive definite. The CGLS algorithm is applied to the factored form of the normal equations written as

$$\mathbf{H}^T(g - \mathbf{H}f) = 0. \tag{3}$$

The convergence of CGLS is dependent upon the eigenvalues of the matrix $\mathbf{H}^T\mathbf{H}$. If the eigenvalues of $\mathbf{H}^T\mathbf{H}$ cluster around one, convergence to a solution can be expected to be rapid. The CGLS solution to the 2D systems explored in this paper is given by the following algorithm.

ALGORITHM: **Conjugate Gradient Least Squares**

$r_0 = g;\ \hat{f} = g$
$d = \mathbf{H_{eig}}^T g$
$\| r_0 \|_2^2 = \| d \|_2^2$
`for` $k = 0, 1, 2, \ldots$
  $\mathbf{H_d} = \mathbf{H_{eig}} d$
  $\alpha = \frac{\|r_k\|_2^2}{\|\mathbf{H_d}\|_2^2}$
  $r_{k+1} = r_k - \alpha \mathbf{H_d}$
  $s = \mathbf{H_{eig}}^T r$
  $\| r_k \|_2^2 = \| s \|_2^2$
  $\beta = \frac{\|r_{k+1}\|_2^2}{\|r_k\|_2^2}$
  $\| r_k \|_2^2 = \| r_{k+1} \|_2^2$
  $d = s + \beta d$
  $k = k + 1$
`end`

Here some stopping criterion for the iterations must be determined, and $\mathbf{H_{eig}}$ is an $n^2 \times n^2$ array containing the eigenvalues of the blurring operator [9, 10]. These eigenvalues are obtained by taking the 2D FFT of a reordered $h(x, y)$ so that h(1,1) = maximum eigenvalue. Matrix-vector multiplies are performed component-wise in the frequency domain. Proper selection of a stopping criterion is usually accomplished by some error metric of the restored image and is relative to the noise present. In theory, if all aspects of the system are known explicitly (including the noise), then the exact iteration number for the most optimal solution can be computed [10]. Using the 2D FFT, the complexity of this algorithm is given by $\mathcal{O}(n^2 \log n)$ and the constant depends upon the number of iterations used [10]. In our experience, the number of iterations is usually relatively small, e.g. less than 1 percent of the number of unknowns.

2.1.1 *Generating* $\mathbf{H_{eig}}$

The matrix $\mathbf{H_{eig}}$ found in the CGLS algorithm is a dense $n^2 \times n^2$ matrix containing the eigenvalues of $h(x, y)$. $\mathbf{H_{eig}}$ is determined by computing the eigenvalues of a spatially transformed blurring matrix $h_k(x, y)$ as

$$h_k = \mathcal{T}\{h\}, \tag{4}$$

where $\mathcal{T}$ is the spatial transformation operator which shifts $h(x, y)$ so that its center (that is, maximum value) is in position $(1, 1)$ of $h_k(x, y)$. This is done to ensure that the maximum eigenvalue will also be located in the $(1, 1)$ position after Fourier transforming (using the 2D FFT) as

$$\mathbf{H_{eig}}(u, v) = \mathcal{F}\{h_k(x, y)\}. \tag{5}$$

$\mathbf{H_{eig}}$ is now a 2D multi-level Toeplitz [10] array. The ordering of $\mathbf{H_{eig}}$ allows for regularization via truncation of the smaller eigenvalues usually associated with the noise subspace.

2.2 *Modified Residual Norm Steepest Descent*

Enforcing a nonnegativity constraint on solutions to linear systems is computationally nontrivial. In this section we consider a recent improvement to CGLS which incorporates a nonnegativity constraint to the volumetric restoration task because the true volume is known to contain intensity values of zero and above. We have extended the so-called modified residual norm steepest descent (MRNSD) algorithm introduced by Nagy and Strakos [9]. The notation follows that given in Section 2.1 for CGLS, except that the unknown 2D image pixel values are now denoted by x for convenience. Here, g is the observed intensity function, f is the (unknown) actual intensity function, and $\mathbf{H_{eig}}$ denotes the $n^2 \times n^2$ array containing the eigenvalues of the PSF. As before, the matrix-vector computations are performed using the 2D FFT.

ALGORITHM: **Modified Residual Norm Steepest Descent (MRNSD)**

$x = g$
$w = \mathbf{H_{eig}}^T(\mathbf{H_{eig}}x - g)$
$\mathbf{X} = \mathrm{diag}(x)$
$\gamma = w^T\mathbf{X}w$
`for` $k = 0, 1, 2, \ldots$
  $p = -\mathbf{X}w$
  $u = \mathbf{H_{eig}}p$
  $\alpha = \min(\gamma/u^Tu, \min_{p_i<0}(-x_i/p_i))$
  $x = x + \alpha p$
  $\mathbf{X} = \mathrm{diag}(x)$
  $z = \mathbf{H_{eig}}^Tu$
  $w = w + \alpha z$
  $\gamma = w^T\mathbf{X}w$
`end`

This algorithm is similar to an algorithm called EMLS introduced by Kaufman [7], and it follows that nonnegativity of the voxel values is preserved. It is interesting to observe that each step of this algorithm can be viewed as applying a Jacobi-like preconditioned system, where the "preconditioner", $\mathbf{X}$, is step dependent [7]. The algorithm thus enforces nonnegativity constraints at each iteration, and has been demonstrated to converge much more quickly than other similar methods [7, 9] for 2D reconstructions applied to optical imaging problems such as those discussed in [10].

2.3 *Preconditioning*

It is well known that conjugate gradient and related iterative methods may often converge very slowly. Preconditioning is often used to accelerate convergence of iterative algorithms, especially conjugate gradient type methods. Suppose one is solving a symmetric positive definite linear system $Ax = b$. In general, preconditioning amounts to finding a nonsingular matrix C, such that $C \approx A$ and such that C can be easily "inverted" (that is, linear systems of the form $Cw = r$ can be easily solved). The iterative method is then applied to the preconditioned system, $C^{-1}Ax = C^{-1}b$. In the case of CGLS and MRNSD, we obtain preconditioned algorithms which we denote as PCGLS and PMRNSD, respectively.

For image restoration problems, many good fast Fourier transform (FFT) based preconditioners have been developed, especially for spatially invariant blurs. The type of preconditioner we use here is based on work of Hanke, Nagy, and Plemmons [6], and is effectively a regularizing preconditioner [4]. It is formed by modifying the Fourier spectrum of C in such a way so that the noise present in the data is not significantly magnified in the iterations. See [10] for further details and numerical experiments.

3 Numerical Tests

We consider an object at a single focal plane to concentrate on restorability rather than focus invariance. The object is artificially blurred with a PSF for a circular aperture that includes the cubic phase mask. In this first numerical example we use a simulated blur rather than lab data so that we have access to the true object and can quantify the fidelity. Also, we only consider an object at a single focal plane to concentrate on restorability rather than focus invariance. Figure 1 shows the true image, the PSF and the simulated blurred, noisy image. The noise is white Gaussian scaled to .5% of the maximum pixel value of the true image. This amount of noise was chosen to roughly comply with that observed in lab data.

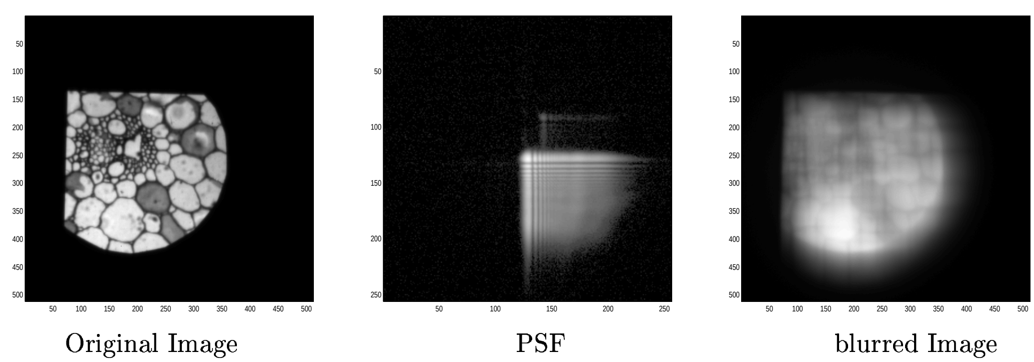


Fig. 1. Simulated blurring experiment with a 512×512 image. The PSF was acquired in the laboratory by imaging a point source through a cubic phase imaging system and was used in this numerical blurring experiment.

It should be noted that this restoration problem has 262,144 unknown pixel values. We have compared the convergence histories of CGLS and MRNSD with their preconditioned versions PCGLS and PMRNSD, respectively. Due to the ill-posed nature of the restoration problem, the relative errors at each iteration decrease initially, and after a certain point they begin to increase, as expected. Table 1 gives relative errors and iteration numbers of the optimal solutions for all four methods. The preconditioner is a “regularized” block circulant matrix with circulant blocks, with eigenvalue truncation tolerance 0.02 (see [10] for further details).

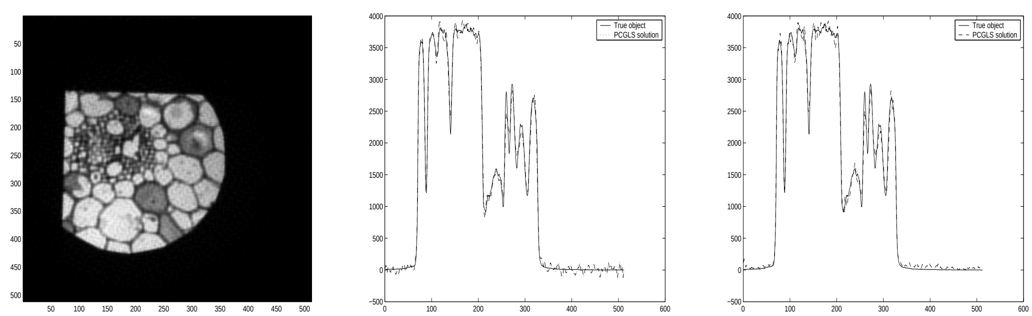
	rel. error	iteration k
CGLS	5.9147e-02	56
PCGLS	6.3643e-02	35

	rel. error	iteration k
MRNSD	-	> 500
PMRNSD	5.9975e-02	21

Table 1. Relative errors $||O_{\text{true}} - O_k||_2/||O_{\text{true}}||_2$ and iteration numbers for optimal solutions.

We note that PCGLS starts converging rapidly but has some difficulty before it eventually reaches its optimal solution at iteration $k = 35$. However, PMRNSD converges very quickly, reaching its optimal solution at iteration $k = 21$. For this example, the iterations converge smoothly and do not behave erratically as is the case for PCGLS. In each case, the optimal solutions had approximately the same relative error, and an examination shows no difference in the overall visual appearance.

Figure 2 shows the computed PMRNSD solution at iteration 21 as well as line plots of a slice through the images. Little difference in overall visual appearance of the optimal solutions for all cases was observed. However, close inspection of line plots reveals less ringing near the object boundaries for the restorations produced by the nonnegative constrained PMRNSD method compared to that produced by the unconstrained PCGLS method. This behavior likely depends on factors such as the object, PSF and noise level and will receive more attention in a later paper.



PMRNSD solution   PCGLS slice   PMRNSD slice

Fig. 2. Computed solution using PMRNSD (iteration 21) and line plots of PCGLS and PMRNSD solutions (at row 350).

4 Summary and Comments

From these preliminary results, we see that a nonnegatively constrained algorithm can produce very good restorations of images blurred by cubic phase PSFs. Moreover, the preconditioned, nonnegative constrained, nonlinear PMRNSD method exhibits a much more stable and fast convergence behavior than the preconditioned linear PCGLS method. In a later paper we will additionally present results using experimental lab data with objects in several different focal planes.

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