Optimization for Economics, a Visual Approach

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Note to the Reader

For the last several years I have taught a course in optimization at Emory University for seniors majoring in mathematics and economics. While mathematics and economics is the standard preparation for a doctoral program in economics, the vast majority of students in this course are bound for careers in finance. One of the challenges in designing this course has been balancing the interests and needs of students on both tracks. These notes have evolved from my own teaching materials. Their goal is to present methods for optimization and the reasoning behind them. This reasoning makes students resilient to the myriad modifications that these tools endure at the hands of economists.

As an outsider to economics education, I was impressed to see the extent to which economics has embraced visual methods of instruction. More than any other field whose foundation lies in mathematics, the standard economics curriculum provides students not only with descriptions and formulas, but with diagrams to depict core principles. Microeconomics students are inundated with supply curves, budget lines, indifference curves, and graphs in marginal space. This is certainly a lifeline for visual learners, but I suspect it produces deeper understanding for all.

As a geometer, my first inclination is to lean heavily upon visual reasoning when presenting the methods of optimization. In our efforts to find a textbook for our admittedly niche curriculum, we have not found a economics-focused text that takes this approach. I have made these notes available in the hope that this visual viewpoint can become an effective supplement to existing techniques. Like with the graphical arguments in undergraduate microeconomics, I think it can bring a complete and rigorous understanding of optimization methods to a broader set of learners.

The audience for this text is anyone who wants to understand the mathematical methods for finding maximizers in economic theory. The best-prepared reader will have mastered the techniques of differentiation, including partial derivatives. They will also be familiar with the foundations of mathematical logic: set notation, functions, and methods of proof like contradiction and the contrapostive. Finally, they will need a basic proficiency in vector and matrix operations: sums, products and determinants. A competent high-school treatment may suffice for this requirement.

Anyone who sets out to teach or learn the methods of optimization from this text should be aware of its limitations. It does not contain the economic applications that my economist colleagues present each semester. These must be provided in order to create genuine enthusiasm for the material. This is a deficiency that I would like to rectify eventually, if there is some consensus about which examples to include.

We have used one widely-available source of economic examples: *Optimization for Economic Theory* by Avinash Dixit. This excellent book has a rigorous, traditional treatment of most of the material here and good explanations of some advanced applications. Dixit's book has proved too difficult for all but the strongest undergraduates to read independently, but the examples are compelling and comprehensible with some extra exposition.

Finally I would like to acknowledge my economist colleagues Blake Allison and Teddy Kim who were essential partners in developing this course. Three students, Alexia Witthaus, Jacob Sugarmann and Aarya Aamer, carefully read my early drafts and asked many questions. Their feedback allowed me to identify the most opaque exposition (and clarify it, I hope). There is still a long road of revision and improvement ahead for this document. I would appreciate any feedback you have.

Mike Carr
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Chapter 0

Calculus Reference

0.1 Graphs of Functions

Goals:

1. Graph algebraic functions.
2. Graph transformations of functions.

0.1.1 Graphs of Basic Functions

Definition 0.1

The **graph** of a function $f(x)$ is the set of points (x, y) whose coordinates satisfy the equation $y = f(x)$.

In this section we review several basic functions and their graphs. These will be important as examples and counterexamples for our methods of optimization. In addition, knowing the shape of a graph is an efficient way to memorize the behavior of functions that appear frequently in economic models.

Definition 0.2

Linear functions can be written in **slope-intercept form**:

$$f(x) = mx + b.$$

- The graph of a linear function is a line.
- m is the slope, which is the change in y over the change in x between any two points on the line.
- $(0, b)$ is the y intercept.
- If we have the slope and a known point (a, b) on a line. We can write its equation in **point-slope** form.

$$y - b = m(x - a)$$

Definition 0.3

A **monomial** is a function of the form:

$$f(x) = x^n$$

where n is an integer greater than 0.

- For $n \geq 2$ the graph $y = x^n$ curves upward over the positive values of x.
- Greater values of n have lower values of $f(x)$ when $0 < x < 1$ but higher values when $x > 1$.
- For even values of n the graph is symmetric across the y-axis, curving up when x is negative.
- For odd values of n the graph curves down when x is negative. It is **anti-symmetric** across the $x = 0$.

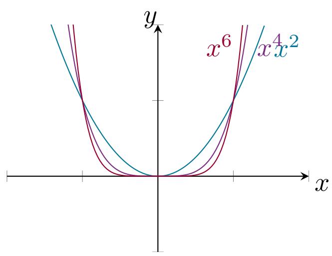


Figure 1: Graphs of even-powered monomials

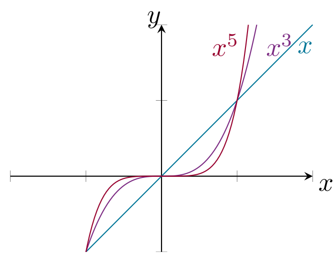


Figure 2: Graphs of odd-powered monomials

Monomials of Negative Power

Monomials of negative power have the form $f(x) = x^{-n}$. They are also commonly written

$$f(x) = \frac{1}{x^n}$$

.

- The graph $y = \frac{1}{x^n}$ has a vertical asymptote at $x = 0$.
- The graph approaches the x-axis, $y = 0$ as x gets large.
- For even values of n, the graph is above the x-axis.
- For odd values of n, the graph is above the x-axis for positive x and below it for negative x.
- A greater choice of n makes the function approach the x-axis more quickly.

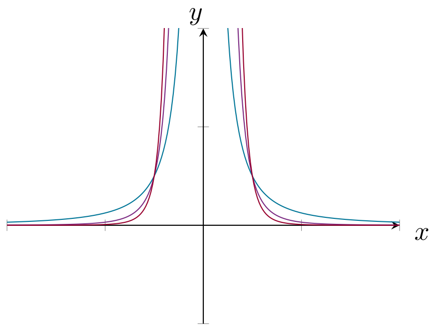


Figure 3: Graphs of $y = x^{-2}$, $y = x^{-4}$ and $y = x^{-6}$

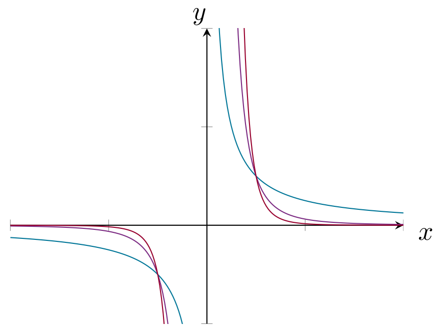


Figure 4: Graphs of $y = x^{-1}$, $y = x^{-3}$ and $y = x^{-5}$

Definition 0.4

A **root function** is a function of the form:

$$f(x) = \sqrt[n]{x}$$

where n is an integer greater than 0.

- The domain of $\sqrt[n]{x}$ is $[0, \infty)$ if n is even and all real numbers if n is odd.
- The x and y intercept of $y = \sqrt[n]{x}$ is at $(0, 0)$.
- Root functions are increasing. At $x = 0$, they travel straight up.

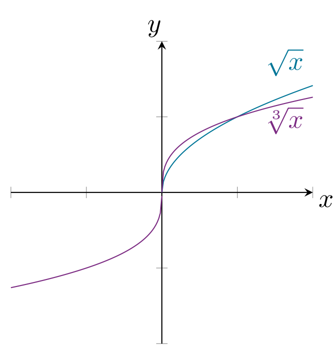


Figure 5: Graphs of root functions

Definition 0.5

An **exponential** function has the form:

$$f(x) = a^x$$

where a is a number greater than 0.

- a is called the **base** of the exponential function.
- The graph $y = a^x$ passes through $(0, 1)$.
- If $a > 1$ then $f(x)$ increases quickly as x takes on positive values. Greater values of a give a steeper increase. $f(x)$ approaches 0 as x goes to $-\infty$. Greater values of a give a faster approach. The graph does not touch or cross the x-axis.
- If $a < 1$, then the above is reversed.
- e is a commonly used base. e is approximately 2.718.

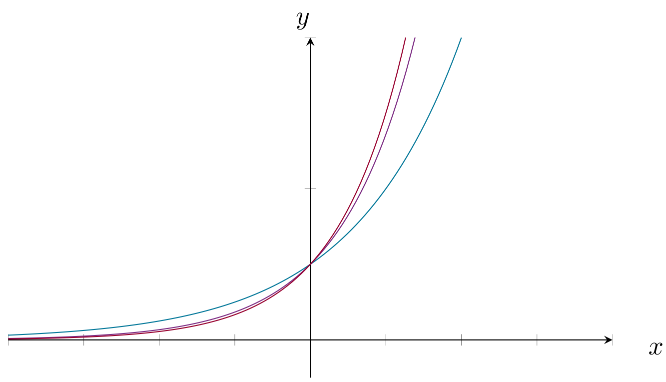


Figure 6: Graphs of $y = 2^x$, $y = e^x$ and $y = 3^x$

Definition 0.6

A **logarithmic function** has the form:

$$f(x) = \log_a x$$

where the **base** a is a number greater than 1. $\log_a x$ is the number b such that $a^b = x$. The **natural logarithm** is the logarithm with base e. It is denoted $f(x) = \ln x$.

- a^b can never be 0 or less. The domain of $f(x) = \log_a x$ is $(0, \infty)$.
- As x approaches 0, $\log_a x$ goes to $-\infty$.
- $y = \log_a x$ has an x intercept at $(1, 0)$.

- $y = \log_a x$ grows more and more slowly as x increases. This effect is more pronounced for larger values of a.

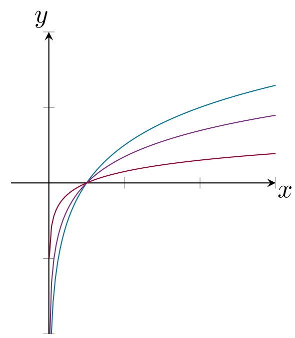


Figure 7: Graphs of $y = \log_2 x$, $y = \ln x$ and $y = \log_{10} x$

Logarithms and exponents are inverse functions. We solve exponential equations by applying a logarithm to both sides. We solve logarithm equations by exponentiating both sides.

$$a^x = c \longrightarrow x = \log_a c$$
$$\log_a x = c \longrightarrow x = a^c$$

0.1.2 Graphs of Transformations

Suppose we would like to transform the graph $y = f(x)$. Here are four ways we can.

- The graph of $y = af(x)$ is stretched by a factor of a in the y direction.
- The graph of $y = f(x) + b$ is shifted by b in the positive y direction.
- The graph of $y = f(cx)$ is compressed by a factor of c in the x direction.
- The graph of $y = f(x + d)$ is shifted by d in the negative x direction.

We can perform multiple transformations on a single function.

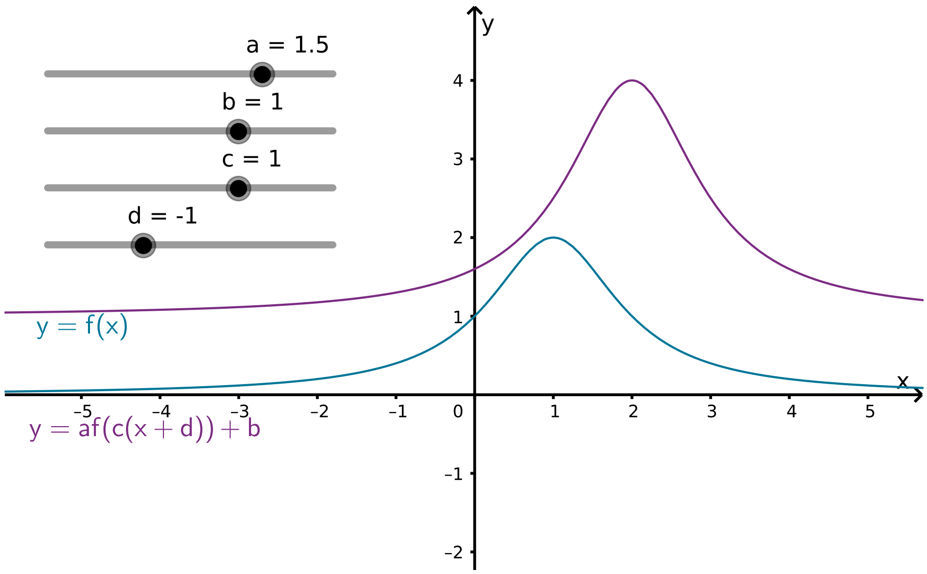


Figure 8: The graph of $y = f(x)$ and its transformation $y = af(cx + d) + b$

0.2 Limits and Derivatives

Goals:

1. Verify that a function is continuous.
2. Compute derivatives.
3. Use derivatives to understand graphs and vice versa.

0.2.1 The Limit of A Function

Calculus is the study of change. Our most important rate of change cannot be computed directly, but exists only as a limit of rates.

Definition 0.7

- The **limit** as x approaches a (or $x \to a$) of a function $f(x)$ is denoted $\lim\limits_{x \to a} f(x)$.
- $\lim\limits_{x \to a} f(x) = L$ means that we can make $f(x)$ arbitrarily close to L by restricting x to a small enough neighborhood surrounding a.
- If there is no L such that $\lim\limits_{x \to a} f(x) = L$, we say that $\lim\limits_{x \to a} f(x)$ **does not exist**.

Remark

- **arbitrarily close** means any amount of closeness demanded. We need to be able to make $f(x)$ within $\frac{1}{10}$ of L, within $\frac{1}{1000}$ of L, within $\frac{1}{10000000}$ of L and so on. When proving that a limit exists, mathematicians traditionally model this closeness with the variable ϵ. We indicate or verify arbitrary closeness with the inequality $|f(x) - L| < \epsilon$.
- By a **neighborhood** we mean an open interval that contains a. The set $\{a\}$ is not a neighborhood. If it were, then every function would limit to $f(a)$ as $x \to a$. Mathematicians generally restrict to neighborhoods of the form $(a - \delta, a + \delta)$, then they need a way to produce a valid, positive δ for any given positive ϵ.

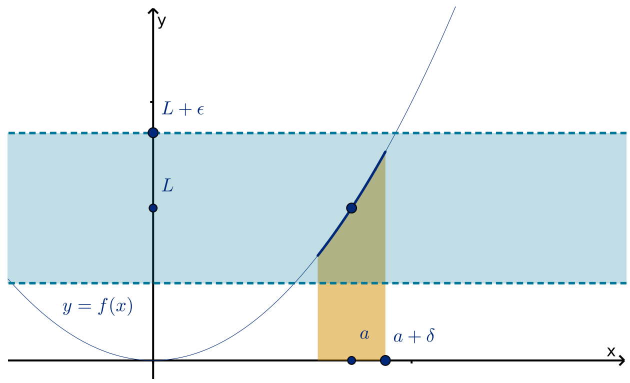


Figure 9: A neighborhood of a that keeps $f(x)$ within ϵ of L

0.2.2 Continuity

Limits give us a formal approach to defining continuity. Many of our results will rely on the fact that a function is continuous.

Definition 0.8

A function $f(x)$ is **continuous** at a, if

$$\lim_{x \to a} f(x) = f(a).$$

$f(x)$ can also be continuous on an interval or other set of points if it is continuous at each a in that set. If it is continuous on $\mathbb{R}$, we say $f(x)$ is a continuous function.

Proving that a function is continuous requires us to verify its limit at every point a. This is too much work for a case-by-case basis. Instead mathematics adopts a constructive approach. First we show that a few basic functions are continuous. Next we prove that sums, differences, products and other combinations preserve continuity.

Theorem 0.9

The following functions are continuous on their domains

1. Constant functions
2. Linear functions
3. Polynomials
4. Roots
5. Exponential functions
6. Logarithms
7. Trigonometric functions
8. $f(x) = |x|$

Theorem 0.10

If $f(x)$ and $g(x)$ are continuous on their domains, then the following are also continuous on their domains

1. $f(x) + g(x)$
2. $f(x) - g(x)$
3. $f(x)g(x)$
4. $\frac{f(x)}{g(x)}$ (note that any x where $g(x) = 0$ is not in the domain)
5. $f(x)^{g(x)}$ as long as $f(x) > 0$
6. $f(g(x))$

We can use these theorems together to argue that complicated functions are continuous.

Example

The function $f(x) = \sqrt[4]{3x^2 - 17x + 2} - \frac{e^x}{\log_5 x}$ is the difference of two functions. The first is a composition of a root function and polynomial (both continuous on their domains). The second is a quotient of an exponential and a logarithm (both continuous on their domains). Thus $f(x)$ is the difference of two continuous functions and is continuous on its domain.

Remark

Just about any function we can write using algebraic expressions is continuous on its domain. This does not mean it is continuous everywhere. $f(x) = \frac{1}{x}$ is not continuous at $x = 0$, for example.

0.2.3 The Intermediate Value Theorem

One early intuition for continuity is that the graph of a continuous function can be drawn without any breaks. There are many ways to formalize this idea. One of the most important is the following theorem.

Theorem 0.11 [The Intermediate Value Theorem]

If f is a continuous function on $[a, b]$ and K is a number between $f(a)$ and $f(b)$, then there is some number c between a and b such that $f(c) = K$.

Intuitively, a continuous graph cannot get from one side of the line $y = K$ to the other without intersecting $y = K$. Notice that this theorem does not say exactly where this intersection must occur, only that it must occur somewhere in the interval (a, b). It also does not rule out the possibility of more than one such c existing.

Example

Show that $f(x) = e^x - 3x$ has a root between 0 and 1.

0.2.3 The Intermediate Value Theorem

Solution

A root is a number c such that $f(c) = 0$. To prove such a root exists, we check the conditions of the intermediate value theorem.

- $f(x)$ is a sum of continuous functions, so it is continuous on its domain.
- $f(0) = 1$
- $f(1) = e - 3 < 0$
- 0 is between $f(0)$ and $f(1)$

We conclude there is some c between 0 and 1 such that $f(c) = 0$.

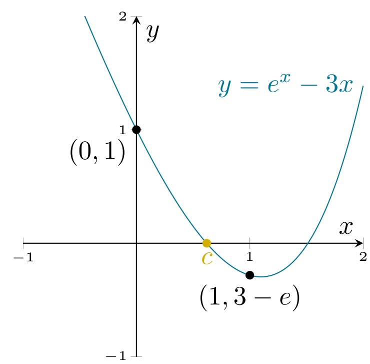


Figure 10: A root of $y = e^x - 3x$

0.2.4 The Derivative

The derivative is a method for measuring the rate of change of a function.

Definition 0.12

Given a function $f(x)$, the **derivative** of $f(x)$ at a is the number

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

The **derivative function** of $f(x)$ is the function

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Here are two different notations for the derivative at a.

1. $\frac{df}{dx}(a)$ (Leibniz)
2. $f'(a)$ (Lagrange)

The ratio $\frac{f(a+h)-f(a)}{h}$ can be interpreted two ways

1. The average rate of change of f between a and $a+h$
2. The slope of a secant line of $y=f(x)$ from $(a,f(a))$ to $(a+h,f(a+h))$

Since the derivative is the limit of these, we interpret $f'(a)$ as

1. The instantaneous rate of change of f at a
2. The slope of a tangent line to $y=f(x)$ at $(a,f(a))$

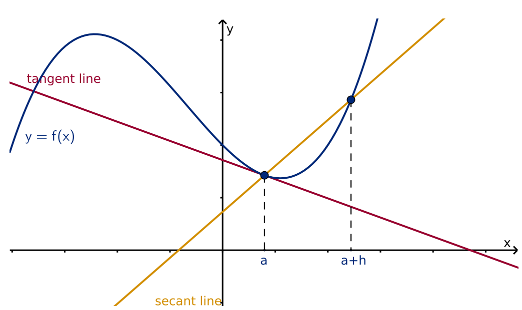


Figure 11: A secant line and a tangent line to $y=f(x)$

0.2.4 The Derivative

We can take **higher order derivatives** by taking derivatives of derivatives. The derivative function of f in this context is called the **first derivative**. Its derivative function is the **second derivative**. The second derivative's derivative function is the **third derivative** and so on.

Notation

The following notations are used for higher order derivatives

name	Lagrange notation	Leibniz notation
first derivative	$f'(x)$	$\frac{df}{dx}$
second derivative	$f''(x)$	$\frac{d^2f}{dx^2}$
third derivative	$f'''(x)$	$\frac{d^3f}{dx^3}$
fourth derivative	$f^{(4)}(x)$	$\frac{d^4f}{dx^4}$
fifth derivative	$f^{(5)}(x)$	$\frac{d^5f}{dx^5}$

The sign of a higher order derivative tells us how the derivative of one order lower is changing. For example if $\frac{d^5f}{dx^5} < 0$, then $\frac{d^4f}{dx^4}$ is decreasing.

0.2.5 Computing Derivatives

The limit definition of a derivative is too unwieldy to use every time. Instead calculus students learn the derivatives of some basic functions. They then use theorems to compute derivatives when those functions are combined.

Derivatives of Basic Functions

- $\frac{d}{dx}c = 0$ (derivative of a constant is 0)
- $\frac{d}{dx}x^n = nx^{n-1}$ for any $n \neq 0$ (The Power Rule)
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}a^x = a^x \ln a$ for $a > 0$
- $\frac{d}{dx}\ln x = \frac{1}{x}$
- $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$

The following rules allow us to differentiate functions made of simpler functions whose derivative we already know.

Differentiation Rules

- Sum Rule: $(f(x) + g(x))' = f'(x) + g'(x)$
- Constant Multiple Rule: $(cf(x))' = cf'(x)$
- Product Rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$
- Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$ unless $g(x) = 0$
- Chain Rule: $(f(g(x)))' = f'(g(x))g'(x)$

0.2.6 Computing a Derivative

Compute $\dfrac{d^2}{dx^2}(4\sqrt{75 - 3x^2} + 10)$

Solution

We begin by computing the first derivative. We use the chain rule with $4x^{1/2}+10$ as the outer function and $75-3x^2$ as the inner function.

$$\begin{aligned}\frac{d}{dx}(4\sqrt{75-3x^2}+10) &= 2(75-3x^2)^{-1/2}(-6x)\\ &= -\frac{12x}{\sqrt{75-3x^2}}\end{aligned}$$

We compute the second derivative by differentiating the first derivative. We use the quotient rule. We need the chain rule again when we differentiate the denominator.

$$\begin{aligned}\frac{d^2}{dx^2}(4\sqrt{75-3x^2}+10) &= -\frac{d}{dx}\frac{12x}{\sqrt{75-3x^2}}\\ &= -\frac{\frac{d}{dx}(12x)\sqrt{75-3x^2}-\frac{d}{dx}(\sqrt{75-3x^2})(12x)}{(\sqrt{75-3x^2})^2}\\ &= -\frac{12\sqrt{75-3x^2}+\frac{3x}{\sqrt{75-3x^2}}(12x)}{75-3x^2}\\ &= -\frac{12\sqrt{75-3x^2}+\frac{36x^2}{\sqrt{75-3x^2}}}{75-3x^2}\end{aligned}$$

We can obtain a common denominator to simplify this expression.

$$\begin{aligned}\frac{d^2}{dx^2}(4\sqrt{75-3x^2}+10) &= -\frac{\frac{12(\sqrt{75-3x^2})^2}{\sqrt{75-3x^2}}+\frac{36x^2}{\sqrt{75-3x^2}}}{75-3x^2}\\ &= -\frac{12(75-3x^2)+36x^2}{(75-3x^2)^{3/2}}\\ &= -\frac{900}{(75-3x^2)^{3/2}}\end{aligned}$$

0.3 Multivariable Functions

Goals:

1. Compute partial derviatives.
2. Recognize continuous multivariable functions.
3. Apply the chain rule to compositions of multivariable functions.

0.3.1 Multi-Variable Functions

Most interesting phenomena are not described by a single variable. We will need to develop methods for optimizing multivariable functions. There are many ways to denote multivariable domains and the functions on them. This is how we will denote them.

Notation

- An n-**vector** is an ordered set of n numbers called **components**. For instance $\vec{a} = (5, -\sqrt{17}, 12.31)$ is a 3-vector.
- We add a vector arrow, as in $\vec{a}$, to indicate that $\vec{a}$ is a vector.
- The components of a vector can be denoted abstractly by subscripts: $\vec{x} = (x_1, x_2, \ldots x_n)$. The x_i do not have arrows because they are numbers, not vectors.
- n-**space**, the set of all n-dimensional vectors, is denoted $\mathbb{R}^n$.
- $\vec{0}$ is the **zero vector**: $(0, 0, 0, \ldots, 0)$. The dimension should be clear from context.
- The vectors $\vec{e}_1$, $\vec{e}_2$, ..., $\vec{e}_n$ are the **standard basis vectors** of $\mathbb{R}^n$. $\vec{e}_i$ has 1 in the ith component and 0 in the others. For example $\vec{e}_2 = (0, 1, 0, \ldots, 0)$.
- An n-variable function $f(x_1, x_2, \ldots, x_n)$ can be written $f(\vec{x})$.

Remark

Using a common letter with an index variable

$$x_1, x_2, x_3 \dots$$

is a good choice for a large or an unknown number of variables. We will use this notation when developing the theory of multivariable optimization. In many economics problems, there is a fixed, small number of variables. In these problems, it is more convenient to use different letters for each variable, to avoid keeping track of subscripts.

$$x, y, z, \dots$$

Even better, we should try to choose descriptive variable names like q for quantity or p for price.

We visualize functions with their graphs. The height of the graph over a point in the domain represents the value of the function at that point. This allows us to detect visually where the function is large or small, increasing or decreasing.

Definition 0.13

Given an n-variable function $f(\vec{x})$, the **graph** of f is the set of points $(x_1, x_2, \dots, x_n, y)$ in $\mathbb{R}^{n+1}$ that satisfy $y = f(x_1, x_2, \dots, x_n)$.

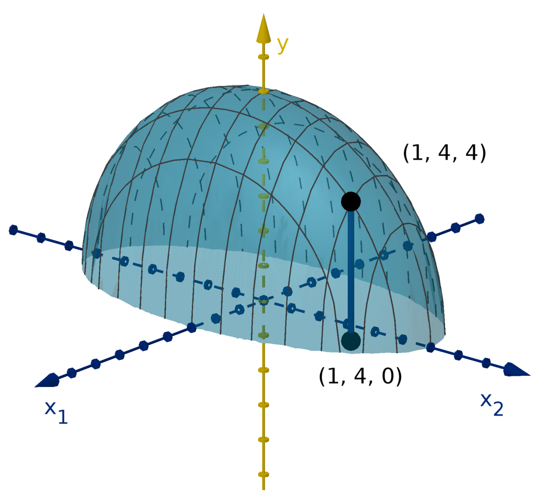


Figure 12: The graph $y = \sqrt{36 - 4x_1^2 - x_2^2}$ and the height showing $f(1,4)$.

Remark

In general, a graph of the form $y = f(x_1, x_2)$ will be hard to visualize. For more than three variables, this visualization becomes impossible. So why bother? Graphs are a useful visual aid to the reasoning behind our methods. As we progress, it is useful to have an prototypical two-variable graph in your head. You can apply our methods to that graph, whether or not you have an algebraic expression to go with it.

0.3.2 Partial Derivatives

Our optimization tools rely on the ability to measure rates of change. For a function of multiple variables, there are many rates of change, because there are many ways in which the input variables can change. The simplest are those where only one variable is changing while the others remain fixed.

Definition 0.14

The **partial derivative** of f with respect to x_i is a function of $\vec{x}$. It measures the rate of change of f at $\vec{x}$ as x_i increases but the other coordinates remain constant. The formula is

$$\lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_i) - f(\vec{x})}{h}$$

Here are two different notations for the partial derivative.

1. $\dfrac{\partial f}{\partial x_i}(\vec{x})$ (Leibniz)
2. $f_{x_1}(\vec{x})$ (Lagrange)

Each has advantages, so we will use both. When it will not cause confusion, we can shorten Lagrange's notation from $f_{x_i}(\vec{x})$ to $f_i(\vec{x})$.

In the two variable case, we can interpret $f_1(\vec{x})$ as the slope of the tangent line to the graph $y = f(x_1, x_2)$ in the x_1-direction. Higher-dimensional partial derivatives are also slopes, but are harder to visualize.

0.3.2 Partial Derivatives

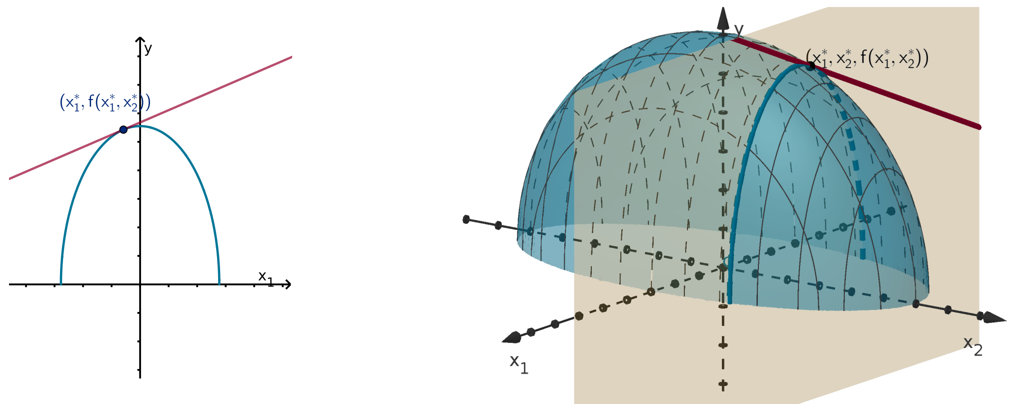


Figure 13: The tangent line to $y = f(x_1, x_2)$ in the x_1-direction

Computing a partial derivative requires us to treat the non-changing variables as constants. Then we can perform ordinary single-variable differentiation with the respect to the variable that is changing.

0.3.3 Computing a Partial Derivative

The profit of a firm with a Cobb-Douglas production function might be modeled by

$$\pi(L, K) = pL^\alpha K^\beta - wL - rK.$$

We can compute the partial derivative $\pi_L(L, K)$, which measures the marginal effect of hiring more labor.

Solution

Since $\pi_L(L, K)$ is a partial derivative, we can treat K as a constant. That means that neither K^β nor rK is changing. We treat K^β as a constant multiple of the monomial function L^α. We treat rK as a constant term with derivative 0.

$$\pi_L(L, K) = p\alpha L^{\alpha-1} K^\beta - w$$

Definition 0.15

A multivariable function $f(\vec{x})$ is **continuous** at $\vec{a}$, if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

In order to verify this limit, we must check that $f(\vec{x})$ can be made arbitrarily close to $f(\vec{a})$ by restricting to a sufficiently small neighborhood of $\vec{a}$. This neighborhood allows for travel in infinitely many directions from $\vec{a}$, rather than just forwards and backwards like a one-variable limit. This makes multivariable limits difficult to compute rigorously and multivariable continuity difficult to verify directly.

Fortunately, we can use the same approach we used for single-variable functions.

Variant of Theorem 0.9

The following multivariable functions are continuous on their domains

1. Constant functions
2. Linear functions
3. Polynomials
4. Roots
5. Exponential functions
6. Logarithms
7. Trigonometric functions

Variant of Theorem 0.10

If $f(\vec{x})$ and $g(\vec{x})$ are continuous on their domains, and c is a constant, then the following are also continuous on their domains

1. $f(\vec{x}) + g(\vec{x})$
2. $f(\vec{x}) - g(\vec{x})$
3. $f(\vec{x})g(\vec{x})$
4. $\frac{f(\vec{x})}{g(\vec{x})}$ (note that any $\vec{x}$ where $g(\vec{x}) = 0$ is not in the domain)
5. $f(\vec{x})^{g(\vec{x})}$ as long as $f(\vec{x}) > 0$
6. $f(g(\vec{x}))$ where $f(x)$ is a one-variable function

Multivariable continuity becomes important when discussing derivatives. Partial derivatives do not use multivariable limits. They use a limit as a single variable h goes to 0. For this reason, we are not guaranteed that partial derivatives reliably model the shape of a function.

Example

Consider the function

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \leq 0 \\ x_1 & \text{if } x_2 > 0 \end{cases}$$

This function is 0 when $x_1 = 0$ or $x_2 = 0$. Thus the partial derivatives at $(0, 0)$ are

$$f_1(0,0) = 0 \qquad\qquad f_2(0,0) = 0$$

If we increase x_1 while holding x_2 constant or increase x_2 while holding x_1 constant, then the function stays constant at 0. This does not reflect the fact that if we increase both x_1 and x_2 at $(0, 0)$, the function will have a positive slope.

Many theorems rely on a function behaving consistently with its partial derivatives no matter which direction we travel. The following property will usually serve that purpose.

Definition 0.16

A function $f(\vec{x})$ is **continuously differentiable**, if all the partial derivative functions $f_i(\vec{x})$ are continuous functions. If instead they are all continuous at a point $\vec{a}$, we say $f(\vec{x})$ is continuously differentiable at $\vec{a}$.

How do model the change of a multivariable function when more than one input variable is changing? We write each input variable as a function of a parameter. For instance, if x_1 and x_2 are both changing we can write each as a function of a parameter t. We can combine these to define a vector function:

$$\vec{x}(t) = (x_1(t), x_2(t)).$$

$f(\vec{x}(t))$ is a composition of functions. If we have defined $x_1(t)$ and $x_2(t)$ to correctly model the change we want in x_1 and x_2, then the derivative of $f(\vec{x}(t))$ will tell us how f is changing as well. Notice $f(\vec{x}(t))$ is a single variable function. The value of t determines its value completely. The multivariable chain rule computes its derivative using $\vec{x}(t)$ and the gradient of $f(\vec{x})$.

Definition 0.17

Given a function $f(\vec{x})$, the **gradient vector** of f at $\vec{x}$ is

$$\nabla f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_n(\vec{x}))$$

Theorem 0.18 [The Multivariable Chain Rule]

Suppose $f(\vec{x})$ is a continuously differentiable, n-variable function. If $\vec{x}(t)$ is differentiable, then the derivative of the composition $f(\vec{x}(t))$ with respect to t is

$$\frac{df}{dt}(t) = \nabla f(\vec{x}(t)) \cdot \vec{x}'(t)$$

$$\text{or } \frac{df}{dt}(t) = \sum_{i=1}^{n} f_i(\vec{x}(t))x_i'(t)$$

We should avoid using the notations $f'(t)$ and $f_t(t)$ for derivatives of compositions. Instead, we use Leibniz notation. This makes the variable of differentiation clear without implying that we are computing a partial derivative.

0.3.6 Applying the Chain Rule

Suppose $f(x_1, x_2) = \frac{\ln x_1}{x_2}$. If $x_1(t) = t^2$ and $x_2(t) = e^t$, compute $\frac{d}{dt} f(x_1(t), x_2(t))$.

Solution

According to the chain rule

$$\begin{aligned}
\frac{d}{dt} f(x_1(t), x_2(t)) &= f_{x_1}(x_1(t), x_2(t))x_1'(t) + f_{x_2}(x_1(t), x_2(t))x_2'(t) \\
&= \frac{1}{x_1(t)x_2(t)} x_1'(t) - \frac{\ln x_1}{x_2^2} x_2'(t) \\
&= \frac{1}{t^2 e^t} 2t - \frac{\ln t^2}{e^{2t}} e^t \\
&= \frac{2}{te^t} - \frac{\ln t^2}{e^t} \\
&= \frac{2 - t\ln t^2}{te^t}
\end{aligned}$$

Remark

The multivariable chain rule is not useful for direct calculations. Substituting the expressions for $x_1(t)$ and $x_2(t)$ would have given us

$$f(x_1(t), x_2(t)) = \frac{\ln t^2}{e^t}.$$

We can differentiate this using single-variable methods to obtain the same answer. The multivariable chain rule will instead serve us well in more abstract situations.

0.3.7 Alternate Notations and the Chain Rule

The multivariable chain rule is easiest to state when we give the component functions names that match the variables of f. When this is not the case, we need to take care with our notation.

Example

Let $f(x, y)$ be a continuously differentiable function. Let x and y be defined by differentiable functions $g(t)$ and $h(t)$ respectively. The chain rule states that

$$\frac{d}{dt} f(g(t), h(t)) = f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t).$$

We do not write f_g or f_h in this example. f is defined as a function of x and y, so those are the only partial derivatives it has.

Some applications use one of the variables of f as the parameter. The simplest example gives an alternate formulation for the partial derivative.

Example

Let $f(x_1, x_2)$ be a continuously differentiable function. Let x_1 be the identity function of itself and let x_2 to be a constant function of x_1.

$$x_1 = x_1 \qquad \frac{dx_1}{dx_1} = 1$$

$$x_2 = c \qquad \frac{dc}{dx_1} = 0$$

The rate of change with this parameterization should reflect that x_1 is changing and x_2 is not. That is exactly what the chain rule tells us.

$$\begin{aligned}\frac{d}{dx_1} f(x_1, c) &= f_{x_1}(x_1, c)\frac{dx_1}{dx_1} + f_{x_2}(x_1, c)\frac{dc}{dx_1} \\ &= f_{x_1}(x_1, c)(1) + f_{x_2}(x_1, c)(0) \\ &= f_{x_1}(x_1, c)\end{aligned}$$

We will build upon this formulation when we compute comparative statics.

Finally, we should note that the chain rule applies when the x_i are multivariable functions of a vector $\vec{t}$. In this case, $f(\vec{x}(\vec{t}))$ is a function of $\vec{t}$ and thus we can compute its partial derivatives.

0.3.7 Alternate Notations and the Chain Rule

Generalization of Theorem 0.18

Suppose $f(\vec{x})$ is a continuously differentiable, n-variable function. If $x_i(\vec{t})$ are differentiable functions of the variables t_j, then the partial derivative of the composition $f(\vec{x}(\vec{t}))$ with respect to t_k is

$$\frac{\partial f}{\partial t_k}(\vec{t}) = \nabla f(\vec{x}(\vec{t})) \cdot \vec{x}_k(\vec{t})$$

$$\text{or } \frac{\partial f}{\partial t_k}(\vec{t}) = \sum_{i=1}^{n} f_i(\vec{x}(\vec{t}))\frac{\partial x_i}{\partial t_k}(\vec{t})$$

This generalization follows immediately from treating each t_j except t_k as a constant.

0.3.8 Proving the Multivariable Chain Rule

The proof of the multivariable chain rule uses the same tools as the single variable chain rule (and product rule). However, multivariable limits are much more difficult to verify than single variable limits. To check that $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$, we have to consider values of $\vec{x}$ in every direction from $\vec{a}$, not just forward or backwards along a line. We will sketch the proof for the case where f is a two-variable function. Even the sketch is quite technical. It contains no arguments that are important enough to commit to memory.

Proof

We apply the definition of a derivative

$$\begin{aligned}\frac{df}{dt} &= \lim_{h\to 0}\frac{f(\vec{x}(t+h)) - f(\vec{x}(t))}{h} \\ &= \lim_{h\to 0}\frac{f(x_1(t+h), x_2(t+h)) - f(x_1(t), x_2(t))}{h}\end{aligned}$$

We break up this limit into a sum of two limits by adding and subtracting a term and regrouping the result (assuming the limit of each summand exists)

$$\begin{aligned}&= \lim_{h\to 0}\frac{f(x_1(t+h), x_2(t+h)) - f(x_1(t), x_2(t+h)) + f(x_1(t), x_2(t+h)) - f(x_1(t), x_2(t))}{h} \\ &= \lim_{h\to 0}\frac{f(x_1(t+h), x_2(t+h)) - f(x_1(t), x_2(t+h))}{h} \\ &\quad + \lim_{h\to 0}\frac{f(x_1(t), x_2(t+h)) - f(x_1(t), x_2(t))}{h}\end{aligned}$$

Next we multiply each limit by 1, represented as a quotient of an expression divided by itself.

$$= \lim_{h\to 0} \frac{f(x_1(t+h), x_2(t+h)) - f(x_1(t), x_2(t+h))}{h} \frac{x_1(t+h) - x_1(t)}{x_1(t+h) - x_1(t)}$$
$$+ \lim_{h\to 0} \frac{f(x_1(t), x_2(t+h)) - f(x_1(t), x_2(t))}{h} \frac{x_2(t+h) - x_2(t)}{x_2(t+h) - x_2(t)}$$

Naturally $\frac{x_i(t+h)-x_i(t)}{x_i(t+h)-x_i(t)}$ evaluates to $\frac{0}{0}$ when $h = 0$. This is not a problem for a limit. However, if it also evaluates to $\frac{0}{0}$ at other values of h, no matter how small a neighborhood of $h = 0$ we choose, then another approach is needed. In this case, the entire term will limit to 0, but we omit the formal argument from this sketch. Instead we assume all is well, and we reorganize each product by swapping denominators.

$$= \lim_{h\to 0} \frac{f(x_1(t+h), x_2(t+h)) - f(x_1(t), x_2(t+h))}{x_1(t+h) - x_1(t)} \frac{x_1(t+h) - x_1(t)}{h}$$
$$+ \lim_{h\to 0} \frac{f(x_1(t), x_2(t+h)) - f(x_1(t), x_2(t))}{x_2(t+h) - x_2(t)} \frac{x_2(t+h) - x_2(t)}{h}$$

We break up the result as a product of limits (assuming the limit of each factor exists)

$$= \lim_{h\to 0} \frac{f(x_1(t+h), x_2(t+h)) - f(x_1(t), x_2(t+h))}{x_1(t+h) - x_1(t)} \lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h}$$
$$+ \lim_{h\to 0} \frac{f(x_1(t), x_2(t+h)) - f(x_1(t), x_2(t))}{x_2(t+h) - x_2(t)} \lim_{h\to 0} \frac{x_2(t+h) - x_2(t)}{h}$$

The second factor of each product now looks like a derivative. To make first factors look more like derivatives, we let $j = x_1(t+h) - x_1(t)$ and $k = x_2(t+h) - x_2(t)$. These quantities go to 0 as $h \to 0$. Our limits can be rewritten as

$$= \lim_{(h,j)\to(0,0)} \frac{f(x_1(t) + j, x_2(t+h)) - f(x_1(t), x_2(t+h))}{j} \lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h}$$
$$+ \lim_{(h,k)\to(0,0)} \frac{f(x_1(t), x_2(t) + k) - f(x_1(t), x_2(t))}{k} \lim_{h\to 0} \frac{x_2(t+h) - x_2(t)}{h}$$

At this point we have four limits, each of which looks like the definition of a derivative. We can replace each one with its derivative notation. In general, we cannot evaluate a multivariable limit by handling one variable at a time, but the fact that the partial derivatives are continuous allows us to do so here. The details of this kind of argument are covered in an analysis course.

$$= \lim_{h\to 0} f_1(x_1(t), x_2(t+h))x_1'(t) + \lim_{h\to 0} f_2(x_1(t), x_2(t))x_2'(t)$$
$$= f_1(x_1(t), x_2(t))x_1'(t) + f_2(x_1(t), x_2(t))x_2'(t)$$

This is the dot product

$$= \nabla f(\vec{x}(t)) \cdot \vec{x}'(t)$$

One can adapt this proof to a higher dimension by breaking the limit into more summands. The theorem we gave is even more general, because applies to an n-variable function. To prove that version, we would use a proof by induction.

0.3.8 Proving the Multivariable Chain Rule

Chapter 1

Unconstrained Optimization

1.1 Single-Variable Optimization

Goals:

1. Know the definition of a local or global maximizer.
2. Apply the first- and second-order conditions to calculate maximizers.
3. Distinguish between necessary and sufficient conditions and recognize the role of each in optimization.
4. Understand the role of the derivative in proving the first- and second-order conditions.

Some of the most important methods in calculus are those that identify maximizers and minimizers of functions. This section gives precise theorems to describe those methods. We will also examine the distinct but complementary roles played by necessary conditions and sufficient conditions. Finally, we will give a reasonably compact formal basis to prove the theorems of this section and support the theorems in the sections that follow.

1.1.1 The First-Order Condition

Given a function, we are interested in what inputs of that function will produce the largest or smallest values of that function. These inputs are called maximizers and minimizers. In order to identify maximizer and minimizers, we need to have a rigorous definition that we can verify.

Definition 1.1

Suppose a lies in the domain of a function $f(x)$.

1. a is a **maximizer** of f if $f(a) \geq f(x)$ for all other x in the domain of f. In this case $f(a)$ is called the **maximum** of f.
2. a is a **local maximizer** of f if $f(a) \geq f(x)$ for all other x in some neighborhood of a. It is a **strict local maximizer** if $f(a) > f(x)$ instead. In either case $f(a)$ is called a **local maximum** of f.
3. a is a **minimizer** of f if $f(a) \leq f(x)$ for all other x in the domain of f. In this case $f(a)$ is called the **minimum** of f.
4. a is a **local minimizer** of f if $f(a) \leq f(x)$ for all other x in some neighborhood of a. It is a **strict local minimizer** if $f(a) < f(x)$ (than 0) to the left of a and smaller values (than 0) to the right. Thus, travelinstead. In either case $f(a)$ is called a **local minimum** of f.

Remark

When we are trying to draw a contrast with a local maximizer, sometimes we use **global maximizer** or **absolute maximizer** to refer to a maximizer of a function.

The difficulty in finding maximizers is that the domain has infinitely many points. Using only the definition, we would need to evaluate them all, one by one, to find a maximizer. This is obviously impossible. Thankfully, calculus gives us a way to narrow down the search.

Derivatives measure the rate of change of a function. Knowing whether the rate is positive or negative should tell us how $f(a)$ compares to nearby values. Here is a way to formally state that relationship.

Lemma 1.2

If $f'(a) > 0$, then for all x in some neighborhood of a,

1. $f(x) > f(a)$ if $x > a$
2. $f(x) < f(a)$ if $x < a$.

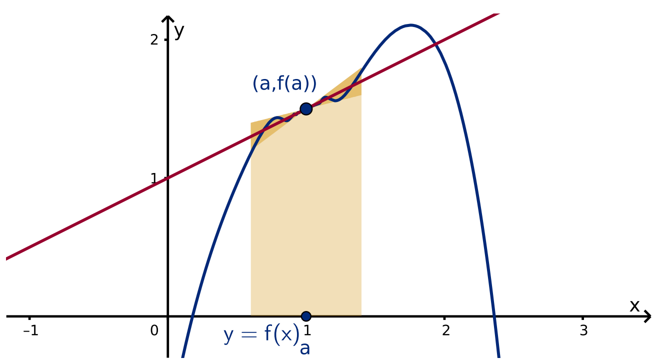


Figure 1.1: The graph $y = f(x)$ and a neighborhood where it stays close to its tangent line

Remark

A **lemma** is a statement that is not particularly interesting on its own. It is used as a step to proving a more important result.

We will provide a formal proof for this lemma later. The main argument is that near a, $y = f(x)$ lies close enough to the tangent line at a to mimic its behavior, attaining larger values for $x > a$ and smaller values for $x < a$. Notice that this behavior does not need to persist for all x. We cannot say how long $y = f(x)$ will stay close to its tangent line. It may be that the neighborhood where it does is quite small.

We can make the same argument for functions with a negative derivative, except the behavior is backwards. Aside from switching the direction of some inequalities, the proof is identical. Rather than treat this result as its own lemma, we present it as a variant.

Variant of Lemma 1.2

If $f'(a) < 0$, then for all x in some neighborhood of a,

1. $f(x) < f(a)$ if $x > a$
2. $f(x) > f(a)$ if $x < a$.

The existence of nearby x such that $f(x) > f(a)$ is inconsistent with the definition of a local maximizer. Lemma 1.2 and its variant guarantee such x, so:

- If $f'(a) > 0$, then a is not a local maximizer.
- If $f'(a) < 0$, then a is not a local maximizer.

We convert these statements to their contrapositives. If a is a local maximizer, then $f'(a)$ is neither positive nor negative. This gives us the following condition.

Theorem 1.3 [The First-Order Condition (FOC)]

Let a be a local maximizer or local minimizer of $f(x)$. Either $f'(a)$ does not exist or $f'(a) = 0$.

Definition 1.4

The values of x that satisfy the first-order condition are called **critical points**.

What does the first-order condition tell us about $f(x) = 8x^3 - x^4$?

Solution

The first-order condition tells us that a local maximizer or minimizer only occurs where the derivative is 0 or undefined. The derivative of this function is $24x^2 - 4x^3$. It is defined for all x, so we solve for where it is 0.

$$\begin{aligned} f'(x) &= 0 \\ 24x^2 - 4x^3 &= 0 \\ 4x^2(6-x) &= 0 \\ x = 0 \text{ or } x &= 6 \end{aligned}$$

This means that no point except $x = 0$ or $x = 6$ can be a local maximizer or minimizer.

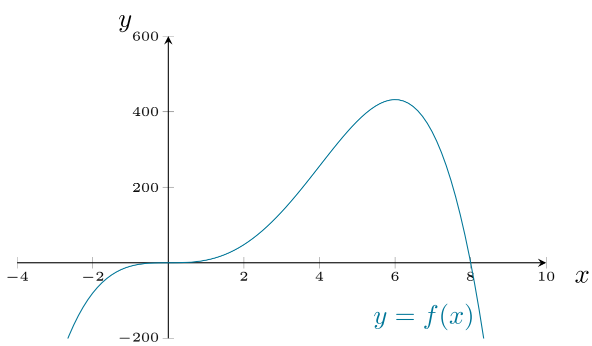


Figure 1.2: The graph of $f(x) = 8x^3 - x^4$

Remark

The first-order condition does not tell us that either of $x = 0$ or $x = 6$ must a local maximizer or a local minimizer. In fact, $x = 6$ is a local maximizer but $x = 0$ is neither.

As the previous example shows, the first-order condition is limited in its conclusion. Knowing the value of $f'(a)$ is not enough to give us certainty about the shape of the graph near a. For that we need to know how the first derivative is changing at a. The change in the first derivative is measured by the second derivative. The second derivative function, denoted $f''(x)$, is the derivative of the function $f'(x)$. The sign of the second derivative allows us to classify some critical points.

Theorem 1.5 [The Second-Order Condition (SOC)]

If $f'(a) = 0$ and $f''(a) < 0$, then a is a strict local maximizer of f.

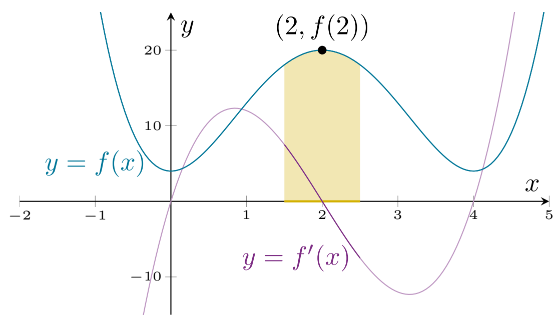


Figure 1.3: A neighborhood where $a = 2$ is the maximizer of $f(x)$

The intuition behind this result relies on the fact that $f''(a)$ is the derivative of $f'(x)$ at a. If $f''(a) < 0$, then $f'(a)$ takes on larger values (than 0) to the left of a and smaller values (than 0) to the right. Traveling left to right, the function increases until it reaches a. After passing a, it decreases.

Naturally, we have the following variant.

Variant of Theorem 1.5

If $f'(a) = 0$ and $f''(a) > 0$, then a is a strict local minimizer of f.

Remark

- This is sometimes called the **local** second-order condition, since it gives information about local maximizers.
- We cannot conclude anything, if $f''(a) = 0$. a may be a maximizer, a minimizer or neither.

1.1.4 Applying the Second-Order Condition

What does the second-order condition tell us about $f(x) = 8x^3 - x^4$?

Solution

The second-order condition requires $f'(x) = 0$. We've already shown that this only occurs at $x = 0$ and $x = 6$. The other part of the condition requires us to compute the second derivative.

$$f'(x) = 24x^2 - 4x^3$$

$$\begin{aligned} f''(x) &= 48x - 12x^2 \\ f''(0) &= 0 \\ f''(6) &= -144 < 0 \end{aligned}$$

This means that $x = 6$ is strict local maximizer. The second-order condition does not tell us anything about $x = 0$.

1.1.5 The Global Second-Order Condition

A firm seeking to increase its profits does not have much use for a local maximizer. The excuse: "Our strategy was superior to all numerically similar strategies" will not impress a board of directors. Nor would any rational actor settle for a mere local maximizer of their utility function. Utility maximizers and profit maximizers want to find the global maximizer. If we know more about the second derivative of the utility function, we can identify such a value.

Theorem 1.6 [The Global Second-Order Condition (GSOC)]

If $f'(x^*) = 0$ and $f''(x) < 0$ for all x, then x^* is the only critical point of f and is the unique global maximizer of f.

Remark

- Unlike the (local) second-order condition, this theorem requires that the second derivative is negative everywhere, not just at x^*.
- In return for a stronger requirement, we obtain a much stronger conclusion.
- Economists traditionally use x^* to denote a global maximizer. Thus we will use x^* to denote a known maximizer or any point that will imminently be identified as a maximizer.

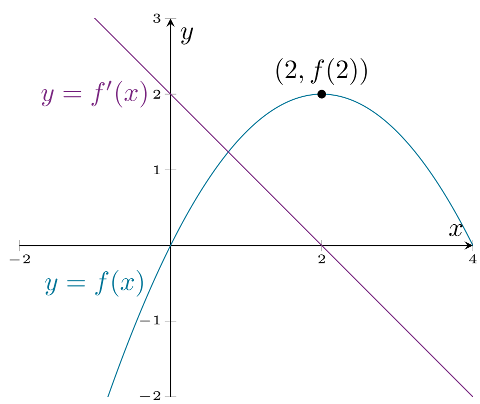


Figure 1.4: A function and its derivative near a maximizer

Unlike in the local case, we can make some use of a zero second derivative here.

Variants of Theorem 1.6

1. If $f'(x^*) = 0$ and $f''(x) > 0$ for all x, then x^* is the only critical point of f and is the global minimizer of f.
2. If $f'(x^*) = 0$ and $f''(x) \leq 0$ for all x, then x^* is a global maximizer of f (not necessarily unique).
3. If $f'(x^*) = 0$ and $f''(x) \geq 0$ for all x, then x^* is a global minimizer of f (not necessarily unique).

The conclusions we draw from the first-order condition are fundamentally different from the conclusions that we draw from the second-order conditions. Neither of them allows us to fully answer the question of which points are local maximizers of f. We can see the difference in their structure

FOC: If a is a local maximizer, then [condition]
SOC: If [condition], then a is a local maximizer

Mathematics has the following vocabulary for describing this difference.

Definition 1.7

Suppose we have a condition P that we are using to detect whether a statement Q is true or false.

1. A statement of the form "If Q then P" indicates that P is a **necessary condition** for Q.
2. A statement of the form "If P then Q" indicates that P is a **sufficient condition** for Q.

Remark

We can find uses for both necessary and sufficient conditions, but we need to be careful to interpret their conclusions correctly.

- If you want to show that Q is true, you need to use a sufficient condition. A necessary condition will not suffice.
- If you want to rule out Q being true, one way is to show that a necessary condition is not satisfied.

Going forward, it is important to identify each new result as being necessary or sufficient (or maybe both). We can begin by seeing how these terms apply to the first- and second-order conditions.

The first-order condition is a necessary condition. You can not have a local maximizer without satisfying it. It is not a sufficient condition. A point can satisfy the first-order condition without being a local maximizer.

Example

Let $f(x) = x^3$. The value $a = 0$ satisfies the FOC but is not a local maximizer (or minimizer).

This means that the first-order condition can only rule out values that are not maximizers. We cannot conclude that a point is a maximizer just by showing that it satisfies the first-order condition.

The second-order condition is a sufficient condition. If $f'(a) = 0$ and $f''(a) < 0$ then a must be a local maximizer. It is not a necessary condition: there may be local maximizers that do not satisfy the second-order condition.

Example

Let $f(x) = -x^4$. The local maximizer $a = 0$ does not satisfy the SOC, because $f''(0) = 0$.

The global second-order condition is also a sufficient condition and not a necessary one. For example, a function can have a global maximizer without satisfying it.

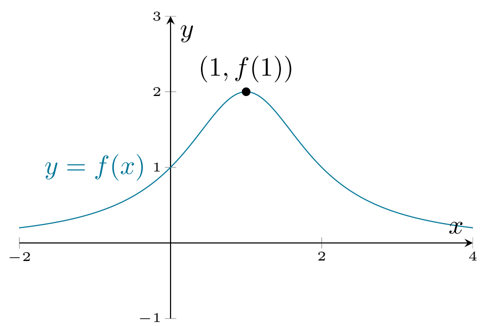


Figure 1.5: The graph of $f(x) = \frac{2}{x^2-2x+2}$, which has a global maximizer and unique critical point at $x^* = 1$, but does not satisfy the GSOC.

Remark

Abstractly, there is no difference between P and Q. There are two ways to view the statement:

If P, then Q.

- P is a sufficient condition for Q.
- Q is a necessary condition for P

However, we like to think of **conditions** as statements that we use to test for what we really care about. So while, abstractly, being a local maximizer is a sufficient condition for $f'(a) = 0$, this does not reflect the way we use the first-order condition in practice.

Generally we will want to have both necessary and conditions to test for properties that we care about. With the tools we have so far, we can determine that certain points are local or global maximizers. We can determine that many points are not local maximizers. Still, there may be points that satisfy the first-order condition but not the second-order condition. We cannot tell whether these are local maximizers or not.

The best condition would be a condition that is both necessary and sufficient. No such tool exists for general optimization, so we are on the lookout for additional conditions to apply when the ones we have are inconclusive. Coming up with new conditions is usually hard work, but we can obtain one easily, if we exploit the relationship between maximizers and minimizers.

A sufficient condition for a minimizer can be turned into a necessary condition for a maximizer. If $f'(a) = 0$ and $f''(a) > 0$ then by a variant of Theorem 1.5, a is a strict local minimizer. Thus it cannot be a maximizer. On the other hand, if $f''(a)$ exists but $f'(a) \neq 0$ then a is still not a local maximizer. This leaves only the following possibilities for the second derivative at a local maximizer a.

Theorem 1.8

If a is a local maximizer of f, then $f''(a) \leq 0$ or $f''(a)$ does not exist.

1.1.7 Proving the First-Order Condition

In general, we expect a positive derivative to mean that greater values of x produce greater values of $f(x)$. The lemma at the heart of the first-order condition described what we can conclude when a derivative is positive at one point.

The derivative is a limit, so any formal argument needs to start there. Proofs about limits can require extensive computations and creative problem solving. Fortunately, we will only need the following lemma about limits.

Lemma 1.9

If $\lim\limits_{x \to a} f(x) = L$ and $L > 0$, then there is a neighborhood of a on which f is positive.

Proof

Since L is positive, $L/2$ is also positive. By definition of a limit there is some neighborhood of a in which $f(x)$ is within $L/2$ of L. We can express that distance with an absolute value.

$$\begin{aligned} |f(x) - L| &< L/2 \\ -L/2 < f(x) - L &< L/2 \\ L/2 < f(x) &< 3L/2 \end{aligned}$$

Since $L/2$ is positive, so is $f(x)$.

Variant of Lemma 1.9

If $\lim_{x \to a} f(x) = L$ and $L < 0$, then there is a neighborhood of a on which f is negative.

We are now in a position to prove Lemma 1.2.

Lemma 1.2

If $f'(a) > 0$, then for all x in some neighborhood of a,

1. $f(x) > f(a)$ if $x > a$
2. $f(x) < f(a)$ if $x < a$.

Proof

Since $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} > 0$, Lemma 1.9 guarantees that there is a neighborhood of h values near 0 where $\frac{f(a+h)-f(a)}{h} > 0$. Let $x = a + h$. In the corresponding neighborhood of x values around a, we can make the following computations.

1. If $x > a$ then $h > 0$ and
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &> 0 \\ \frac{f(x) - f(a)}{h} &> 0 \\ f(x) - f(a) &> 0 \\ f(x) &> f(a) \end{aligned}$$
2. If $x < a$ then $h < 0$ and
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &> 0 \\ \frac{f(x) - f(a)}{h} &> 0 \\ f(x) - f(a) &< 0 \\ f(x) &< f(a) \end{aligned}$$

Remark

- This proof uses an earlier lemma. If you are carefully reading this proof to understand the argument, you may need to look back at the lemma and think about how it is being used here.
- Visually, the argument of this proof is that the secant lines from a have positive slope in some neighborhood of a.

1.1.8 Proving the Second-Order Conditions

In order to prove the second-order condition, we need a stronger conclusion than Lemma 1.2. Calculus teaches that a function with positive derivatives is increasing, while a function with negative derivatives is decreasing. In order to make a rigorous argument, we should know formal definitions of increasing and decreasing.

Definition 1.10

Let $f(x)$ be a function, and let I be an interval in its domain.

- $f(x)$ is **increasing** if for any numbers $a < b$ in the domain of f, $f(a) < f(b)$.
- $f(x)$ is **decreasing** if for any numbers $a < b$ in the domain of f, $f(a) > f(b)$.

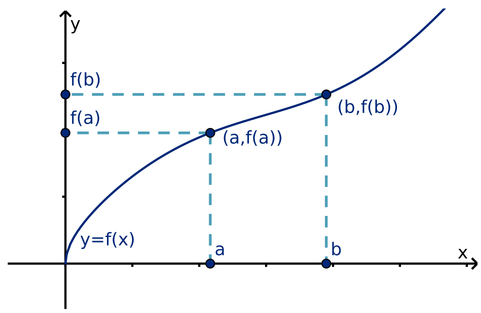


Figure 1.6: The graph of an increasing function

$f'(a) > 0$ is not enough to guarantee that $f(x)$ increases in a neighborhood of a. $f(x)$ may suffer from oscillations that persist arbitrarily close to a (see Figure 1.1). If we want to use a derivative to show that a function is increasing, we need to know the derivative is positive at every point on an interval.

Lemma 1.11

If $f'(x) > 0$ for all x on an interval I, then $f(x)$ is increasing on I.

To prove this lemma, we need to show that when $f'(x) > 0$, the function satisfies the formal definition of increasing that we saw above. This is surprisingly arduous to prove using the definition of a derivative as our starting point. Here are two deceptively short proofs

Proof

Suppose $a < b$. Apply the fundamental theorem of calculus: $f(b) - f(a) = \int_a^b f'(x)\,dx$. The integrand is positive over all of $[a, b]$, so the integral is positive too. Thus $f(b) > f(a)$.

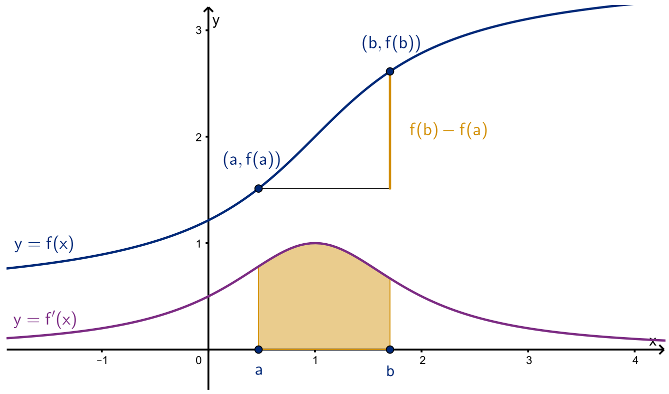


Figure 1.7: The fundamental theorem of calculus applied to a positive derivative

Proof

Suppose $a < b$. Apply the mean value theorem: There is some c between a and b such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. Since $c \in I$, we know that $f'(c) > 0$. Thus $f(b) > f(a)$.

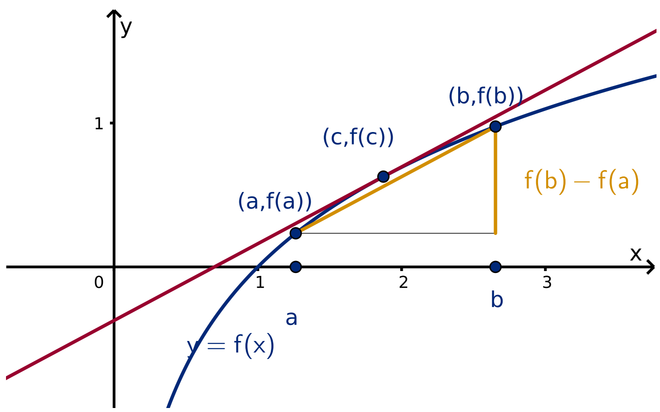


Figure 1.8: The point c, whose tangent line has the same slope as the secant from a to b

You should not be satisfied by these proofs. Each raises a new question.

1. How do you prove the fundamental theorem of calculus?
2. How do you prove the mean value theorem?

A satisfactory argument would either prove the missing result or find another method that eliminates the need for them. Unfortunately, any of these approaches takes us into concepts beyond the scope of these notes. You can expect to find their formal proofs in an analysis course.

Naturally, this lemma has variants. We can switch the direction of the inequality, but we can also relax the **strictness** of the inequality.

Variants of Lemma 1.11

1. If $f'(x) < 0$ on an interval I, then $f(x)$ is decreasing on I.
2. If $f'(x) \geq 0$ on an interval I, then $f(x)$ is **non-decreasing** on I, meaning that if $a < b$, $f(a) \leq f(b)$.
3. If $f'(x) \leq 0$ on an interval I, then $f(x)$ is **non-increasing** on I, meaning that if $a < b$, $f(a) \geq f(b)$.

Moving forward, we will see strict and non-strict variants so frequently that when we don't, it is worth pondering why not.

With these lemmas in hand, we are ready to prove the second-order condition.

Theorem 1.5

If $f'(a) = 0$ and $f''(a) < 0$, then a is a strict local maximizer of f.

Arguments about the second derivative usually rely on the fact that $f''(x)$ is the derivative of $f'(x)$. We will apply Lemma 1.9, letting $f'(x)$ play the role of the original function with $f''(x)$ as its derivative. We can thus use the sign of $f''(a)$ to compare the values of $f'(x)$ and $f'(a)$. Here is the formal argument.

Proof

We suppose $f'(a) = 0$ and $f''(a) < 0$. $f''(x)$ is the derivative of $f'(x)$, and it is negative at a. A variant of Lemma 1.9 applies. There is some neighborhood I of a where

- if $x > a$, then $f'(x) < f'(a) = 0$
- if $x < a$, then $f'(x) > f'(a) = 0$.

The second inequality shows, by Lemma 1.11, that $f(x)$ is increasing to the left of x in I. By definition of increasing this means that $f(x) < f(a)$ for all $x < a$ in I. Similarly, the first inequality shows that $f(x)$ is decreasing to the right of a in I. By definition of decreasing this means that $f(x) < f(a)$ for all $x > a$ in I. Thus $f(a) > f(x)$ for all other x in I. This satisfies the definition of a strict local maximizer, so we conclude that a is a strict local maximizer of f.

The proof of the global second-order condition follows the same logic. The only difference is that we can apply Lemma 1.11 instead of Lemma 1.9 to get a global statement about the first derivative on either side of x^*. Here is the full text for the sake of completeness.

Proof

We suppose $f'(x^*) = 0$ and $f''(x) < 0$ for all x. A variant of Lemma 1.11 tells us that $f'(x)$ is decreasing for all x. This means:

- if $x > x^*$, then $f'(x) < f'(x^*) = 0$
- if $x < x^*$, then $f'(x) > f'(x^*) = 0$.

The second inequality shows, by Lemma 1.11, that $f(x)$ is increasing for all $x < x^*$. By definition of increasing this means that $f(x) < f(x^*)$ for all $x < x^*$. Similarly, the first inequality shows that $f(x)$ is decreasing for all $x > x^*$. By definition of decreasing this means that $f(x) < f(x^*)$ for all $x > x^*$. Thus $f(x^*) > f(x)$ for all other x. This satisfies the definition of a unique maximizer, so we conclude that x^* is a unique maximizer of f.

To show that x^* is the only critical point, we simply note that if there were another critical point, it would also satisfy this condition and be the unique maximizer. Since there can be only one unique maximizer, no such critical point can exist.

1.1.9 Section Summary

The most important definitions and results from this section were

- The first-order condition (Theorem 1.3)
- The second-order condition (Theorem 1.5)
- The global second-order condition (Theorem 1.6)
- Necessary and sufficient conditions (Definition 1.7)

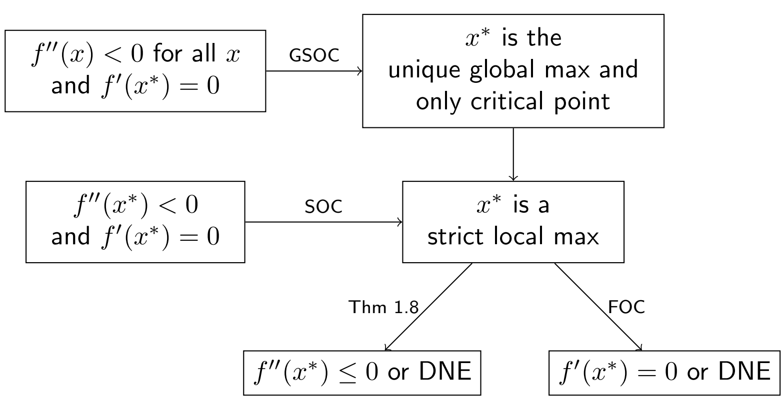


Figure 1.9: Relationships between the conditions of single variable optimization

1.2 Concavity

Goals:

1. Recognize convex sets.
2. Recognize concave and convex functions visually.
3. Understand the difference between strict and non-strict convexity/concavity.
4. Verify concavity using inequalities, tangent lines or second derivatives as appropriate.
5. Use concavity to find maximizers of a function.

1.2.1 Convex Sets

Our goal is to understand what the shape of a graph tells us about the maximizer(s) of a function. We would prefer methods that apply to both single-variable and multivariable functions. For this, we need to be able to describe shapes in any dimension. Most students learn how to recognize a convex polygon, but we can define convexity in a large variety of shapes. Convex regions can be angular or smooth. They can exist in any dimension. To see this, we need the formal definition of convexity.

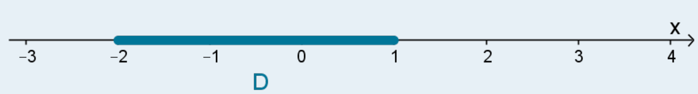
Definition 1.12

A region $D \subseteq \mathbb{R}^n$ is **convex**, if the line segment between any two points $\vec{a}$ and $\vec{b}$ in D entirely lies in D.

There is no restriction on the dimension of D. We can even apply it to regions in the real line.

Example

If D is a subset of $\mathbb{R}^1$, then D is convex if and only if it is connected.



For a polygon, this definition should match our previous intuition of convexity. Convince yourself that any two points in the convex polygons you learned about can be connected with a segment. In a nonconvex polygon there is at least one pair of points whose line segment leaves the polygon. You can convince yourself by picking your favorite nonconvex polygon and finding such a segment.

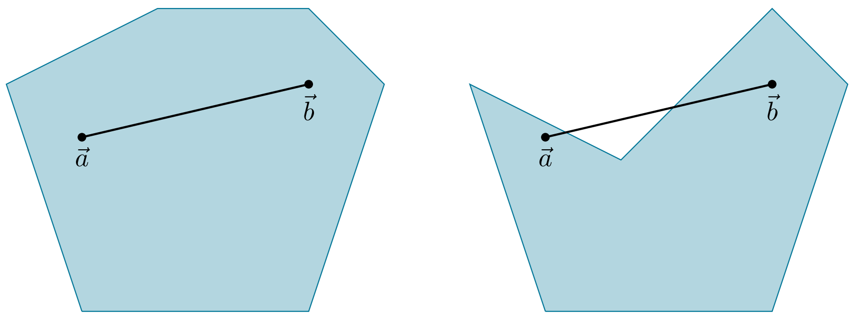


Figure 1.10: A convex polygon and a nonconvex polygon

We cannot make a rigorous case for convexity by drawing segments. There are too many to check. We can instead make convexity more algebraic by parameterizing the line segment from $\vec{a}$ to $\vec{b}$.

$$\vec{x}(t) = (1-t)\vec{a} + t\vec{b} \qquad 0 \leq t \leq 1$$

You can check that this is a line by writing it as

$$\vec{x}(t) = \vec{a} + t(\vec{b} - \vec{a}) \qquad 0 \leq t \leq 1$$

$\vec{x}(0) = \vec{a}$. As t increases, $\vec{x}(t)$ travels in the direction of $(\vec{b} - \vec{a})$ until arriving at $\vec{b}$ when $t = 1$. To check for convexity of D, we just need to test that the points of $\vec{x}(t)$ lie in D.

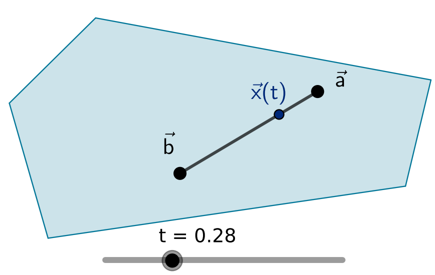


Figure 1.11: The line segment from $\vec{a}$ to $\vec{b}$

Furthermore, we do not need to check every $\vec{a}$ and $\vec{b}$. Any $\vec{a}$ and $\vec{b}$ whose segment leaves D will leave the at some boundary point $\vec{c}$ and reenter at some boundary point $\vec{d}$ (Figure 1.12). Thus if we only checked segments between boundary points, the segment between $\vec{c}$ and $\vec{d}$ would be sufficient to indicate that D is not convex. We summarize this argument in the following theorem.

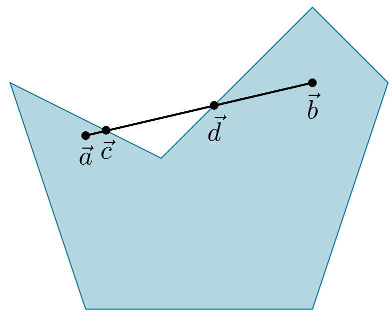


Figure 1.12: A segment that leaves a nonconvex region

Theorem 1.13

A region D is convex if for all $\vec{a}$ and $\vec{b}$ in the boundary of D and all t in $[0, 1]$,

$$(1-t)\vec{a} + t\vec{b} \in D$$

We visually identify convex polygons as polygons where every corner "points outward." However, the corners of the polygon are the only places that point outward. The rest of the boundary is flat, pointing neither inward nor outward. With non-polygonal regions, it is possible to have a boundary that points outward everywhere, not just at a few corners. This will be a useful property to keep track of, so we have a name for such regions.

Definition 1.14

A region D is **strictly convex**, if the segment, not including the endpoints, between any two points $\vec{a}$ and $\vec{b}$ in D entirely lies in the interior of D.

Figure 1.13: Two convex regions, one strictly convex and the other not strictly convex

Remark

Every strictly convex region is also convex, but some convex regions are not strictly convex. We say that strict convexity is a **stronger** condition and convexity is a **weaker** condition.

We are often interested in the intersection of two regions. For instance, if one region is the set of points satisfying one condition, while a second region is the set of points satisfying a second conditions, then their intersection is the points satisfying both conditions. Convexity behaves well in these situations.

Theorem 1.15

If D_1 and D_2 are convex regions, then $D_1 \cap D_2$ is convex.

The proof is a good exercise.

1.2.2 Concave Functions

Our next step is to use convexity to describe the shape of functions or, more precisely, their graphs. Most students encounter convex and concave functions in calculus, though they are sometimes called concave up and concave down. With a rigorous definition of a convex region, we are now in position to define what it means to be a convex function.

Definition 1.16

Let $f(\vec{x})$ be a function whose domain is convex. We say $f(\vec{x})$ is **convex**, if $y \geq f(\vec{x})$, the region above its graph, is convex. It is **concave** if $y \leq f(\vec{x})$, the region below its graph, is convex.

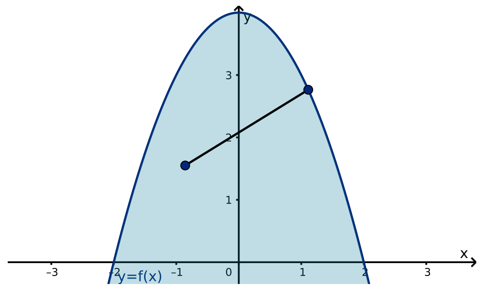


Figure 1.14: A concave function

Remark

Be careful when learning these definitions. Students often expect the definition of concave function to have something to do with a region being nonconvex (which is sometimes called concave), but it does not. Nonconvex regions above and below graphs are unremarkable. It is when one of regions is convex that the graph is special.

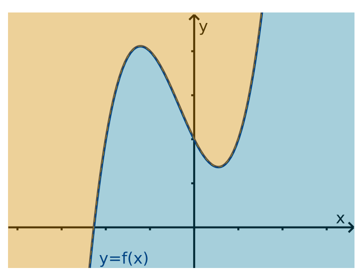


Figure 1.15: The graph of a function that is neither concave nor convex and the nonconvex regions above and below it.

We might ask whether the region above or below $y = f(\vec{x})$ is strictly convex, rather than merely convex. If it is, we can pass the "strictness" onto our description of f.

Definition 1.17

Let $f(\vec{x})$ be a function whose domain is convex. $f(\vec{x})$ is **strictly convex**, if the region $y \geq f(\vec{x})$ is strictly convex. It is **strictly concave** if the region $y \leq f(\vec{x})$ is strictly convex.

Remark

Strict concavity is a stronger condition than concavity. Every strictly concave function is also concave, but some concave functions are not strictly concave.

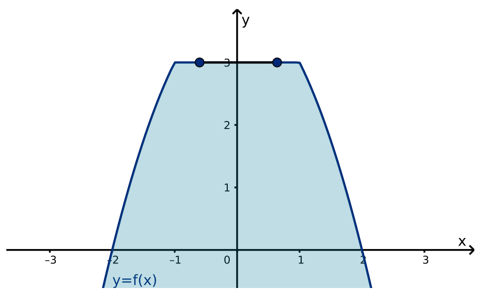


Figure 1.16: A function that is concave but not strictly concave

Notice that the region below $y = f(\vec{x})$ is a mirror image of the region above $y = -f(\vec{x})$. If one is convex, so is the other. We can make the following connection between a function and its negative.

Lemma 1.18

Let $f(\vec{x})$ be a function.

1. $f(\vec{x})$ is concave if and only if $-f(\vec{x})$ is convex.
2. $f(\vec{x})$ is strictly concave if and only if $-f(\vec{x})$ is strictly convex.

We will use this lemma as an excuse to ignore convex functions. Any argument about concave functions becomes an argument about convex functions by introducing a negative sign.

We verify the convexity of the region below $y = f(\vec{x})$ by checking line segments between points on the boundary. The boundary in this case is the graph itself. The line segments are called **secants**.

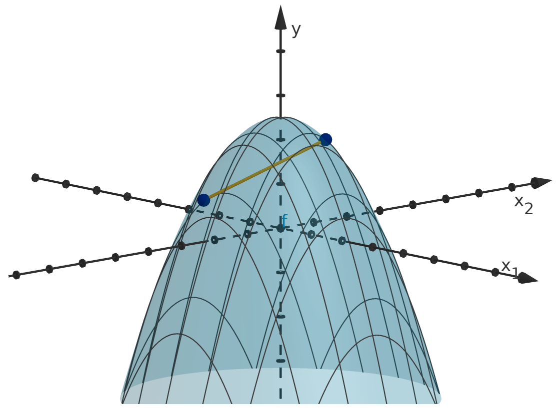


Figure 1.17: A secant below the graph of $y = 5 - x_1^2 - x_2^2$

We take two general points $(\vec{a}, f(\vec{a}))$ and $(\vec{b}, f(\vec{b}))$ on $y = f(\vec{x})$. We parametrize the secant between them.

$$\begin{aligned} \vec{x}(t) &= (1-t)\vec{a} + t\vec{b} \\ y(t) &= (1-t)f(\vec{a}) + tf(\vec{b}) \end{aligned} \qquad 0 \le t \le 1$$

We can write an inequality to express the condition that secant lies below the graph.

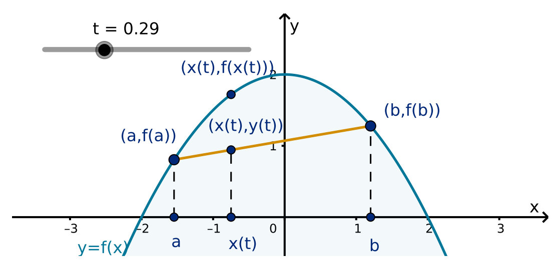


Figure 1.18: A secant below the graph of a single-variable function

Theorem 1.19

A function is concave, if and only if for all $\vec{a}$ and $\vec{b}$ in its domain and any $0 \le t \le 1$ we have

$$\underbrace{(1-t)f(\vec{a}) + tf(\vec{b})}_{\text{height of secant}} \le \underbrace{f((1-t)\vec{a} + t\vec{b})}_{\text{height of } y=f(\vec{x}(t))} \qquad 0 \le t \le 1$$

Remarks

- This is an "if and only if" condition. That means it is both necessary and sufficient for establishing concavity.
- Only a function with a convex domain can meet this condition. Otherwise we cannot always evaluate $f((1-t)\vec{a} + t\vec{b})$, because $(1-t)\vec{a} + t\vec{b}$ may lie outside the domain.

The theorem follows directly from the definitions we have developed so far. Here is a representation of our reasoning.

$(1-t)f(\vec{a}) + tf(\vec{b}) \le f((1-t)\vec{a} + t\vec{b})$
for all $\vec{a}$, $\vec{b}$ and t

The secant between any $\vec{a}$ and $\vec{b}$
lies under the graph $y = f(\vec{x})$

The region below $y = f(\vec{x})$ is convex

f is concave

Applying the same reasoning to the definitions of convexity and strict concavity/convexity gives the following variants. Note that we cannot demand a strict inequality at $t = 0$ or $t = 1$ because the secant will always intersect the graph $y = f(x)$ at a and b.

Variants of Theorem 1.19

1. A function is convex, if and only if for all $\vec{a}$ and $\vec{b}$ in its domain and any $0 \leq t \leq 1$ we have

$$(1-t)f(\vec{a}) + tf(\vec{b}) \geq f((1-t)\vec{a} + t\vec{b}) \qquad 0 \leq t \leq 1$$

2. A function is strictly concave, if and only if for all $\vec{a}$ and $\vec{b}$ in its domain and any $0 < t < 1$ we have

$$(1-t)f(\vec{a}) + tf(\vec{b}) < f((1-t)\vec{a} + t\vec{b}) \qquad 0 < t < 1$$

3. A function is strictly convex, if and only if for all $\vec{a}$ and $\vec{b}$ in its domain and any $0 < t < 1$ we have

$$(1-t)f(\vec{a}) + tf(\vec{b}) > f((1-t)\vec{a} + t\vec{b}) \qquad 0 < t < 1$$

1.2.4 Verifying Concavity with an Inequality

Verify that $f(x) = 5 - x^2$ is concave using the inequality condition.

Solution

Let a and b be any two real numbers. We need to show that for all t in $[0, 1]$:

$$(1-t)f(a) + tf(b) \leq f((1-t)a + tb)$$

To prove this inequality, we first need the AM-GM (arithmetic mean-geometric mean) inequality from algebra. Here it is.

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ a^2 + b^2 &\geq 2ab \end{aligned}$$

We also note that for t in $[0, 1]$, $t(1-t) \geq 0$. We now prove the required inequality, starting with the right side.

$$
\begin{aligned}
f((1-t)a+tb) &= 5-((1-t)a+tb)^2 && \text{definition of } f\\
&= 5-(1-t)^2a^2-2t(1-t)ab-t^2b^2 && \text{distribute}\\
&\geq 5-(1-t)^2a^2-t(1-t)(a^2+b^2)-t^2b^2 && \text{AM-GM}\\
&= 5-(1-t)^2a^2-t(1-t)a^2-t(1-t)b^2-t^2b^2 && \text{distribute}\\
&= 5-(1-t)(t+1-t)a^2-t(1-t+t)b^2 && \text{factor}\\
&= 5-(1-t)a^2-tb^2 && \text{simplify}\\
&= 5(1-t)-(1-t)a^2+5t-tb^2 && \text{break up } 5\\
&= (1-t)(5-a^2)+t(5-b^2) && \text{factor}\\
&= (1-t)f(a)+tf(b) && \text{definition of } f
\end{aligned}
$$

The inequality condition is difficult to verify, even for this simple function. It is much worse for a function involving a square root or an exponential. On the other hand. The inequality condition can be convenient for abstract results.

Theorem 1.20

If $f(\vec{x})$ and $g(\vec{x})$ are both concave, then so is their sum: $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$

Proof

Let $\vec{a}$ and $\vec{b}$ be in the domain of h. Since the domain of h is the intersections of the domains of f and g, h has a convex domain (Theorem 1.15). We know from the concavity of f and g that for all t in $[0,1]$:

$$(1-t)f(\vec{a})+tf(\vec{b}) \leq f((1-t)\vec{a}+t\vec{b})$$

$$(1-t)g(\vec{a})+tg(\vec{b}) \leq g((1-t)\vec{a}+t\vec{b})$$

We now verify that the inequality holds for h:

$$
\begin{aligned}
(1-t)h(\vec{a})+th(\vec{b}) &= (1-t)f(\vec{a})+(1-t)g(\vec{a})+tf(\vec{b})+tg(\vec{b})\\
&\leq f((1-t)\vec{a}+t\vec{b})+g((1-t)\vec{a}+t\vec{b})\\
&= h((1-t)\vec{a}+t\vec{b})
\end{aligned}
$$

We conclude that h is concave.

1.2.4 Verifying Concavity with an Inequality

Main Idea

The inequality condition is sometimes useful at a theoretical level, but is painful to check for any but the simplest specific functions. We will want a condition that is easier to check.

1.2.5 Concavity and Tangent Lines

We sought out the idea of concavity because it applies to functions of many variables. Some results about concave functions are related to their derivatives. We will develop these tools in the setting that is most comfortable and intuitive: one-variable functions. Once we have presented the theory of optimization for multivariable functions, we will revisit and generalize these results.

Derivatives tell us about tangent lines, and tangent lines have an interesting relationship to convex sets. If you draw a nonconvex set, you will be able to find a tangent line that crosses through the set. A convex set, on the other hand, lies entirely on one side of any of its tangent lines. We will not try to prove this for general sets, but we will prove it for the strictly convex region below the graph of a strictly concave function.

Lemma 1.21

If $f(x)$ is a strictly concave function that is differentiable at a, then the rest of the graph $y = f(x)$ lies below its tangent line at $(a, f(a))$.

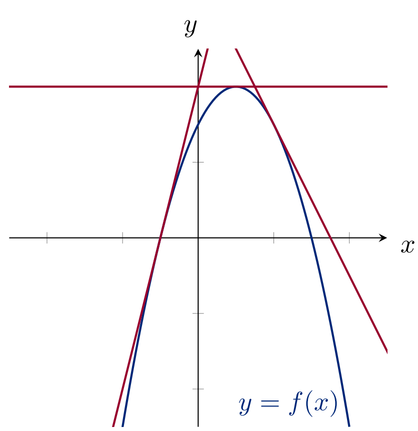


Figure 1.19: The graph of a strictly concave function and some of its tangent lines

This lemma describes the relationship between $y = f(x)$ and a single tangent line. It can apply to a function that is not differentiable at other points in its domain. If the function is differentiable on its entire domain, we can produce a necessary and sufficient condition on the tangent lines.

Theorem 1.22

Let $f(x)$ be a differentiable function on a convex domain. $f(x)$ is strictly concave, if and only if the graph $y = f(x)$ lies below all of its tangent lines (except at the point on tangency).

The proofs of these results are somewhat technical. We will provide them after we have discussed applications.

1.2.6 Concavity Conditions for a Maximizer

Our main use of concavity (for now) is to produce sufficient conditions for maximizers. The following corollary follows from Lemma 1.21.

Corollary 1.23

If $f(x)$ is a strictly concave function and $f'(x^*) = 0$, then x^* is the unique global maximizer of f.

Proof

The assumption that $f'(x^*) = 0$ tells us that f is differentiable at x^*, and the tangent line at x^* is horizontal. By Lemma 1.21, the rest of the graph lies below this line. We conclude that x^* is the unique global maximizer.

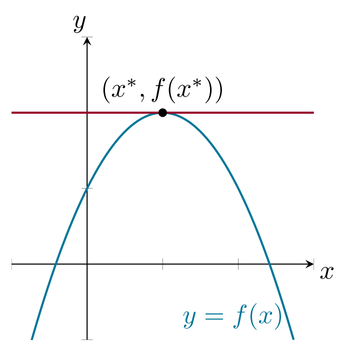


Figure 1.20: The graph of $y = f(x)$, which lies below the tangent line at x^*.

Like in the global second-order condition, we can also argue that x^* is the only critical point. If there were another, then it would also be the unique global maximizer, which is nonsense.

1.2.7 A Second Derivative Test for Concavity

We would still like to have a better way to determine when a function is concave. In calculus, we learn that the sign of the second derivative determines the concavity of a function. We are now in a position to formally argue this result.

Theorem 1.24

If f is a function on a convex domain $D \subseteq R$ and $f''(x) < 0$ for all x in D, then $f(x)$ is strictly concave.

This means that a negative second derivative is a sufficient condition for strict concavity. Is it necessary? No, consider the following.

Example

Consider $f(x) = -x^4$. This function is strictly concave, but $f''(0) = 0$, so the second derivative is not always negative.

Putting this with Corollary 1.23 gives a string of conditions. Each is sufficient and neither is necessary, so the implications can only be followed from left to right.

$$\boxed{\begin{array}{c} f''(x) < 0 \\ \text{for all } x \end{array}} \longrightarrow \boxed{f \text{ strictly concave}} \longrightarrow \boxed{\begin{array}{c} \text{Any } x^* \text{ with } f'(x^*) = 0 \\ \text{is a maximizer} \end{array}}$$

We could claim that Corollary 1.23 and Theorem 1.24 have combined to give an alternate argument for the global second-order condition. However, proving Theorem 1.24 without the global second-order condition would require us to redo much of the work that went into that theorem. Instead we will use the global second-order condition to keep our proof of Theorem 1.24 brief.

Proof

Our goal is to apply Theorem 1.22. Let a be a point in the domain of f. Since $f''(x)$ exists, $f'(a)$ exists. Let $y = \ell(x)$ be the tangent line to $y = f(x)$ at a. Let $g(x) = f(x) - \ell(x)$. We know the following about g:

- $g''(x) = f''(x) - \ell''(x) < 0$ for all x, since $\ell''(x) = 0$.
- $g'(a) = f'(a) - \ell'(a) = 0$, since the derivative is the slope of the tangent line.
- $g(a) = 0$, since $\ell(a) = f(a)$.

g and a satisfy the conditions of the global second-order condition. Thus a is the unique maximizer of g, meaning $g(x) < g(a) = 0$ for all $x \neq a$. Thus $y = f(x)$ lies below $y = \ell(x)$. This reasoning holds for the tangent line at any value a. Theorem 1.22 tells us that f is strictly concave.

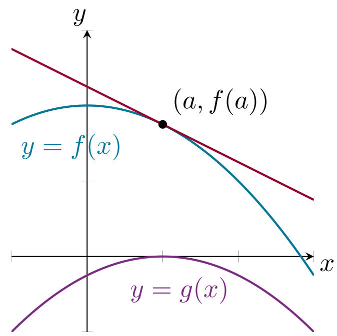


Figure 1.21: The graph of $f(x)$, the tangent line at a, and their difference

1.2.8 Verifying Concavity with the Second Derivative

Verify that $f(x) = 5 - e^x$ is strictly concave.

Solution

$f''(x) = -e^x < 0$ for all x. Therefore by Theorem 1.24, $f(x)$ is strictly concave.

Main Idea

The second derivative is an easier test of strict concavity than the inequality we used earlier, but it is only a sufficient condition. It does not detect every strictly concave function.

1.2.9 Proving the Relationship Between Concavity and Tangent Lines

We will now present the formal reasoning of Lemma 1.21 and Theorem 1.22. We prove Lemma 1.21 by examining the difference between $f(x)$ and its secants.

Proof

Pick any $b \neq a$ on the graph $y = f(x)$. The tangent line to $y = f(x)$ at $(a, f(a))$ has equation $y = f(a) + f'(a)(x-a)$. We will first show that $(b, f(b))$ does not lie above the tangent line by showing that $f(b) \leq f(a) + f'(a)(b-a)$.

Consider the case that $b > a$. Denote the secant from $(a, f(a))$ to $(b, f(b))$ by the equation $y = s(x)$ and let $g(x) = s(x) - f(x)$. The region below $y = f(x)$ is convex, so $y = s(x)$ is below $y = f(x)$ for any x between a and b. Their difference, $g(x)$, is less than or equal to 0 on this interval. By the contrapositive of Lemma 1.2 we know $g'(a)$ cannot be greater than 0.

$s'(a)$ is the slope of the secant: $\dfrac{f(b)-f(a)}{b-a}$. We substitute this into $g'(a) \leq 0$ and solve.

$$\begin{aligned}
g'(a) &\leq 0 \\
s'(a) - f'(a) &\leq 0 && \text{(sum rule)} \\
\frac{f(b)-f(a)}{b-a} - f'(a) &\leq 0 && \text{(substitute)} \\
f(b) - f(a) - f'(a)(b-a) &\leq 0 && (b-a>0) \\
f(b) &\leq f(a) + f'(a)(b-a)
\end{aligned}$$

We conclude that $(b, f(b))$ does not lie above the tangent line.

Now we must also show it does not lie on the tangent line. This is easiest with a contradiction argument. Suppose some $(b, f(b))$ lies on the tangent line at a. Then consider any c between a and b. $(c, f(c))$ also does not lie above tangent line by the argument we gave for b. However, the secant from $(a, f(a))$ to $(b, f(b))$ is part of the tangent line, and $(c, f(c))$ lies above this by the definition of strict concavity. This is a contradiction. Thus there is no $(b, f(b))$ that lies on the tangent line.

The case where $b < a$ can be proved with a similar argument. We conclude that $(b, f(b))$ lies below the tangent line to $y = f(x)$ at $(a, f(a))$.

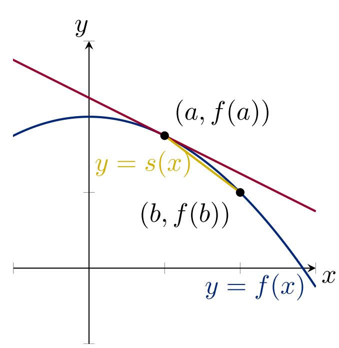


Figure 1.22: A tangent line and a secant to a concave function

Theorem 1.22 is an "if and only if" statement, so it requires two arguments to prove. We must show that if f is strictly concave, then its graph lies below its tangent lines. We must also show that if its graph lies below its tangent lines, then it is strictly concave. Fortunately, Lemma 1.21 has already shown that if $f(x)$ is strictly concave, then $y = f(x)$ lies below the tangent line at a.

Proof

Suppose we know f is concave. For each a in the domain of f, Lemma 1.21 states that $y = f(x)$ lies below its tangent line at a. Since this holds for all a, $y = f(x)$ lies below all of its tangent lines.

Now suppose that we know $y = f(x)$ lies below all its tangent lines. We will verify that the region under $y = f(x)$ is strictly convex. Let a and b be any points in the domain of f. Let c be any point between them. We know that $(a, f(a))$ and $(b, f(b))$ both lie below the tangent line to $y = f(x)$ at $(c, f(c))$. Thus the tangent line lies above the entire secant between $(a, f(a))$ and $(b, f(b))$. $(c, f(c))$ is on this tangent line, so $(c, f(c))$ is above the secant as well.

Since this argument holds for any such c, we conclude that the entire graph between a and b lies above the secant. Since this is true for any choice of a and b, we conclude that the region below $y = f(x)$ is strictly convex and $f(x)$ is a strictly concave function.

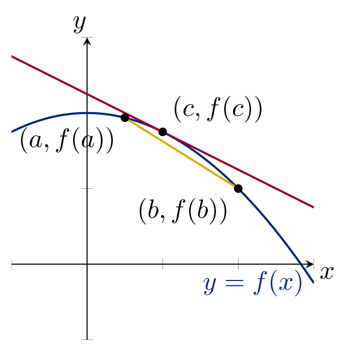


Figure 1.23: A secant with both endpoints below the tangent line at $(c, f(c))$

1.2.10 Non-Strict Concavity and Convexity

We have primarily worked with strictly concave functions, because they give the strongest results about maximizers. The properties of non-strict concavity and convexity give similar results. The results for convexity are obtained by flipping the directions of inequalities. They are probably not worth memorizing. On the other hand, economists frequently need to work with non-strictly concave functions. It is worth understanding how utility maximization works for these.

Graphs of non-strictly concave functions can intersect their own tangent lines. For instance, a linear function is concave, and its graph is identical to its tangent line. While the graph of a concave function can intersect the tangent lines, it cannot go above them.

Variants of Lemma 1.21

Let $f(x)$ be a function on a convex domain.

1. If $f(x)$ is a concave function that is differentiable at a, then the graph $y = f(x)$ has no points above its tangent line at $(a, f(a))$.
2. If $f(x)$ is a strictly convex function that is differentiable at a, then the rest of the graph $y = f(x)$ lies above its tangent line at $(a, f(a))$.
3. If $f(x)$ is a convex function that is differentiable at a, then the graph $y = f(x)$ has no points below its tangent line at $(a, f(a))$.

The proof for non-strict concavity is identical to the proof for strict concavity, with the contradiction argument omitted. The necessary and sufficient conditions are unsurprising.

Variants of Theorem 1.22

Let $f(x)$ be a differentiable function on a convex domain.

1. $f(x)$ is concave if and only if the graph $y = f(x)$ has no points above any of its tangent lines.
2. $f(x)$ is strictly convex if and only if the graph $y = f(x)$ lies above each of its tangent lines (except at the point of tangency).
3. $f(x)$ is convex if and only if the graph $y = f(x)$ has no points below any of its tangent lines.

These results can also be extended to well-behaved non-differentiable functions. For instance, whatever "tangent lines" you might reasonably add to the function in Figure 1.16 will not stray below the graph. Rigorously defining tangent lines in these situations would require a detailed argument involving limits, so we will not pursue it here.

The variants of Lemma 1.21 give rise to their own corollaries about maximizers and minimizers.

Variants of Corollary 1.23

1. If $f(x)$ is a concave function and $f'(x^*) = 0$, then $\vec{x}^*$ is a global maximizer (but other points may be as well).
2. If $f(x)$ is a strictly convex function and $f'(x^*) = 0$, then $\vec{x}^*$ is the unique global minimizer of f.
3. If $f(x)$ is a convex function and $f'(x^*) = 0$, then $\vec{x}^*$ is a global minimizer (but other points may be as well).

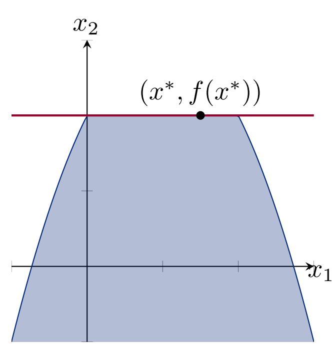


Figure 1.24: A global but non-unique maximizer of a not strictly concave function

Finally, we can use the Variants of Theorem 1.22 to produce second-derivative tests for the other forms of concavity and convexity.

Variants of Theorem 1.24

1. If f is a function on a convex domain $D \subset R$ and $f''(x) \leq 0$ for all x in D, then $f(x)$ is concave.
2. If f is a function on a convex domain $D \subset R$ and $f''(x) > 0$ for all x in D, then $f(x)$ is strictly convex.
3. If f is a function on a convex domain $D \subset R$ and $f''(x) \geq 0$ for all x in D, then $f(x)$ is convex.

In the non-strict case, the condition is necessary as well as sufficient.

1.2.10 Non-Strict Concavity and Convexity

Corollary 1.25

A twice differentiable function $f(x)$ on a convex domain D is concave, if and only if $f''(x) \leq 0$ for all x in D.

The sufficiency of $f''(x) \leq 0$ is found in the preceding variant. We establish necessity with a contrapositive argument. It should look familiar to anyone who read our proof of Theorem 1.24. Notice that it is making a local rather than a global argument.

Proof

Suppose $f''(x) \leq 0$ for all x. A variant of Theorem 1.24 tells us that f is concave.

On the other hand, suppose $f''(a) > 0$ for some a. First note than since $f''(x)$ exists, f is a differentiable function. Let $y = \ell(x)$ be the tangent line to $y = f(x)$ at a. Consider $g(x) = f(x) - \ell(x)$. We know the following about g:

- $g''(a) = f''(a) - \ell''(a) > 0$, since $\ell''(x) = 0$.
- $g'(a) = f'(a) - \ell'(a) = 0$, since the derivative is the slope of the tangent line.
- $g(a) = 0$, since $f(a) = \ell(a)$.

Thus by a variant of the (local) second-order condition, Theorem 1.5, a is a strict local minimizer of g, meaning $g(x) > g(a) = 0$ for all x in a neighborhood of a. Thus $y = f(x)$ lies above $y = \ell(x)$ in this neighborhood. By a variant of Theorem 1.22, f is not concave.

The contrapositive of this argument is that if f is concave, then there is no a such that $f''(a) > 0$. Equivalently, $f''(x) \leq 0$ for all x.

This result means that we not only have a way to show that $f(x)$ is concave, but also that it is not concave. If $f''(a) > 0$ for any a, then f is not a concave function.

1.2.11 Section Summary

The most important definitions and results from this section were

- The definition of a (strictly) convex region (Definitions 1.12 and 1.14)
- The definition of a (strictly) concave function (Definitions 1.16 and 1.17)
- The inequality condition for concave functions (Theorem 1.19)
- The tangent line condition for strictly concave functions (Theorem 1.22)

- The sufficient condition for a maximizer using strict concavity (Corollary 1.23)
- The second derivative tests for strictly concave and concave functions (Theorem 1.24 and Corollary 1.25)

Here is a summary of which conditions and statements from this section imply which others. Some of these conditions only apply to differentiable, or even twice differentiable functions.

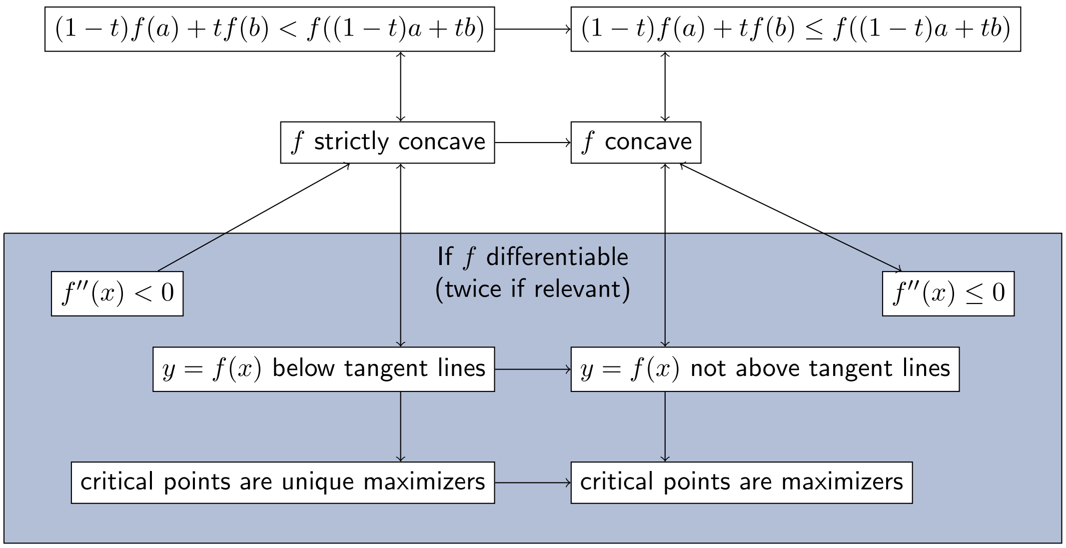


Figure 1.25: Conditions and applications of concave functions

1.3 Multivariable Optimization

Goals:

1. Apply the first- and second-order conditions to multivariable functions
2. Calculate the Hessian of a function
3. Understand the definition and significance of positive or negative definite matrices
4. Test whether a matrix is positive or negative definite
5. Show that a multivariable function is strictly concave

1.3.1 Parametrizations and Compositions

In this section we develop conditions to find maximizers of multivariable functions. Our optimization methods so far rely on using rates of change to compare a potential maximizer to the points around it. This is a more daunting task in the multivariable situation, because the comparison points can lie in any direction. The function can be changing differently depending on which direction we travel. We avoid the need to develop now tools from scratch by considering paths through a potential maximizer $\vec{a}$.

A path through $\vec{a}$ is a function $\vec{x}(t)$ such that for some t_0, $\vec{x}(t_0) = \vec{a}$. We can think of the variable t as representing time. Then $\vec{x}(t)$ represents the position on our path at time t.

A path breaks down into coordinate functions

$$\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

If these are differentiable, the derivative

$$\vec{x}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t))$$

represents the instantaneous rate of change in position of $\vec{x}$ with respect to t. It is tangent to the path. In physics it is called the velocity vector.

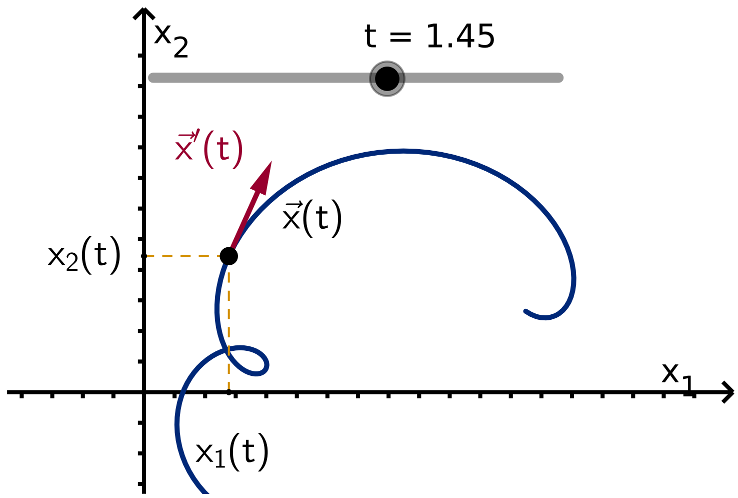


Figure 1.26: A path and its tangent vector

We will study a function $f(\vec{x})$ by studying compositions of the form $f(\vec{x}(t))$. At each value of t, we compute the corresponding position on the path and then evaluate f at that position. Geometrically, this is the height of the graph $y = f(\vec{x})$ above the curve $\vec{x}(t)$.

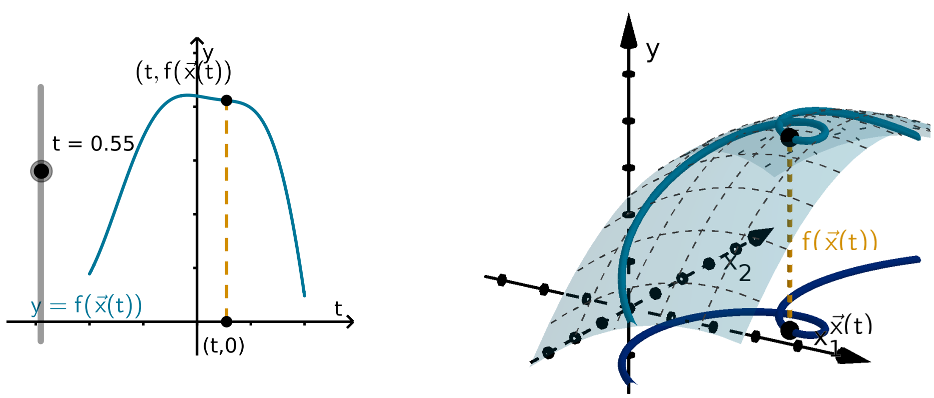


Figure 1.27: The graph of the composition $f(\vec{x}(t))$ and the corresponding points in the original graph $y = f(\vec{x})$

The derivative of $f(\vec{x}(t))$ with respect to t is computed using the multivariable chain rule.

Theorem 0.18

Suppose $f(\vec{x})$ is an continuously differentiable n-variable function. If $\vec{x}(t)$ is a differentiable path, then the derivative of the composition $f(\vec{x}(t))$ with respect to t is

$$\frac{df}{dt}(t) = \nabla f(\vec{x}(t)) \cdot \vec{x}'(t)$$
$$\text{or } \frac{df}{dt}(t) = \sum_{i=1}^{n} f_i(\vec{x}(t))x_i'(t)$$

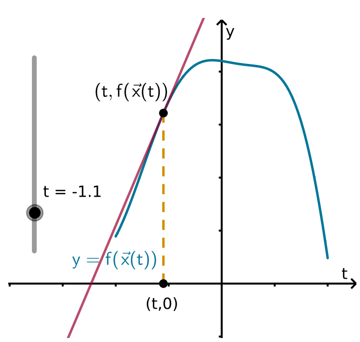
As with single-variable optimization, the sign of a derivative is important to our arguments. We can obtain the sign of $\frac{df}{dt}$ visually, using the following fact about dot products.

The Sign of the Dot Product

Given nonzero vectors $\vec{u}$ and $\vec{v}$, the sign of $\vec{u} \cdot \vec{v}$ is

positive	if $\vec{u}$ and $\vec{v}$ make an acute angle
zero	if $\vec{u}$ and $\vec{v}$ are orthogonal
negative	if $\vec{u}$ and $\vec{v}$ make an obtuse angle

The angle between ∇f and $\vec{x}'(t)$ determines the sign of $\frac{df}{dt}$. This indicates whether $f(\vec{x})$ increases or decreases as $\vec{x}$ travels along the path $\vec{x}(t)$.



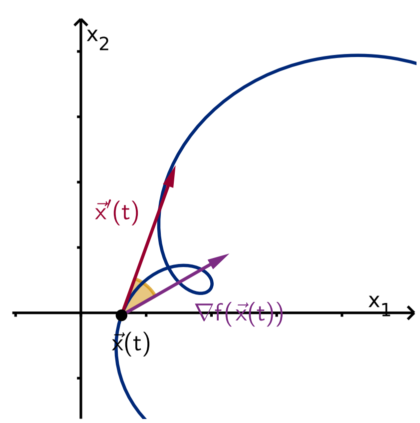


Figure 1.28: The tangent line to the graph $y = f(\vec{x}(t))$ and the angle between $\nabla f(\vec{x}(t))$ and $\vec{x}'(t)$

The value of $\frac{df}{dt}$ is realized geometrically as the slope of the tangent line to the graph of the composition function: $y = f(\vec{x}(t))$.

Using compositions to find maximizers has two main consequences. The first is helpful. The second creates difficulties.

1. $f(\vec{x}(t))$ is a one-variable function. No matter the dimension of $\vec{x}$, we can graph $y = f(\vec{x}(t))$ in the ty-plane and look for maximizers there.
2. One path will not cover the whole domain of $f(\vec{x})$. We have to study many different paths $\vec{x}(t)$, if we want to compare a potential maximizer to every other point in the domain.

We could study all paths in the domain of f, but that would be needlessly difficult. Instead we restrict ourselves to paths that are lines. We will express these lines using the following notation.

Notation

Every line has a **direction vector** $\vec{v} = (v_1, v_2, \ldots, v_n)$. A line through $\vec{a}$ in the direction of $\vec{v}$ has the equation

$$\vec{x}(t) = \vec{a} + t\vec{v}$$

In a convex domain, any two values $f(\vec{a})$ and $f(\vec{b})$ can be attained as values of the composition $f(\vec{a} + t\vec{v})$, where $\vec{v} = \vec{b} - \vec{a}$. If we want to compare these values (for instance to argue that $\vec{a}$ is or is not a maximizer), we compare $f(\vec{a} + 0\vec{v})$ and $f(\vec{a} + 1\vec{v})$.

Lemma 1.26

Suppose $f(\vec{x})$ has a convex domain. A point $\vec{a}$ is a (local or global) maximizer, if and only if 0 is a (local or global) maximizer of $f(\vec{a} + t\vec{v})$ for all direction vectors $\vec{v}$.

This lemma tells us that a necessary condition for 0 to be a maximizer of $f(\vec{a} + t\vec{v})$ for all $\vec{v}$ is a necessary condition for $\vec{a}$ to be a maximizer of $f(\vec{x})$. Similarly, a sufficient condition for 0 to be a maximizer of $f(\vec{a} + t\vec{v})$ for all $\vec{v}$ is a sufficient condition for $\vec{a}$ to be a maximizer of $f(\vec{x})$.

The compositions $f(\vec{a} + t\vec{v})$ are realized as cross sections of the graph $y = f(\vec{x})$ above the line $\vec{x}(t) = \vec{a} + t\vec{v}$.

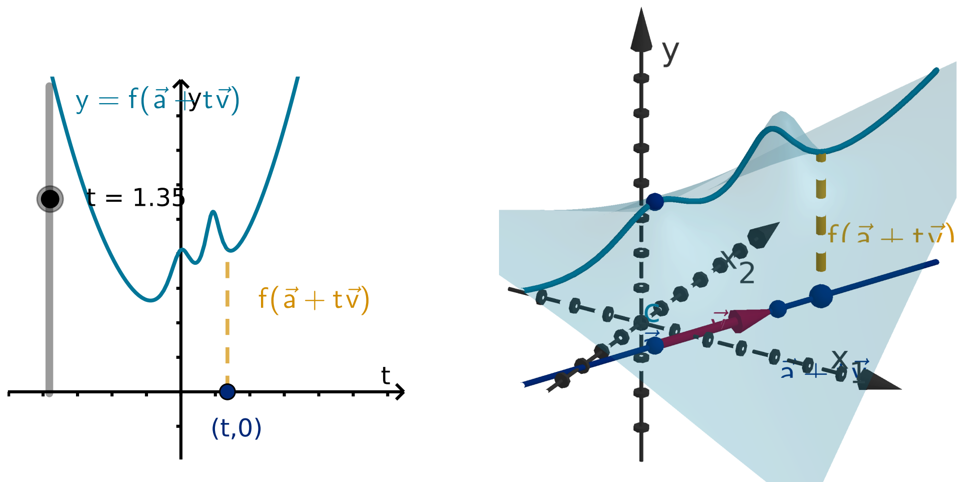


Figure 1.29: The graph of a two-variable function and its cross section over a line

1.3.2 The Multivariable First-Order Condition

Our starting point for multivariable optimization is Lemma 1.26. $\vec{a}$ is a local maximizer or minimizer of $f(\vec{x})$, if and only if 0 is a local maximizer or minimizer of $f(\vec{a}+t\vec{v})$ for all vectors $\vec{v}$. A condition on the compositions $f(\vec{a}+t\vec{v})$ at $t=0$ becomes a condition on $\vec{a}$.

The first-order condition on $f(\vec{a}+t\vec{v})$ at $t=0$ is

$$\begin{aligned} \frac{df}{dt}(0) &= 0 \\ \nabla f(\vec{a}+0\vec{v})\cdot\vec{v} &= 0 \\ \nabla f(\vec{a})\cdot\vec{v} &= 0 \end{aligned}$$

If $\vec{a}$ is a local maximizer, this must hold for all directions $\vec{v}$. There is only one value of $\nabla f(\vec{a})$ that satisfies this requirement.

Theorem 1.27 [The Multivariable First-Order Condition]

Suppose $\vec{a}$ lies in the domain of $f(\vec{x})$. If $\vec{a}$ is a local maximizer or minimizer of f, then either $\nabla f(\vec{a})=\vec{0}$ or $\nabla f(\vec{a})$ does not exist.

Like in the single variable case, we call points where $\nabla f(\vec{x})$ is $\vec{0}$ or undefined **critical points**. Finding critical points is often the first step in a multivariable optimization.

Remark

If $\vec{a}$ lies at the boundary of the domain, then it may not be possible to travel in all directions $\vec{v}$ from $\vec{a}$ and still evaluate $f(\vec{a}+t\vec{v})$. Fortunately, the first-order condition still holds. If $\nabla f(\vec{a})$ exists, then the partial derivatives $f_i(\vec{a})$ exist. Thus $f(\vec{a}+h\vec{e}_i)$ exists for h near 0. That means we can at least travel in the standard basis directions from $\vec{a}$ without immediately leaving the domain of $f(\vec{x})$. At a local maximizer, the derivatives in these directions must be 0.

1.3.3 Computing Critical Points

What does the multivariable first-order condition say about $f(x_1,x_2)=x_1^4-4x_1x_2+x_2^4$?

Solution

We compute the gradient vector by taking the partial derivatvies with respect to x_1 and x_2.

$$\begin{aligned}\nabla f(\vec{x}) &= (f_1(\vec{x}), f_2(\vec{x}))\\ &= (4x_1^3-4x_2, 4x_1+4x_2^3)\end{aligned}$$

This is never undefined, so we solve for where it is $(0,0)$.

$$\begin{aligned} 4x_1^3-4x_2 &= 0 & -4x_1+4x_2^3 &= 0 & \\ x_1^3 &= x_2 & & & \text{(solve for } x_2\text{)}\\ & & -4x_1+4x_1^9 &= 0 & \text{(substitute)}\\ & & 4x_1(x^8-1) &= 0 & \text{(factor)}\\ & & x_1 &= 0 \text{ or } \pm 1 & \end{aligned}$$

Subsituting back into $x_1^3=x_2$ gives lets us solve for the corresponding values of x_2. We obtain the critical points $(0,0)$, $(1,1)$ $(-1,-1)$. We conclude that no point other than $(0,0)$, $(1,1)$ $(-1,-1)$ can be a local maximizer or a local minimizer. We cannot conclude whether each of these points is a local maximizer, a local minimzer or neither.

1.3.4 The Second Derivative of a Composition

We would also like to apply the local or global second-order condition along each line. In order to do this, we need a way to compute $\frac{d^2}{dt^2} f(\vec{a} + t\vec{v})$. The chain rules allows us to write the first derivative as a dot product, or a sum. We will write it as a sum.

$$\frac{d}{dt} f(\vec{a} + t\vec{v}) = \sum_{i=1}^{n} f_i(\vec{a} + t\vec{v}) v_i$$

To obtain the second derivative we differentiate both sides. We differentiate each term of the sum. We can treat each v_i as a constant multiple, but $f_i(\vec{a} + t\vec{v})$ is a composition of functions. The derivative of each term will need the chain rule. We again write this as a sum, using a different index variable to avoid ambiguity. The partial derivatives of f_i are f_{ij}.

$$\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) = \frac{d}{dt} \sum_{i=1}^{n} f_i(\vec{a} + t\vec{v}) v_i$$

$$\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) = \sum_{i=1}^{n} \frac{d}{dt} \left(f_i(\vec{a} + t\vec{v}) \right) v_i$$

$$\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} f_{ij}(\vec{a} + t\vec{v}) v_j \right) v_i$$

$$\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(\vec{a} + t\vec{v}) v_j v_i$$

We could use this formula to compute the second derivative of a particular composition $f(\vec{a} + t\vec{v})$. We could test whether it is positive or negative. What we need, though, is to test the sign of the second derivative for all direction vectors $\vec{v}$. While this is possible in small dimensions, leaving the v_i as variables, the algebra quickly becomes daunting as n increases.

1.3.5 The Hessian Matrix

A theorem in linear algebra provides a shortcut. To do linear algebra, we need to write our vectors like matrices.

Point Form and Column Form

When we want to emphasize that a vector $\vec{x}$ represents a point in $\mathbb{R}^n$, we write it as $\vec{x} = (x_1, x_2, \dots x_n)$. If we want to do matrix multiplication, we can write it as a column vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The expression

$$\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(\vec{a} + t\vec{v}) v_j v_i$$

that we obtained can be written as a matrix product.

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} f_{11}(\vec{a}+t\vec{v}) & f_{12}(\vec{a}+t\vec{v}) & \cdots & f_{1n}(\vec{a}+t\vec{v}) \\ f_{21}(\vec{a}+t\vec{v}) & f_{22}(\vec{a}+t\vec{v}) & \cdots & f_{2n}(\vec{a}+t\vec{v}) \\ \vdots & \vdots & & \vdots \\ f_{n1}(\vec{a}+t\vec{v}) & f_{n2}(\vec{a}+t\vec{v}) & \cdots & f_{nn}(\vec{a}+t\vec{v}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

You may want to convince yourself that this algebra is correct with a 2 or 3 dimensional example. What do we make of the individual factors in this product? The vectors on either end of this expression are just $\vec{v}$ and its **transpose** ($\vec{v}$ flipped sideways). The matrix in the middle seems important. We should have a name for it.

Definition 1.28

Given a function $f(\vec{x})$ of n variables, the **Hessian** is a $n \times n$ matrix function of $\vec{x}$ whose entries are the second partial derivatives of f at $\vec{x}$. Its formula is

$$Hf(\vec{x}) = \begin{bmatrix} f_{11}(\vec{x}) & f_{12}(\vec{x}) & \cdots & f_{1n}(\vec{x}) \\ f_{21}(\vec{x}) & f_{22}(\vec{x}) & \cdots & f_{2n}(\vec{x}) \\ \vdots & \vdots & & \vdots \\ f_{n1}(\vec{x}) & f_{n2}(\vec{x}) & \cdots & f_{nn}(\vec{x}) \end{bmatrix}$$

Notice that for well-behaved functions, $f_{ij} = f_{ji}$. This means the Hessian will be a symmetric matrix.

Our Hessian notation allows us to write $\dfrac{d^2}{dt^2}f(\vec{a}+t\vec{v}) = \vec{v}^T Hf(\vec{a}+t\vec{v})\vec{v}$. The following vocabulary captures exactly which Hessian matrices will produce a negative (or positive) second derivative in all directions.

Definition 1.29

A symmetric $n \times n$ matrix M is **negative definite**, if for all nonzero n-vectors $\vec{v}$,

$$\vec{v}^T M\vec{v} < 0.$$

It is **positive definite**, if for all nonzero n-vectors $\vec{v}$,

$$\vec{v}^T M\vec{v} > 0.$$

Remark

We may think, based on positive and negative numbers, that most matrices are either positive definite or negative definite. In fact, a randomly chosen matrix is most likely to be neither. For most matrices, $\vec{v}^T M\vec{v}$ is positive for some $\vec{v}$ and negative for others.

Connecting this back to our original goal, we can apply the definition of negative definite to describe the sign of the second derivatives of the compositions $f(\vec{a}+t\vec{v})$. For a critical point $\vec{a}$, we can draw the following conclusions.

- If $Hf(\vec{a})$ is negative definite, then $\frac{d^2f}{dt^2}(0) < 0$ for all $\vec{v}$. Since $\frac{df}{dt}(0) = 0$, $t = 0$ satisfies the second-order condition for all $\vec{v}$. We conclude $\vec{a}$ is a strict local maximizer of f.
- If $Hf(\vec{x})$ is negative definite for all $\vec{x}$, then $\frac{d^2f}{dt^2}(t) < 0$ for all t and all $\vec{v}$. Since $\frac{df}{dt}(0) = 0$, $t = 0$ satisfies the global second-order condition for all $\vec{v}$. We conclude $\vec{a}$ is the unique global maximizer of f.

Without an efficient way to test whether a matrix is negative definite, we have only managed to restate the problem. We can say that we are checking whether $Hf(\vec{a})$ is negative definite, or we can say that we are checking whether $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(\vec{a})v_jv_i < 0$ for all $v_1, v_2, \ldots, v_n$. The computations are the same, and they are daunting.

Fortunately, there is an easier way to show that a matrix is negative or positive definite.

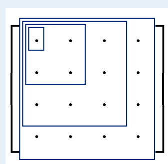
Theorem 1.30

A symmetric $n \times n$ matrix M is negative definite if the upper left square minors M_i have determinants that satisfy the **alternating condition**:

$$(-1)^i |M_i| > 0 \text{ for all } 1 \leq i \leq n$$

Example

For a 4×4 matrix, here are the M_i and what we would check to apply Theorem 1.30.



$$|M_1| < 0 \qquad |M_3| < 0$$

$$|M_2| > 0 \qquad |M_4| > 0$$

Variant of Theorem 1.30

A $n \times n$ matrix M is positive definite if the upper left square minors have positive determinants.

1.3.7 The Multivariable Second-Order Conditions

Now that we have a check for negative definite and positive definite matrices, it is worth stating the results of our investigation as theorems. These are our most important sufficient conditions for a maximizer of a multi-variable function.

Theorem 1.31 [The Multivariable Second-Order Condition]

Given a function $f(\vec{x})$ with $\vec{a}$ it its domain. If

- $\nabla f(\vec{a}) = \vec{0}$
- $Hf(\vec{a})$ is negative definite

then $\vec{a}$ is a strict local maximizer of f.

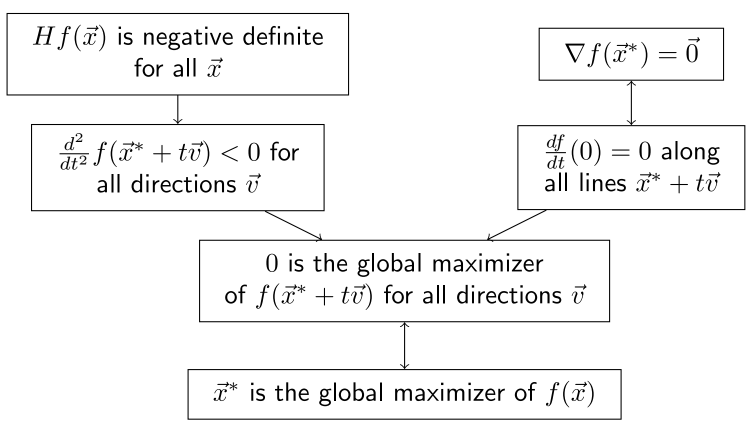
Theorem 1.32 [The Multivariable Global Second-Order Condition]

Suppose f is a function with a convex domain D and $\vec{x}^*$ is in D. If

- $\nabla f(\vec{x}^*) = \vec{0}$
- $Hf(\vec{x})$ is negative definite for all $\vec{x} \in D$.

then $\vec{x}^*$ is the only critical point of f and the unique global maximizer.

We can summarize the reasoning behind the global second-order condition as follows:



These theorems have variants for minimizers.

Variants of Theorems 1.31 and 1.32

Suppose f is a function with domain D and $\vec{a}$ is in D.

1. If
   - $\nabla f(\vec{a}) = \vec{0}$
   - $Hf(\vec{a})$ is positive definite

   then $\vec{a}$ is a strict local minimizer of f.
2. If D is convex and
   - $\nabla f(\vec{a}) = \vec{0}$
   - $Hf(\vec{x})$ is positive definite for all $\vec{x}$ in D

   then $\vec{a}$ is the only critical point of f and the unique global minimizer.

Minor determinants with the wrong sign produce the wrong sign of $\vec{v}^T M \vec{v}$ for some $\vec{v}$. This in turn means the wrong second derivative in the direction of $\vec{v}$. This relationship is formalized in the following necessary condition.

Corollary 1.33

Let $f(\vec{x})$ be a twice-differentiable function on a convex domain D. Let $M = Hf(\vec{a})$.

1. If $\vec{a}$ is a local maximizer, then
$$(-1)^i |M_i| \geq 0 \text{ for all } 1 \leq i \leq n.$$
2. If $\vec{a}$ is local minimizer, then
$$|M_i| \geq 0 \text{ for all } 1 \leq i \leq n.$$

1.3.8 Applying the Multivariable Second-Order Condition

Let $f(x_1, x_2, x_3) = 3\sqrt[3]{x_1 x_3} + \ln x_2 - \frac{x_1}{4} - \frac{x_2}{5} - 2x_3 + 15$ on the domain $D = \{(x_1, x_2, x_3) : x_1, x_2, x_3 > 0\}$. Find the critical point of f. What does the Global Second-Order Condition say about this point?

Solution

We take all three partial derivatives of f. Since they are always defined on D, the gradient vector exists, so the critical points must be where they all equal 0.

$$
\begin{array}{ccc}
\dfrac{\sqrt[3]{x_3}}{\sqrt[3]{x_1^2}} - \dfrac{1}{4} = 0 & \dfrac{1}{x_2} - \dfrac{1}{5} = 0 & \dfrac{\sqrt[3]{x_1}}{\sqrt[3]{x_3^2}} - 2 = 0 \\
\dfrac{\sqrt[3]{x_3}}{\sqrt[3]{x_1^2}} = \dfrac{1}{4} & \dfrac{1}{x_2} = \dfrac{1}{5} & \dfrac{\sqrt[3]{x_1}}{\sqrt[3]{x_3^2}} = 2 \\
\sqrt[3]{x_3} = \dfrac{\sqrt[3]{x_1^2}}{4} & x_2 = 5 & \\
 & & \dfrac{16\sqrt[3]{x_1}}{\sqrt[3]{x_1^4}} = 2 \\
 & & \dfrac{16}{x_1} = 2 \\
 & & 8 = x_1 \\
\sqrt[3]{x_3} = \dfrac{\sqrt[3]{8^2}}{4} & & \\
x_3 = 1 & &
\end{array}
$$

The critical point is $(8, 5, 1)$. We compute the Hessian by taking the second partial derivatives of f.

$$
Hf(x_1, x_2, x_3) = \begin{bmatrix} -\frac{2\sqrt[3]{x_3}}{3\sqrt[3]{x_1^5}} & 0 & \frac{1}{3\sqrt[3]{x_1^2 x_3^2}} \\ 0 & -\frac{1}{x_2^2} & 0 \\ \frac{1}{3\sqrt[3]{x_1^2 x_3^2}} & 0 & -\frac{2\sqrt[3]{x_1}}{3\sqrt[3]{x_3^5}} \end{bmatrix}
$$

To check that $Hf(\vec{x})$ is negative definite for all $\vec{x}$, we apply Theorem 1.30. There are three determinants to calculate.

$$
\begin{aligned}
(-1)^1|M_1| &= (-1)\left| -\frac{2\sqrt[3]{x_3}}{3\sqrt[3]{x_1^5}} \right| = \frac{2\sqrt[3]{x_3}}{3\sqrt[3]{x_1^5}} \\
(-1)^2|M_2| &= (1)\begin{vmatrix} -\frac{2\sqrt[3]{x_3}}{3\sqrt[3]{x_1^5}} & 0 \\ 0 & -\frac{1}{x_2^2} \end{vmatrix} = \frac{2\sqrt[3]{x_3}}{3x_2^2\sqrt[3]{x_1^5}} \\
(-1)^3|M_3| &= -|Hf(x_1, x_2, x_3)| \\
&= -\left(\left(-\frac{2\sqrt[3]{x_3}}{3\sqrt[3]{x_1^5}}\right)\begin{vmatrix} -\frac{1}{x_2^2} & 0 \\ 0 & -\frac{2\sqrt[3]{x_1}}{3\sqrt[3]{x_3^5}} \end{vmatrix} - 0 + \left(\frac{1}{3\sqrt[3]{x_1^2 x_3^2}}\right)\begin{vmatrix} 0 & -\frac{1}{x_2^2} \\ \frac{1}{3\sqrt[3]{x_1^2 x_3^2}} & 0 \end{vmatrix}\right) \\
&= \frac{4}{9x_2^2\sqrt[3]{x_1^4 x_3^4}} - \frac{1}{9x_2^2\sqrt[3]{x_1^4 x_3^4}} \\
&= \frac{1}{3x_2^2\sqrt[3]{x_1^4 x_3^4}}
\end{aligned}
$$

Since x_1, x_2 and x_3 are all positive in this domain, so are these three quantities. We conclude that $Hf(x_1, x_2, x_3)$ is negative definite. By the global second-order condition, we conclude that $(8, 5, 1)$ is the only critical point (which we already knew from the FOC) and that it is the unique maximizer of f.

1.3.9 The Hessian and Concavity

Linear compositions play well with secants. The height of the secant above each t is the same whether we are looking in ty-plane or in $\mathbb{R}^{n+1}$. We can state this relationship formally as a lemma.

Lemma 1.34

If $y = s(t)$ is a secant of $y = f(\vec{a} + t\vec{v})$, then

$$\begin{aligned} \vec{x}(t) &= \vec{a} + t\vec{v} \\ y(t) &= s(t) \end{aligned}$$

is a secant of $y = f(\vec{x})$.

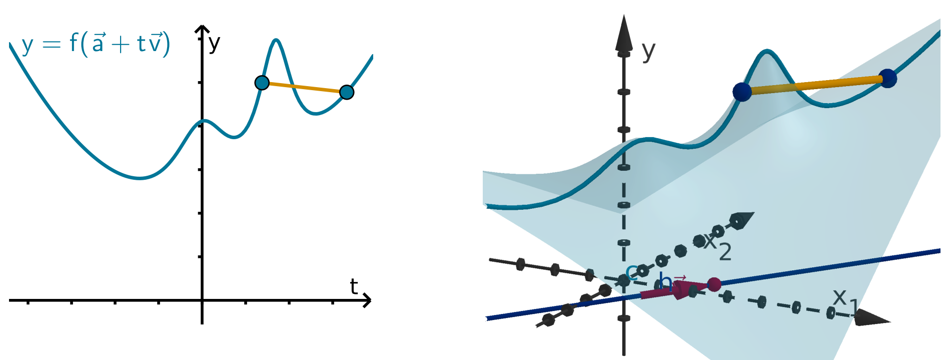
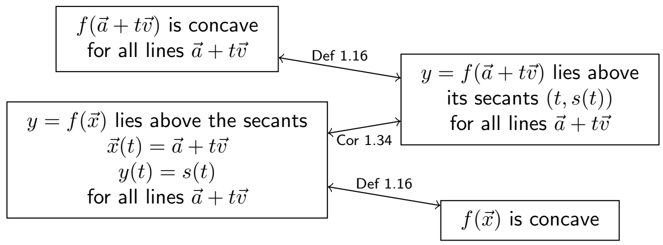


Figure 1.30: A secant of $y = f(\vec{a} + t\vec{v})$ and the corresponding secant of $y = f(\vec{x})$ over $\vec{x}(t) = \vec{a} + t\vec{v}$

We determine whether a function $f(\vec{x})$ is concave by whether its graph lies above its secant lines. Every such secant line lies above some line $\vec{a} + t\vec{v}$. This secant stays below the graph $y = f(\vec{x})$, if and only if the corresponding secant stays below $y = f(\vec{a} + t\vec{v})$.

Lemma 1.35

A function $f(\vec{x})$ on a convex domain is concave, if and only if $f(\vec{a}+t\vec{v})$ is concave for every $\vec{a}$ in the domain of f and every direction vector $\vec{v}$.



Variants of Lemma 1.35

Suppose $f(\vec{x})$ is a function on a convex domain

1. $f(\vec{x})$ is strictly concave if and only if $f(\vec{x}(t))$ is strictly concave for every line $\vec{x}(t)$ in the domain of f.
2. $f(\vec{x})$ is convex if and only if $f(\vec{x}(t))$ is convex for every line $\vec{x}(t)$ in the domain of f.
3. $f(\vec{x})$ is strictly convex if and only if $f(\vec{x}(t))$ is strictly convex for every line $\vec{x}(t)$ in the domain of f.

The result of this lemma is that any test for concavity or convexity of the single-variable compositions $f(\vec{a}+t\vec{v})$ becomes a test for the concavity of $f(\vec{x})$ itself. If every composition passes, then f does. If at least one composition fails, then so does f. Our natural next step is to take our theorems from single-variable concavity and apply them to these compositions,

Theorem 1.22 showed that the graphs single-variable strictly concave functions lie below their tangent lines. Tangent lines to a composition have the same heights as the tangent lines of the multivariable graph. We can formalize this relationship with a lemma.

Lemma 1.36

If the line $y=\ell(t)$ is tangent to $y=f(\vec{a}+t\vec{v})$, then

$$\vec{x}(t)=\vec{a}+t\vec{v}$$
$$y(t)=\ell(t)$$

is a tangent line to $y=f(\vec{x})$.

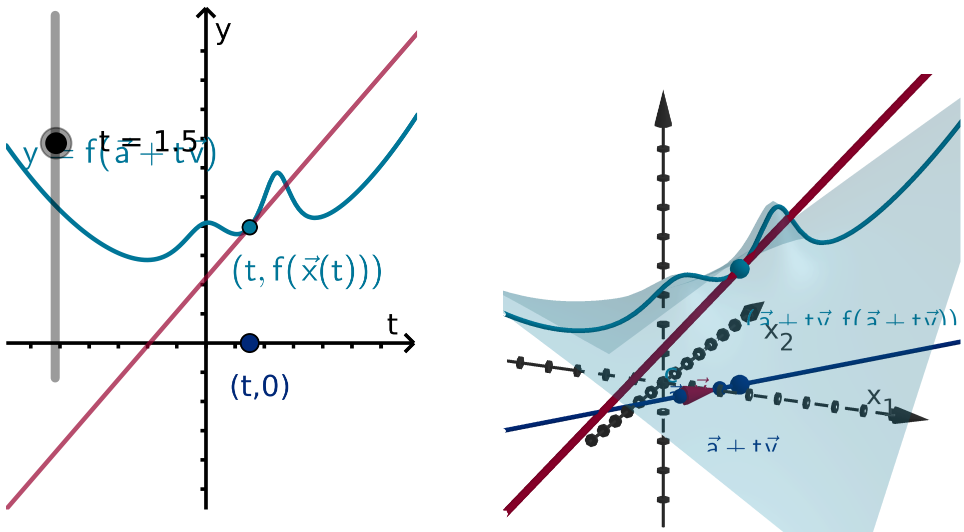
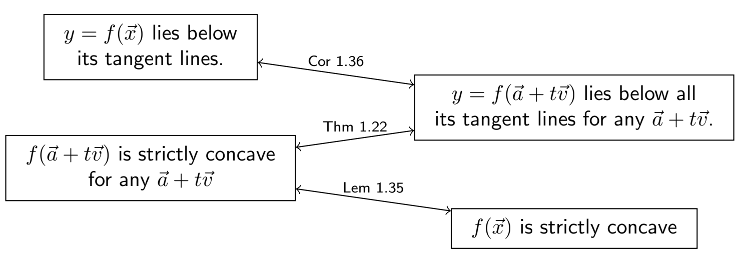


Figure 1.31: A tangent line of $y = f(\vec{a} + t\vec{v})$ and the corresponding tangent line of $y = f(\vec{x})$ over $\vec{x}(t) = \vec{a} + t\vec{v}$

We can check whether tangent lines lie above or below a graph $y = f(\vec{x})$ by checking whether tangent lines lie above or below the compositions $y = f(\vec{a} + t\vec{v})$.

Theorem 1.37 [Multivariable version of Theorem 1.22]

A differentiable function f on a convex domain is strictly concave if and only if the graph $y = f(\vec{x})$ lies below each of its tangent lines (except at the point of tangency).



This theorem allows us to update our corollary on critical points of strictly concave functions.

Corollary 1.38 [Multivariable Version of Corollary 1.23]

If $\vec{x}^*$ is a critical point of a differentiable, strictly concave function $f(\vec{x})$, then it is the only critical point and is the unique global maximizer.

We can also use compositions to update our theorems about second derivatives and concavity.

Theorem 1.39 [Multivariable Version of Theorem 1.24]

If f is a function on a convex domain $D \subset \mathbb{R}^n$ and $\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) < 0$ for all lines $\vec{a} + t\vec{v}$ in D, then $f(\vec{x})$ is strictly concave.

Corollary 1.40 [Multivariable Version of Corollary 1.25]

A twice differentiable function $f(\vec{x})$ on a convex domain D is concave, if and only if $\frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) \leq 0$ on all lines $\vec{a} + t\vec{v}$ in D.

As we have seen, the sign of these second derivatives depends on the Hessian matrix. This gives us our most useful computational test for the concavity of a multivariable function.

Theorem 1.41

Let $f(\vec{x})$ be a twice-differentiable function on a convex domain D. If $Hf(\vec{x})$ is negative definite for all $\vec{x}$ in D, then f is strictly concave.

This means that concavity plays the same role in multivariable optimization as in single-variable optimization. For some functions, we can identify a maximizer by concavity even though they do not satisfy the second-order condition. On the other hand, the most convenient way to identify concave functions is still the second derivatives.

Changing the condition $f''(x) < 0$ to $f''(x) \leq 0$ was a natural approach to generalizing Theorem 1.24. As a bonus, we obtained Corollary 1.25, which is both a necessary and sufficient condition for concavity. The same is possible for multivariable functions, but the condition on the Hessian is surprisingly complicated.

We might expect the multivariable analogue to be that $(-1)^i|M_i| \geq 0$ for each i. Consider a function like $f(x_1, x_2, x_3) = x_3^2$. In the x_3 direction this function is a parabola which is strictly convex. Unfortunately, its Hessian passes our expected test for concavity.

$$Hf(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

This has $|M_i| = 0$ for each i. The determinant cannot detect the curvature of later variables, if the earlier variables produce a row of zeroes in $Hf(\vec{x})$. Our expected test is not a correct test. To avoid this problem, we must consider different orders of variables. Which orders? All of them.

Theorem 1.42

Let $f(\vec{x})$ be a twice-differentiable function on a convex domain $D \subseteq \mathbb{R}^n$. let σ be any reordering of the coordinates of $\mathbb{R}^n$. Let $\sigma M = Hf(\sigma\vec{x})$.

1. f is concave if and only if

$$(-1)^i|\sigma M_i| \geq 0 \text{ for all } \sigma \text{ and all } 1 \leq i \leq n.$$

If so, then $\vec{v}^T Hf(\vec{x})\vec{v} \leq 0$ for all nonzero $\vec{v}$. We say $Hf(\vec{x})$ is **negative semidefinite**

2. f is convex if and only if

$$|\sigma M_i| \geq 0 \text{ for all } \sigma \text{ and all } 1 \leq i \leq n.$$

If so, then $\vec{v}^T Hf(\vec{x})\vec{v} \geq 0$ for all nonzero $\vec{v}$. We say $Hf(\vec{x})$ is **positive semidefinite**

To give a sense of the amount of computation required, we will consider an example. Suppose

$$Hf(\vec{x}) = \begin{bmatrix} 0 & 2 & 5 \\ 2 & -6 & 0 \\ 5 & 0 & -3 \end{bmatrix}.$$

We check the determinants

$$|M_1| = |0| \leq 0 \qquad |M_2| = \begin{vmatrix} 0 & 2 \\ 2 & -6 \end{vmatrix} \geq 0 \qquad |M_3| = \begin{vmatrix} 0 & 2 & 5 \\ 2 & -6 & 0 \\ 5 & 0 & -3 \end{vmatrix} \leq 0$$

1.3.10 Negative Semi-Definite Matrices

But then we must do the same for all possible reorderings of x_1, x_2 and x_3 meaning we must check the same three minors for each of

$$\begin{bmatrix} 0 & 2 & 5 \\ 2 & -6 & 0 \\ 5 & 0 & -3 \end{bmatrix} \text{original } Hf(\vec{x}) \qquad \begin{bmatrix} -6 & 2 & 0 \\ 2 & 0 & 5 \\ 0 & 5 & -3 \end{bmatrix} \sigma \text{ switches } x_1 \text{ and } x_2$$

$$\begin{bmatrix} -3 & 0 & 5 \\ 0 & -6 & 2 \\ 5 & 2 & 0 \end{bmatrix} \sigma \text{ switches } x_1 \text{ and } x_3 \qquad \begin{bmatrix} 0 & 5 & 2 \\ 5 & -3 & 0 \\ 2 & 0 & -6 \end{bmatrix} \sigma \text{ switches } x_2 \text{ and } x_3$$

$$\begin{bmatrix} -6 & 0 & 2 \\ 0 & -3 & 5 \\ 2 & 5 & 0 \end{bmatrix} \sigma : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1) \qquad \begin{bmatrix} -3 & 5 & 0 \\ 5 & 0 & 2 \\ 0 & 2 & -6 \end{bmatrix} \sigma : (x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$$

This is a significant amount of work, though linear algebra knowledge lets us avoid some. For instance, all the $|\sigma M_3|$ are equal. As the dimension increases, there are more symmetries to exploit. With these shortcuts, the main driver in complexity is computing the $n \times n$ determinant, not the number of re-ordered smaller determinants. Checking whether a matrix is negative semidefinite takes about twice as many operations as checking that it is negative definite.

1.3.11 Section Summary

The most important definitions and results from this section were

- The multivariable first-order condition (Theorem 1.27)
- The Hessian matrix (Definition 1.28)
- Positive and negative definite (Definition 1.29)
- The determinant test for a negative definite matrix (Theorem 1.30)
- The multivariable global second-order condition (Theorem 1.32)
- The Hessian test for strict concavity (Theorem 1.41)

Here is a summary of which conditions and statements from this section imply which others.

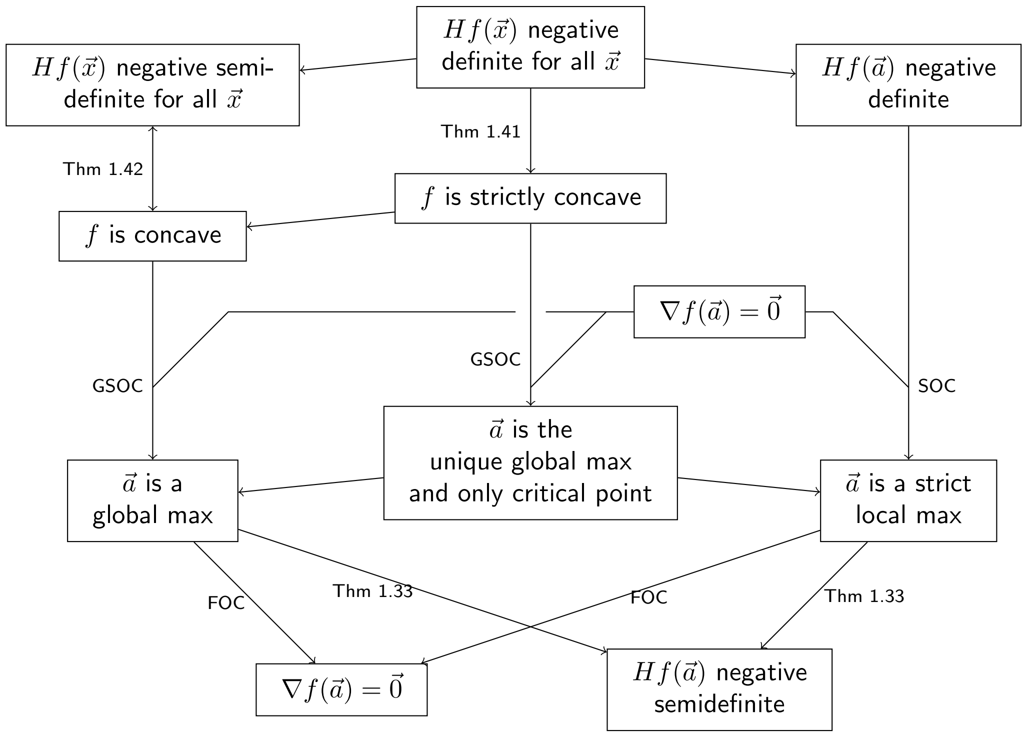


Figure 1.32: Relationships between the conditions of multivariable optimization

1.3.11 Section Summary

Chapter 2

Constrained Optimization

2.1

Equality Constraints

Goals:

1. Use logic to reason about the effects of a constraint on optimization problems.
2. Visually identify maximizers and minimizers using level sets.
3. Compute the maximizer subject to an equality constraint using the Lagrangian.
4. Use paths and the chain rule to understand the reasoning behind the Lagrangian.

2.1.1

Constrained Optimization

Given a function $f(\vec{x})$ with domain D, we have so far performed unconstrained optimization, trying to compute

$$\max_{\vec{x}\in D} f(\vec{x}) \text{ or } \min_{\vec{x}\in D} f(\vec{x})$$

In a constrained optimization problem, we restrict our attention to a subset of the domain. For some subset $S \subseteq D$, we want to know

$$\max_{\vec{x}\in S} f(\vec{x}) \text{ or } \min_{\vec{x}\in S} f(\vec{x})$$

A maximizer in S is a vector $\vec{a}$ such that $f(\vec{a}) \geq f(\vec{x})$ for all $\vec{x} \in S$. Geometrically, it is the highest point on the part of the graph $y = f(\vec{x})$ that lies above S. In constrained optimization, the function we are maximizing is called a **objective function**. The set S is called the **feasible set**.

The notation above is flexible enough to apply to many different kinds of constraints, producing many different kinds of subset S.

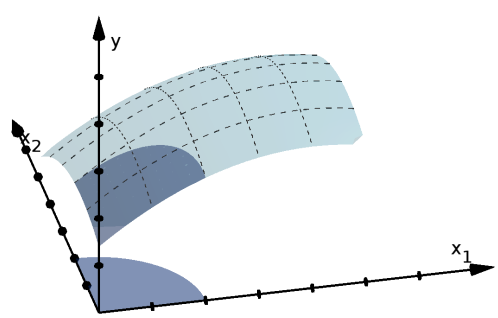


Figure 2.1: The graph $y = f(\vec{x})$ over a two-dimensional subregion S

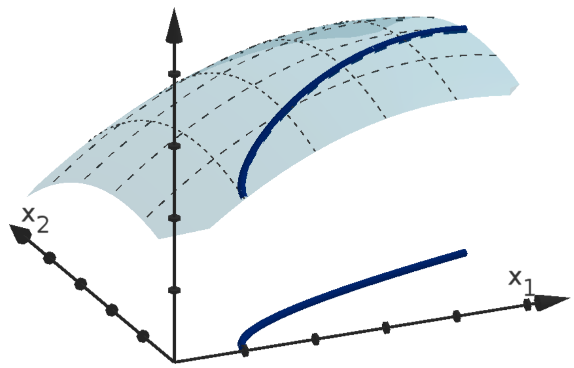


Figure 2.2: The graph $y = f(\vec{x})$ over a curve S

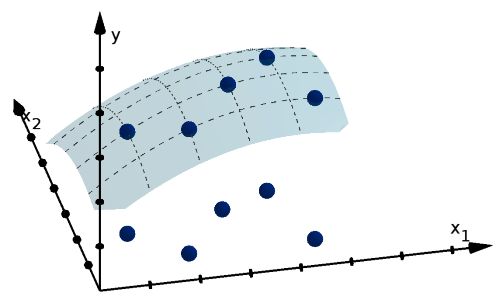


Figure 2.3: The graph $y = f(\vec{x})$ over a finite set S

Constraints arise naturally in economics. Here are two familiar examples recast in the abstract form that we just introduced.

Example

What is the maximum utility one can attain given a budget constraint?

$$\max_{\vec{x} \in S} u(\vec{x})$$

where S is the set of purchases one can afford.

Example

How cheaply can a firm produce 500 units?

$$\min_{\vec{x}\in S} c(\vec{x})$$

Here, S is the set of inputs that produce 500 units.

Without knowing the specific nature of the constraint, we can still deduce some basic facts about constrained optimization using logic and set theory.

Tightening a constraint can only reduce the value of the maximum.

Lemma 2.1

For a function $f(\vec{x})$ with domain D and a two subsets $T \subseteq S \subseteq D$,

$$\max_{\vec{x}\in T} f(\vec{x}) \leq \max_{\vec{x}\in S} f(\vec{x})$$

Sometimes finding a maximum over S is computationally easier than finding a maximum over T. If we are lucky, the maximizer of $f(\vec{x})$ in S might also lie in T. We can apply the following reasoning.

Corollary 2.2

Suppose $\vec{x}^*$ is a maximizer of $f(\vec{x})$ over a set S. If $T \subseteq S$ and $\vec{x}^* \in T$ then $\vec{x}^*$ is also a maximizer of $f(\vec{x})$ over T and

$$\max_{\vec{x}\in T} f(\vec{x}) = \max_{\vec{x}\in S} f(\vec{x})$$

Remark

Why do we make the distinction between the domain and a subset? If we want to constrain $\vec{x}$ to values in S, we could redefine f to have a domain of S. This is generally not a good idea. We will want to use the derivatives of f to solve constrained optimization problems. For example, we may need to know

$$\lim_{h\to 0} \frac{f(\vec{a}+h\vec{e}_i) - f(\vec{a})}{h}$$

If $\vec{a}$ is on the boundary of S and we declare everything outside S to no longer be in the domain of f, then we may not be able to evaluate $f(\vec{a}+h\vec{e}_i)$ in a neighborhood of $h = 0$.

The first type of constraint we will examine is an equality constraint.

Definition 2.3

An **equality constraint** is a constraint of the form $g(\vec{x}) = 0$. With such a constraint we are solving

$$\max_{\vec{x} \in D} f(\vec{x}) \text{ subject to } g(\vec{x}) = 0 \text{ or}$$

$$\max_{\vec{x} \in S} f(\vec{x}) \text{ where } S = \{\vec{x} \in D \ : \ g(\vec{x}) = 0\}$$

To contrast g with the objective function f, we call it a **constraint function**.

Remark

We can rewrite any equation in the variables of $\vec{x}$ to have the form $g(\vec{x}) = 0$. For example:

$$x_2 = \frac{16}{x_1} \qquad \Longrightarrow \qquad 16 - x_1 x_2 = 0$$

An equality constraint is an equation, a condition that a point can satisfy or fail to satisfy. In order to maximize f among such points, we want to be able to visualize the entire set of points that satisfy the constraint. Mathematics has a term for this set.

Definition 2.4

Given a function $g(\vec{x})$ a **level set** of g is the set of points that, for some number c, satisfy $g(\vec{x}) = c$.

Example

Given the function $g(x_1, x_2) = x_1 x_2$, here are four different level sets of g:

$$\{(x_1, x_2) \ : \ x_1 x_2 = 9\}$$
$$\{(x_1, x_2) \ : \ x_1 x_2 = 17.32\}$$
$$\{(x_1, x_2) \ : \ x_1 x_2 = 0\}$$
$$\{(x_1, x_2) \ : \ x_1 x_2 = -4\}$$

Remark

Commonly, people refer to a level set by its defining equation, rather than using set notation. Thus "the level set $x_1^2 + x_2^2 = 16$" refers to the set $\{(x_1, x_2) \ : \ x_1^2 + x_2^2 = 16\}$.

Level sets have several applications in economics you may be familiar with:

Example

If we are choosing to buy quantities x_1 and x_2 of goods priced at p_1 and p_2, then the goods we can buy with a budget constraint I is the level set (often called a budget line):

$$p_1 x_1 + p_2 x_2 = I$$

Example

Given a utility function $u(x_1, x_2)$, the level set $u(x_1, x_2) = c$ is called an **indifference curve**. The chooser will be equally happy with any point on the curve.

For a two-variable function, a level set is the intersection of the graph $y = g(x_1, x_2)$ with a horizontal plane $y = c$. These are often curves, so we call them **level curves**.

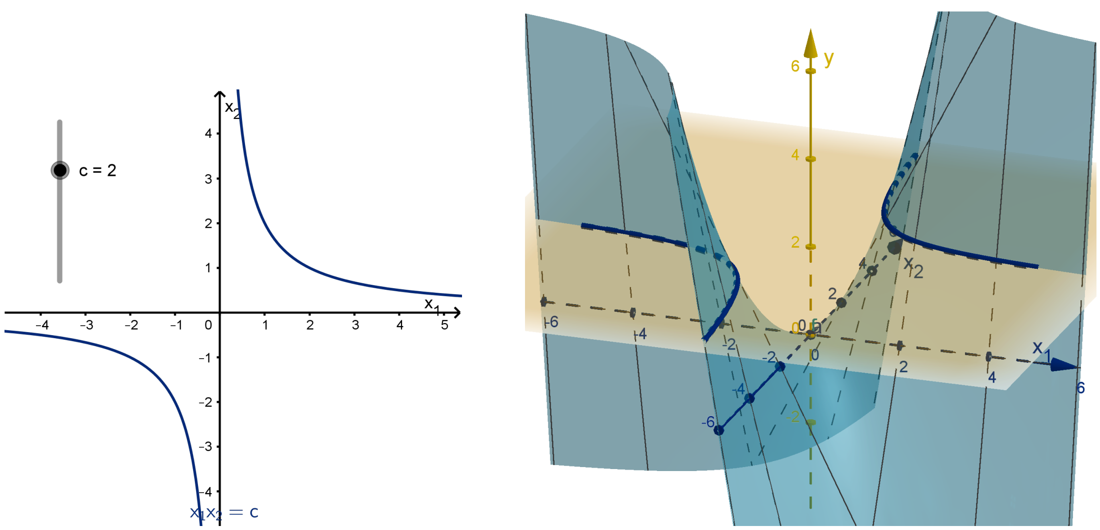


Figure 2.4: The level curves $x_1x_2 = c$

Notice that $x_1x_2 = c$ is not always a curve. When $c = 0$, the level set is the x_1- and x_2-axes. At the origin, this is two curves (lines) intersecting.

For a three-variable function, the level sets are called **level surfaces** because for most values of c, they are 2-dimensional.

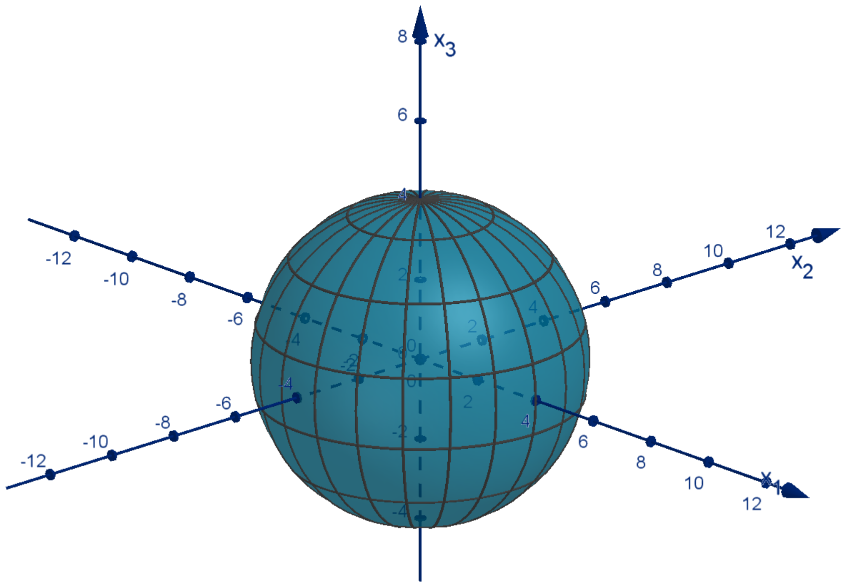


Figure 2.5: The level surfaces $x_1^2 + x_2^2 + x_3^2 = c$

Notice $x_1^2 + x_2^2 + x_3^2 = 0$ is not a surface, it is a point. $x_1^2 + x_2^2 + x_3^2 = -5$ is the empty set.

It can be difficult to determine the shape and structure of a level set. A powerful mathematical theorem can tell us when to expect good behavior.

Corollary 2.5 [Corollary to the Implicit Function Theorem]

Let $g(\vec{x})$ be a continuously differentiable function at $\vec{a}$. If $\vec{a}$ lies on the level set $g(\vec{x}) = c$ and $\nabla g(\vec{a}) \neq \vec{0}$, then the level set $g(\vec{x}) = c$ is a $(n-1)$-dimensional shape in some neighborhood of $\vec{a}$. Specifically, it is the graph of a differentiable function of $n-1$ of the variables of $\mathbb{R}^n$.

This means that for smooth functions we can generally expect the level set to be a space of one dimension lower than the domain, except at the occasional points where $\nabla g(\vec{x}) = \vec{0}$. We can see this in our previous examples.

- For $g(x_1, x_2) = x_1 x_2$ the nonempty level sets are curves, except for $x_1 x_2 = 0$. Even $x_1 x_2 = 0$ looks like a curve in any neighborhood that does not include $(0, 0)$.
- For $g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ the nonempty level sets are surfaces (spheres), except for $x_1^2 + x_2^2 + x_3^2 = 0$

We will give an argument for this corollary once we have seen the implicit function theorem in chapter 4.

2.1.3 Finding the Maximizer and Minimizer Graphically

The level set $g(\vec{x}) = 0$ is obviously important to a constrained optimization problem. We can also use level sets to understand the shape of the objective function. Level sets have the advantage that they live in $\mathbb{R}^n$, not $\mathbb{R}^{n+1}$, making them easier to draw than the graph $y = f(\vec{x})$,

Consider the objective function $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $x_1^2 - x_2 - 4 = 0$.

a What are the level curves of $f(x_1, x_2)$?

b How can we use a diagram of the level curves of f to argue that $(1, -3)$ is not a local maximizer or minimizer?

c Where do the local maximizer(s) and local minimizer(s) of f on $x_1^2 - x_2 - 4 = 0$ appear to lie?

d Are any of the local maximizers and minimizers also global maximizers or minimizer?

Solution

a The level sets have the form $x_1^2 + x_2^2 = c$ these are circles centered at the origin, with higher values the farther from the origin we go.

b $(1, -3)$ lies on the level set $x_1^2 + x_2^2 = 10$. We can see from a sketch that the curve $x_1^2 - x_2 - 4 = 0$ passes through this level set from higher values to lower values. Thus $(1, -3)$ is not a local maximizer or minimizer.

c $(0, -4)$ appears to be a local maximizer, because $x_1^2 - x_2 - 4 = 0$ meets the $x_1^2 + x_2^2 = 16$ and travels back to smaller-values in both directions. There also appear to be local minimizers in the third and fourth quadrant where $x_1^2 - x_2 - 4 = 0$ touches a smallest level set and then travels back to larger-valued level sets.

d There is no global maximizer. As we travel upwards in $x_1^2 - x_2 - 4 = 0$, we cross larger and larger valued level sets. There is no upper bound to the values of $f(x_1, x_2)$ that we can attain on the constraint. On the other hand, the local minimizers are global minimizers. The only smaller-valued level curves are inside that circle, and the constraint does not enter that region.

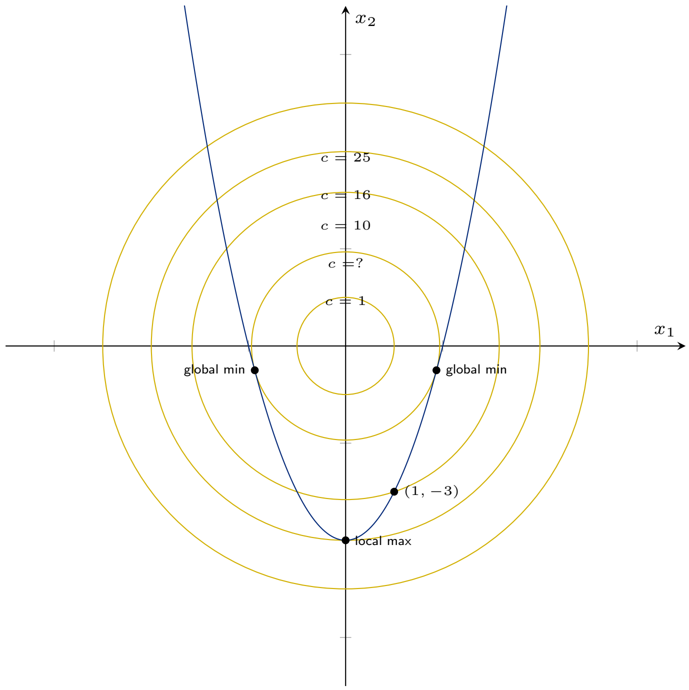


Figure 2.6: Level curves of $x_1^2 + x_2^2$ and the level curve $x_1^2 - x_2 - 4 = 0$

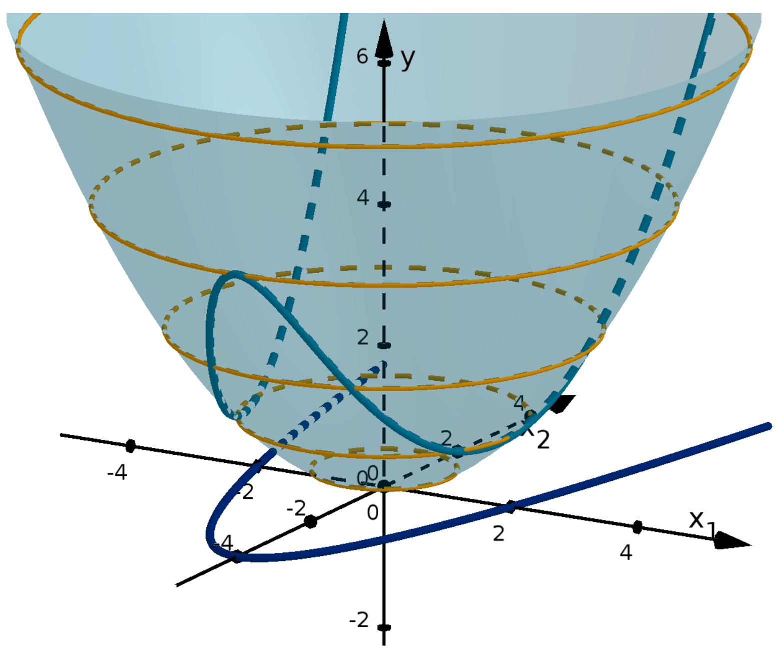


Figure 2.7: The graph of $y = x_1^2 + x_2^2$ over the constraint $x_1^2 - x_2 - 4 = 0$

When the level set $g(\vec{x}) = 0$ crosses a level set $f(\vec{x}) = c$ at $\vec{a}$, this usually means that it intersects both larger-valued and smaller-valued level sets on either side. If it does, then $\vec{a}$ cannot be a local maximizer or minimizer of f on the constraint.

Main Idea

We expect to find local maximizer and minimizers only where the level set $g(\vec{x}) = 0$ is tangent to a level set of f.

We can base our intuition on the following narrative: $g(\vec{x}) = 0$ crosses higher and higher-valued level sets of f as it approaches $f(\vec{x}) = c$. When it reaches $f(\vec{x}) = c$, rather than crossing it to even higher values, it brushes against it (tangency) and backs away toward lower-valued level sets.

This does not describe every possible version of tangency. The curve $x_2 - x_1^3 = 0$ is tangent to the level set $x_2 = 0$, but still crosses onto both sides. Furthermore level sets can behave unpredictably when their gradient vector is $\vec{0}$. A rigorous argument for a maximizer using tangency generally needs to appeal to the algebraic properties of $f(\vec{x})$ and $g(\vec{x})$ in some way.

2.1.4 The Lagrangian

Looking for tangency is a valuable method for approximating the maximizer, but we cannot expect it to be precise enough to determine exact coordinates. For an exact position, we need a set of equations to solve. These equations take the form of a necessary condition for a maximizer subject to an equality constraint. We first form a new function from both the objective function and the constraint.

Definition 2.6

Given an n-variable objective function $f(\vec{x})$ and a constraint $g(\vec{x}) = 0$, the **Lagrangian** is an $(n+1)$-variable function of the x_i and λ (lambda). Its equation is

$$\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) + \lambda g(\vec{x})$$

Theorem 2.7

Let $f(\vec{x})$ and $g(\vec{x})$ be continuously differentiable functions. If $\vec{a}$ is a local maximizer or minimizer of $f(\vec{x})$ subject to the constraint $g(\vec{x}) = 0$, then either

1. there is some number λ such that $(\vec{a}, \lambda)$ satisfies the first-order condition of $\mathcal{L}$ or
2. $\nabla g(\vec{a}) = 0$.

We will call these the points that satisfy these conditions **critical points** or **stationary points** of the constrained optimization problem. The value of λ is called a **Lagrange multiplier**.

Remark

Notice that this theorem is a necessary condition for a maximizer or minimizer, not a sufficient one.

2.1.5 Using the Lagrangian

We expect that the objective function $f(x_1, x_2) = x_1^2 + x_2^2$ has a minimizer on the constraint $x_1^2 - x_2 - 4 = 0$ somewhere in the fourth quadrant. Find it.

Solution

Letting $g(x_1, x_2) = x_1^2 - x_2 - 4$ we have the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(x_1^2 - x_2 - 4)$$

$\nabla g(x_1, x_2) = (2x_1, -1)$. This is never $\vec{0}$, so any minimizer must satisfy the first-order condition of $\mathcal{L}$.

2.1.5 Using the Lagrangian

We set the partial derivatives equal to 0 and solve.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 0 & \frac{\partial \mathcal{L}}{\partial x_2} &= 0 & \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \\
2x_1 + 2x_1\lambda &= 0 & 2x_2 - \lambda &= 0 & x_1^2 - x_2 - 4 &= 0 \\
2x_1(1+\lambda) &= 0 & & & & \\
\text{if } x_1 &= 0 & & & 0^2 - x_2 - 4 &= 0 \\
& & & & x_2 &= -4 \\
\text{if } \lambda &= -1 & 2x_2 + 1 &= 0 & & \\
& & x_2 &= -\frac{1}{2} & & \\
& & & & x_1^2 + \frac{1}{2} - 4 &= 0 \\
& & & & x_1 &= \pm\sqrt{\frac{7}{2}}
\end{aligned}$$

The only possible local maximizers and minimizer are $(0,-4)$, $\left(\sqrt{\frac{7}{2}}, -\frac{1}{2}\right)$ and $\left(-\sqrt{\frac{7}{2}}, -\frac{1}{2}\right)$. Since we are looking for a minimizer in the fourth quadrant, it must be $\left(\sqrt{\frac{7}{2}}, -\frac{1}{2}\right)$.

2.1.6 Proving the First-Order Condition of the Lagrangian

The partial derivatives of the Lagrangian are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_i} &= f_i(\vec{x}) + \lambda g_i(\vec{x}) \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= g(\vec{x})
\end{aligned}$$

Any solution $(\vec{a}, \lambda)$ to the first-order condition on $\mathcal{L}$ satisfies

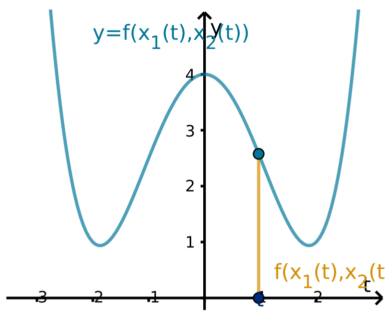
1. $f_i(\vec{a}) = -\lambda g_i(\vec{a})$ for all i, so $\nabla f(\vec{a}) = -\lambda \nabla g(\vec{a})$.
2. $g(\vec{a}) = 0$, so $\vec{a}$ lies on the constraint.

The necessity of 2 is easy to explain. If $\vec{a}$ does not lie on the constraint, then it cannot be a maximizer or minimizer subject to that constraint.

In order to understand 1, we need to investigate the relationship between gradient vectors and level sets. Like with multivariable unconstrained optimization, we use compositions over paths to study maximizers on constraints. We need to alter our approach in three important ways.

- We are only comparing $f(\vec{a})$ to $f(\vec{b})$ for other $\vec{b}$ in the level set $g(\vec{x}) = 0$. Thus we only consider paths $\vec{x}(t)$ that stay in the level set.
- Unless the level set is linear, we cannot assume the paths $\vec{x}(t)$ are straight lines. They likely need to curve to stay in the level set.
- The tangent vectors $\vec{x}'(t)$ generally cannot point in all directions. Most directions of travel would cause the path to leave the level set.

Now we examine the composition $f(\vec{x}(t))$ where $\vec{x}(t)$ is a path in the level set $g(\vec{x}) = 0$.



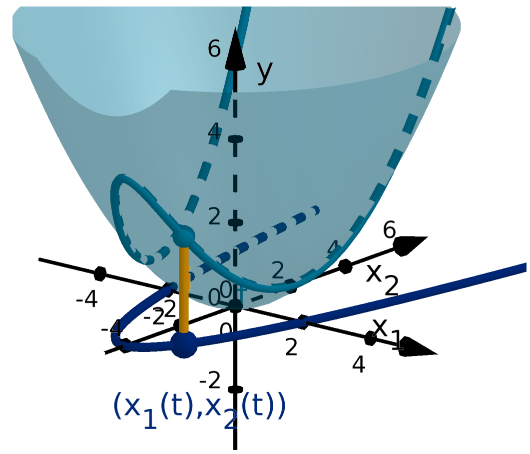


Figure 2.8: The composition $f(x_1(t), x_2(t))$ for a path $(x_1(t), x_2(t))$ in the level set $g(x_1, x_2) = 0$ and its realization as the heights of $y = f(x_1, x_2)$

If $\vec{a}$ is a local maximizer or minimizer of $f(\vec{x})$ among the points on $g(\vec{x}) = 0$ then it must be a local maximizer or minimizer of the composition $f(\vec{x}(t))$ for all paths $\vec{x}(t)$ in the level set $g(\vec{x}) = 0$. This means the t-value corresponding to $\vec{a}$ (we will call it t_0) must satisfy the first-order condition of $f(\vec{x}(t))$. To compute the derivative of the composition, we apply the chain rule.

$$\begin{aligned} \frac{df}{dt}(t_0) &= 0 && \text{(first-order condition)} \\ \nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t_0) &= 0 && \text{(chain rule)} \\ \nabla f(\vec{a}) \cdot \vec{x}'(t_0) &= 0 \end{aligned}$$

In the unconstrained case, $\nabla f(\vec{a}) \cdot \vec{v} = 0$ for all $\vec{v}$ allowed us to conclude that $\nabla f(\vec{a}) = 0$. Our conclusion cannot be so restrictive here. $\vec{x}(t)$ must be a path in the level set. We are thus only considering vectors $\vec{x}'(t_0)$ that are tangent to the level set at $\vec{a}$.

To make a conclusion about $\nabla f(\vec{a})$, we appeal to the geometric properties of the dot product. $\nabla f(\vec{a})$ and $\vec{x}'(t_0)$ must be orthogonal for all paths in the level set that pass through $\vec{a}$ at t_0. We introduce the following vocabulary to describe this relationship between $\nabla f(\vec{a})$ and the level set.

Definition 2.8

A vector $\vec{v}$ is **normal** to a set at $\vec{a}$, if $\vec{v}$ orthogonal to all tangent vectors of that set at $\vec{a}$.

To avoid having to mention it as a separate case, we consider $\vec{0}$ to be orthogonal to every vector.

Lemma 2.9

If $\vec{a}$ is a local maximizer or local minimizer of a continuously differentiable function $f(\vec{x})$ subject to $g(\vec{x}) = 0$, then $\nabla f(\vec{a})$ is normal to the level set $g(\vec{x}) = 0$.

This gives us a good characterization of the gradient of the objective function at a local maximizer or minimizer. We can now visually rule out a potential maximizer, if its gradient does not point in the right direction. We now need to connect this to our computational tool. The Lagrangian compares ∇f to ∇g. What can we say about ∇g?

Again, we consider a path $\vec{x}(t)$ in the level set $g(\vec{x}) = 0$. This time we study the composition $g(\vec{x}(t))$. If you think carefully about what we have done, you will notice that this is a silly composition. We can evaluate this composition at any value of t, and we will always get 0 (make sure you see why).

We can now apply the chain rule to $g(\vec{x}(t))$, the zero function. This time the derivative is zero because we are differentiating a constant function.

$$\begin{aligned} \frac{dg}{dt}(t_0) &= \nabla g(\vec{x}(t_0)) \cdot \vec{x}'(t_0) \\ 0 &= \nabla g(\vec{a}) \cdot \vec{x}'(t_0) \end{aligned}$$

We conclude that $\nabla g(\vec{x}(t))$ is orthogonal to $\vec{x}'(t)$ for all paths $\vec{x}(t)$ in the level set $g(\vec{x}) = 0$.

Remark

This argument did not rely on $\vec{a}$ being any special point on the level set. Nor did it matter that the level set had value 0. Any level set $g(\vec{x}) = c$ will have this relationship to the gradient, since the composition will still be a constant function.

The result of this argument is the one of the most important geometric facts about gradient vectors. We will use it over and over again.

Theorem 2.10

If g is a continuously differentiable function, and $g(\vec{a}) = c$, then $\nabla g(\vec{a})$ normal to the level set $g(\vec{x}) = c$.

In $\mathbb{R}^2$, this means that $\nabla g(\vec{a})$ points either 90° degrees clockwise or 90° counterclockwise from $\vec{x}'(t_0)$.

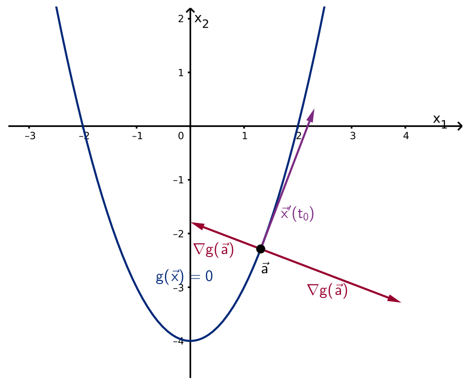


Figure 2.9: Two possible $\nabla g(\vec{x}(a))$ orthogonal to $\vec{x}'(t_0)$

In the case of a level surface, there are many paths through $\vec{a}$. In order to be normal to the level set, there are only two possible directions for $\nabla g(\vec{a})$.

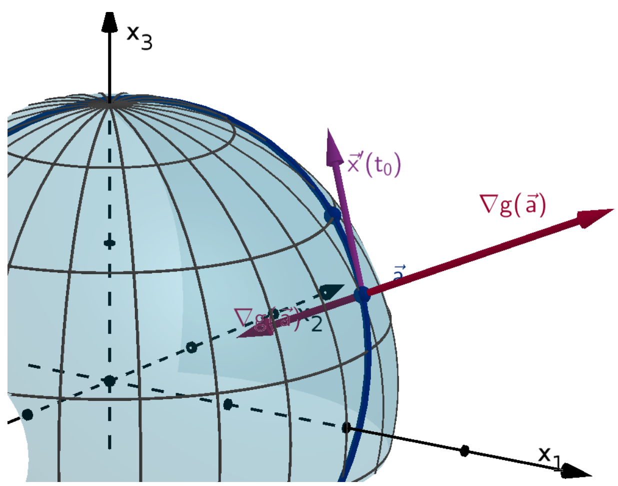


Figure 2.10: Two possible $\nabla g(\vec{a})$ orthogonal to all possible tangent vectors $\vec{x}'(t_0)$ in the level set $g(\vec{x}) = 0$.

We can use Corollary 2.5 to generalize this to an n-variable function. The level set is an $(n-1)$ dimensional graph in n-space (as long as the gradient is nonzero). It has two normal directions, pointing in opposite directions.

We can summarize what we have learned about the gradients of f and g as follows.

- $\nabla f(\vec{a})$ is normal to the level set $g(\vec{x}) = 0$ at a local maximizer $\vec{a}$.
- $\nabla g(\vec{x})$ is normal to the level set $g(\vec{x}) = 0$ at all points on the level set.
- If $\nabla g(\vec{a}) \neq \vec{0}$, then there are only two normal directions to the level set $g(\vec{x}) = 0$, pointing in opposite directions.

Vectors in the same or opposite directions are **parallel**. Algebraically, they are scalar multiples of each other. We conclude that at any local maximizer or local minimizer, $f(\vec{a}) = -\lambda \nabla g(\vec{x})$ for some λ. This is exactly what the first-order condition of the Lagrangian requires.

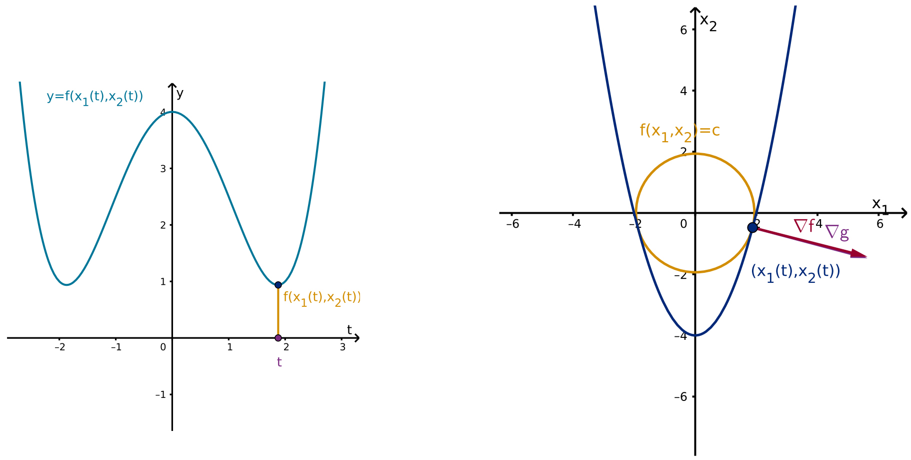
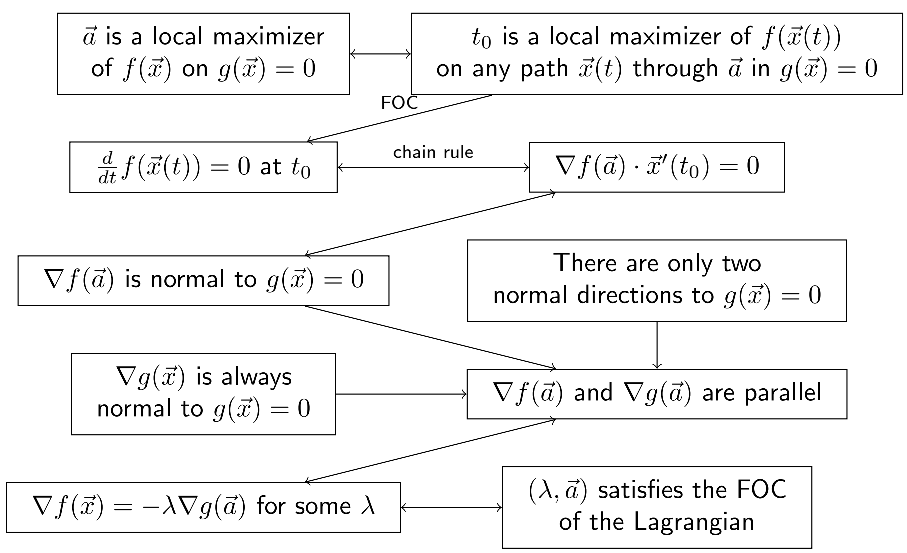


Figure 2.11: Parallel vectors $\nabla f(\vec{x})$ and $\nabla g(\vec{x})$ at a minimizer of $f(\vec{x})$ subject to $g(\vec{x}) = 0$

This line of reasoning is complex. It is also important. We will revisit and generalize it later. Here is a reminder of the steps of our argument.



2.1.7 The Tangency Method

This argument also validates the tangency method of locating a maximizer. Theorem 2.10 applies to f as well as to g. $\nabla f(\vec{a})$ is normal to the level set $f(\vec{x}) = c$ where $c = f(\vec{a})$. At a local maximizer, where $\nabla f(\vec{a})$ and $\nabla g(\vec{a})$ are parallel, it must follow that the level sets $g(\vec{x}) = 0$ and $f(\vec{x}) = c$ have the same normal directions. This implies that $g(\vec{x}) = 0$ and $f(\vec{x}) = c$ are tangent to each other.

Corollary 2.11

Suppose both $\nabla f(\vec{a})$ and $\nabla g(\vec{a})$ exist and are not $\vec{0}$. If $\vec{a}$ is a maximizer of $f(\vec{x})$ subject to $g(\vec{x}) = 0$, then the level sets of f and g that contain $\vec{a}$ are tangent to each other at $\vec{a}$.

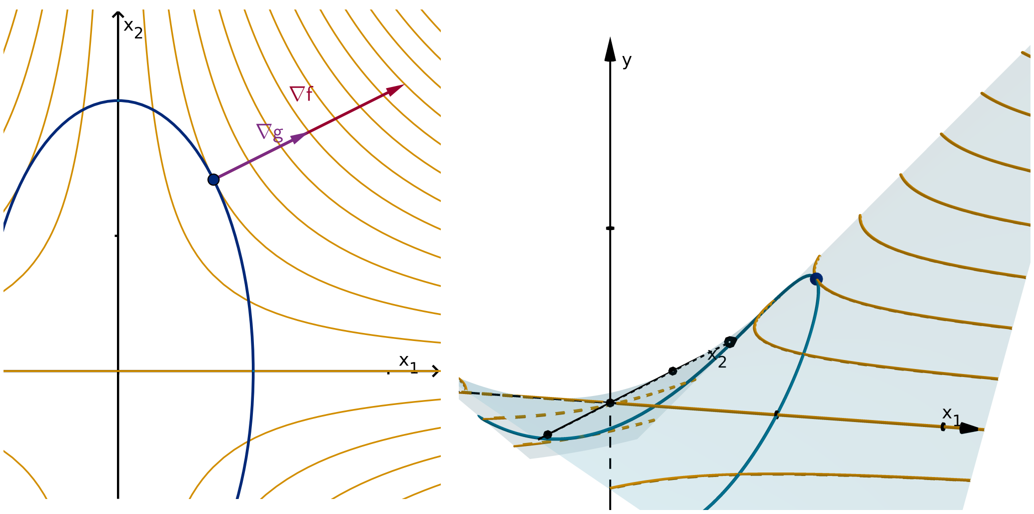


Figure 2.12: Parallel ∇f and ∇g and their tangent level curves at a local maximizer

2.1.8 Section Summary

The important definitions and results from this section were

- The effect of strengthening a constraint (Lemma 2.1)
- The definition of a level set (Definition 2.4)
- The Lagrangian (Definition 2.6)
- The first-order condition of the Lagrangian (Theorem 2.7)
- The direction of the gradient (Theorem 2.10)

2.1.8 Section Summary

- The tangency of level curves (Corollary 2.11)

2.2 Inequality Constraints

Goals:

1. Identify maximizers subject to inequality constraints.
2. Use the Lagrange multiplier to identify points that cannot be maximizers or minimizers.
3. Recognize several notations for complementary slackness.

2.2.1 Inequality Constraints and Upper Level Sets

Definition 2.12

An **inequality constraint** is a constraint of the form $g(\vec{x}) \geq 0$. With such a constraint we are solving

$$\max_{\vec{x} \in D} f(\vec{x}) \text{ subject to } g(\vec{x}) \geq 0 \text{ or}$$

$$\max_{\vec{x} \in S} f(\vec{x}) \text{ where } S = \{\vec{x} \in D \ : \ g(\vec{x}) \geq 0\}$$

Remark

Like with an equality constraint, just about any non-strict inequality can be rewritten in the form $g(\vec{x}) \geq 0$. Even an expression $g(\vec{x}) \leq k$ is equivalent to $k - g(\vec{x}) \geq 0$.

Here are two prototypical applications of inequality constraints in economics.

Example

If you have a budget to buy two types of goods, but you are not required to spend all of it, then your constraint is

$$I - p_1 x_1 - p_2 x_2 \geq 0$$

Example

If x_1 denotes a physical quantity then it is often appropriate to assume

$$x_1 \geq 0.$$

Inequalities like these define the familiar shape of the budget set.

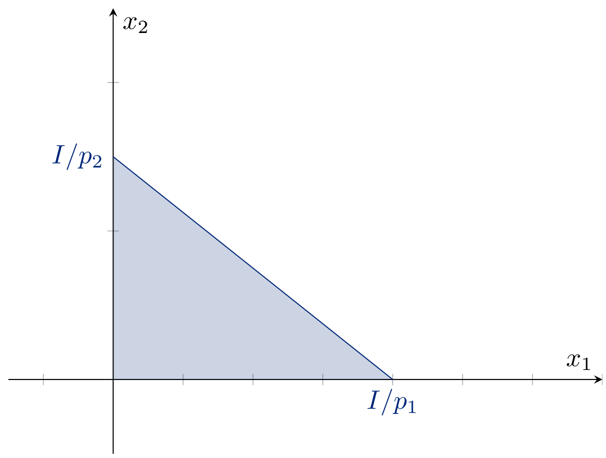


Figure 2.13: The region constrained by $I - p_1x_1 - p_2x_2 \geq 0$, $x_1 \geq 0$ and $x_2 \geq 0$

The set of points that satisfies an inequality constraint generally lies on one side of the level set $g(\vec{x}) = 0$. Graphically, it is the subset of the domain over which the graph $y = g(\vec{x})$ is above the plane $y = 0$. We therefore adopt the following terminology.

Definition 2.13

Given a multivariate function $f(\vec{x})$ a **upper level set** of f is the set of points that, for some number c, satisfy $f(\vec{x}) \geq c$. The points that satisfy $f(\vec{x}) \leq c$ are a **lower level set**.

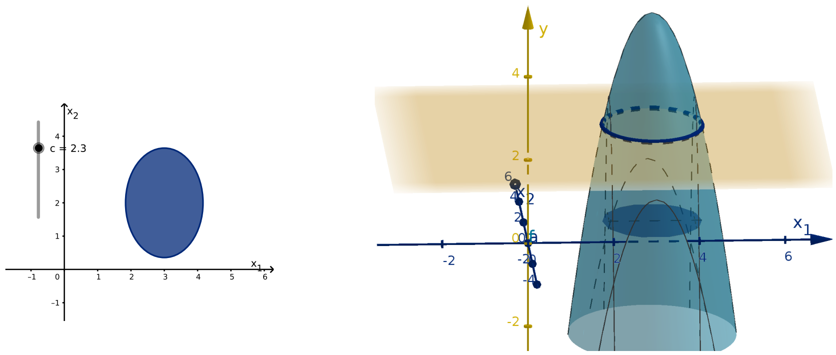


Figure 2.14: The upper level set where $y = f(\vec{x})$ lies above $y = c$

We can use the chain rule to determine whether a path travels into an upper level set or lower level set.

Lemma 2.14

Suppose $\vec{a}$ lies in the level set $f(\vec{x}) = c$, $\nabla f(\vec{a}) \neq \vec{0}$, and $\vec{x}(t)$ is a path that passes through $\vec{a}$ at t_0.

- If $\vec{x}'(t_0)$ makes an acute angle with $\nabla f(\vec{a})$, then immediately after t_0, $\vec{x}(t)$ travels into the upper level set $f(\vec{x}) \geq c$.
- If $\vec{x}'(t_0)$ makes an obtuse angle with $\nabla f(\vec{a})$, then immediately after t_0, $\vec{x}(t)$ travels into the lower level set $f(\vec{x}) \leq c$.

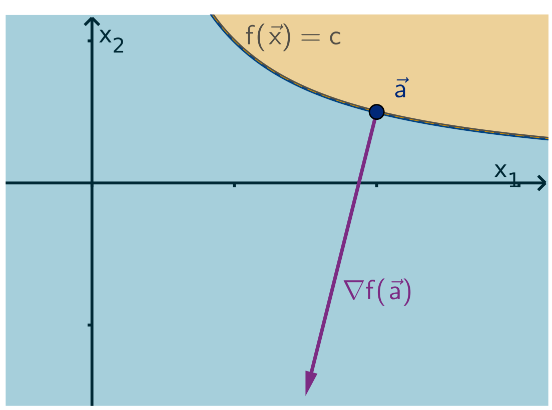


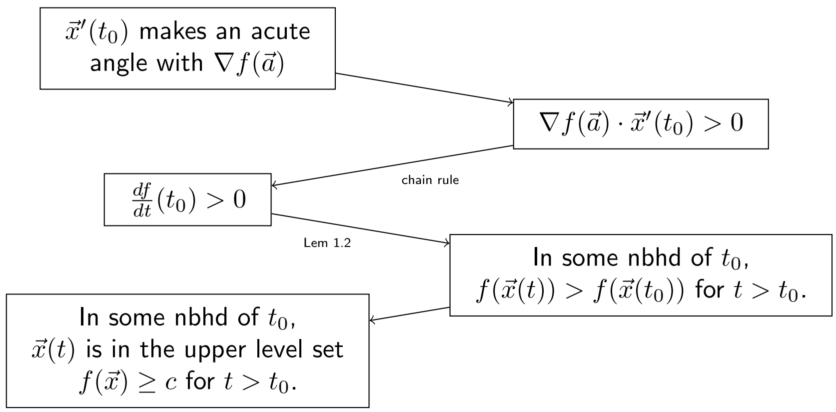
Figure 2.15: The gradient vector, the upper level set and the lower level set of f

The gradient vector makes an acute angle with itself, so we obtain the following characterization immediately.

Corollary 2.15

Suppose $f(\vec{a}) = c$. If $\nabla f(\vec{a}) \neq \vec{0}$, then $\nabla f(\vec{a})$ points into the upper level set $f(\vec{x}) \geq c$.

We can visualize the reasoning behind Lemma 2.14. Here is the case where the angle is acute.



2.2.2 Binding and Nonbinding Constraints

We already have the tools to impose necessary conditions on a maximizer subject to an inequality constraint. Our strategy is to consider two cases and to apply the appropriate first-order conditions to each one.

Suppose $\vec{a}$ is a local maximizer of $f(\vec{x})$ subject to $g(\vec{x}) \geq 0$.

1. If $g(\vec{a}) = 0$ we say that the constraint is **binding**.

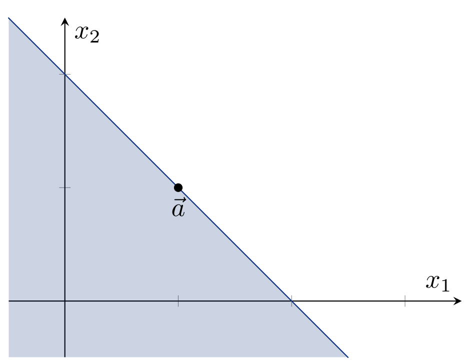


Figure 2.16: A maximizer $\vec{a}$ on $g(\vec{x}) = 0$

If $\vec{a}$ is a maximizer subject to $g(\vec{x}) \geq 0$, then it also maximizes f subject to $g(\vec{x}) = 0$. It must satisfy the FOC of the Lagrangian.

2 If $g(\vec{a}) > 0$ then the constraint is **non-binding**.

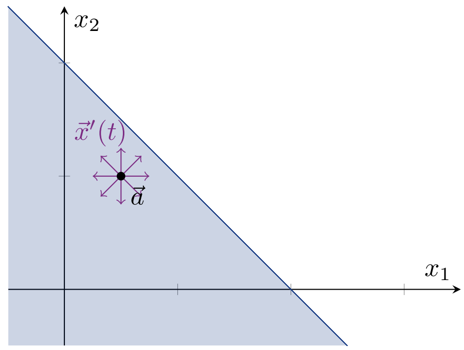


Figure 2.17: A maximizer $\vec{a}$ in $g(\vec{x}) > 0$ and some directions of travel through $\vec{a}$

We can travel in all directions $\vec{x}'(t)$ through $\vec{a}$ without immediately leaving the upper level set. To ensure $\nabla f(\vec{a}) \cdot \vec{x}'(t) = 0$, $\nabla f(\vec{a})$ must be the zero vector.

At an interior point, the boundary is not relevant to local measurements like the derivative. The conditions of optimization are those of an unconstrained optimization, as if the constraint did not exist.

2.2.3 The Sign of λ at a Maximizer

If the inequality constraint $g(\vec{x}) = 0$ binds at local maximizer $\vec{a}$, then $\vec{a}$ will satisfy the first-order condition of the Lagrangian

$$\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) + \lambda g(\vec{x})$$

What can we say about the λ that goes with $\vec{a}$?

Any path in the direction of $\nabla f(\vec{a})$ points toward higher values of f. If $\nabla f(\vec{a})$ points into the region $g(x) \geq 0$, then $\vec{a}$ cannot be a local maximizer subject to $g(\vec{x}) \geq 0$. This can be tested by the sign of λ.

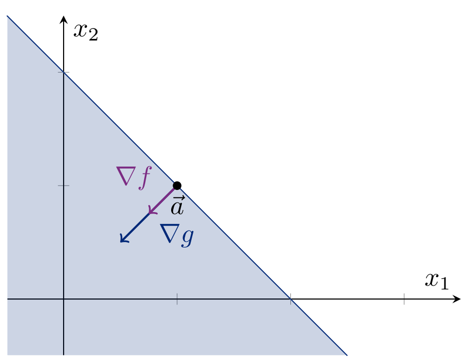


Figure 2.18: A non-maximizing critical point $\vec{a}$ on $g(\vec{x}) = 0$

The first-order condition of the Lagrangian requires that $\nabla f(\vec{a}) = -\lambda \nabla g(\vec{a})$. $\nabla g(\vec{a})$ points into the upper level set $g(\vec{x}) \geq 0$. If λ is negative then $\nabla f(\vec{a})$ points into this set as well.

Lemma 2.16

Suppose that $(\vec{a}, \lambda)$ satisfies the first-order condition of the Lagrangian of f subject to $g(\vec{x}) = 0$. If $\vec{a}$ is a local maximizer of $f(\vec{x})$ subject to $g(\vec{x}) \geq 0$, then either

1. $\lambda \geq 0$ or
2. $\nabla g(\vec{a}) = \vec{0}$

Understanding the intuition behind this result is probably more important than knowing a formal proof. Still, here is a proof. Like the argument we have already given, it is a proof of the contrapositive.

Proof

If $\nabla g(\vec{a}) = \vec{0}$ then the lemma is satisfied. We will consider the case where $\nabla g(\vec{a}) \neq \vec{0}$ and $\lambda < 0$. We will show that $\vec{a}$ is not a local maximizer.

Consider the line through $\vec{a}$ in the direction of $\nabla g(\vec{a})$. Its equation is $\vec{x}(t) = \vec{a} + t\nabla g(\vec{a})$ and $\vec{x}'(0) = \nabla g(\vec{a})$. By Lemma 2.14, since $\nabla g(\vec{a}) \cdot \vec{x}'(0) = \nabla g(\vec{a}) \cdot \nabla g(\vec{a}) > 0$, this line travels into the upper level set $g(\vec{x}) \geq 0$ at $t = 0$.

What happens to values of f along this line? Consider the composition $f(\vec{a} + t\nabla g(\vec{a}))$. Its derivative at $t = 0$ is

$$\begin{aligned} \frac{df}{dt}(0) &= \nabla f(\vec{a}) \cdot \nabla g(\vec{a}) \\ &= (-\lambda \nabla g(\vec{a})) \cdot \nabla g(\vec{a}) \\ &= -\lambda(\nabla g(\vec{a}) \cdot \nabla g(\vec{a})) \\ &> 0 \end{aligned}$$

By Lemma 1.2, there is a neighborhood of 0 where for $t > 0$, $f(\vec{x}(t)) > f(\vec{x}(0)) = f(\vec{a})$. We have seen that these points lie in the upper level set, so $\vec{a}$ cannot be a maximizer of $f(\vec{x})$ subject to $g(\vec{x}) \geq 0$.

Since $\vec{a}$ cannot be a maximizer when $\lambda < 0$, we conclude that if $\vec{a}$ is a maximizer, then $\lambda \geq 0$.

Remark

Showing that $\vec{a}$ is not a maximizer required us to compare it to points in the upper level set $g(\vec{x}) \geq 0$. This reasoning does not work when maximizing subject to an equality constraint. The line we used does not lie in $g(\vec{x}) = 0$. We can thus pout no restriction on λ when maximizing subject to $g(\vec{x}) = 0$. λ can be positive, negative, or zero at a maximizer.

Whenever we have an inequality constraint, there are two cases to check: it can be binding or non-binding. We could keep track of this manually, but there is an easier way to organize our search. We can use the Lagrangian

$$\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) + \lambda g(\vec{x})$$

to test for both the binding and non-binding case.

$g(\vec{x}) \geq 0$ binding:

- A maximizer $\vec{a}$ satisfies the first-order condition of the Lagrangian.
- This automatically checks $g(\vec{a}) = 0$.
- We can check that $\lambda \geq 0$.

$g(\vec{x}) \geq 0$ nonbinding:

- We can set $\lambda = 0$
- Now a maximizer $\vec{a}$ satisfies the first-order condition of $\mathcal{L}(\vec{x}, \lambda) = f(\vec{x})$.
- We can check that $g(\vec{a}) \geq 0$.

There is an interesting algebraic symmetry between the role of λ and $g(\vec{a})$. The following definition allows us to combine both these cases into one statement.

Definition 2.17

A point satisfies two non-strict inequalities **with complementary slackness**, if at least one of them holds with equality.

We can cover both cases of an inequality constraint by demanding that $g(\vec{x}) \geq 0$ and $\lambda \geq 0$ hold with complementary slackness. Here are a few different ways to denote complementary slackness.

Notations for Complementary Slackness

1. $g(\vec{x}) \geq 0$ and $\lambda \geq 0$ hold with complementary slackness
2. $g(\vec{x}) \geq 0$, $\lambda \geq 0$ and $\lambda g(\vec{x}) = 0$
3. $\lambda \geq 0$ and $g(\vec{x}) \geq 0$, $= 0$ if $\lambda > 0$

2.2.5 Conditions for a Maximizer Subject to an Inequality Constraint

With complementary slackness in hand, we can write a necessary condition for a maximizer on an inequality constraint. This theorem is written using notation 2, but we could rewrite it with one of the other notations.

Theorem 2.18

Let $f(\vec{x})$ and $g(\vec{x})$ be continuously differentiable functions. If $\vec{a}$ is a local maximizer of $f(\vec{x})$ subject to the constraint $g(\vec{x}) \geq 0$, then one of the following must be true

1. there is some number λ such that $(\vec{a}, \lambda)$ satisfies
   - $\frac{\partial \mathcal{L}}{\partial x_i}(\vec{a}, \lambda) = 0$ for all i
   - $g(\vec{a}) \geq 0$, $\lambda \geq 0$, and $\lambda g(\vec{a}) = 0$.
2. $g(\vec{a}) = 0$ and $\nabla g(\vec{a}) = 0$.

2.2.6 Opimization Subject to an Inequality Constraint

Find the maximizer of $f(x_1, x_2) = 10 - x_1^2 - x_2^2$ subject to $3x_1 + 4x_2 \geq 25$.

Solution

First we write our constraint in the correct form. $g(x_1, x_2) = 3x_1 + 4x_2 - 25 \geq 0$. Note that $\nabla g(x_1, x_2)$ is never $\vec{0}$, so any critical point must satisfy our complementary slackness conditions. The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = 10 - x_1^2 + x_2^2 + \lambda(3x_1 + 4x_2 - 25)$$

The conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 0$$

$$\lambda \geq 0 \qquad 3x_1 + 4x_2 - 25 \geq 0 \qquad \lambda(3x_1 + 4x_2 - 25) = 0$$

First consider the $\lambda = 0$ case. The constraint is nonbinding. We solve the equations then check the inequality.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 0 & \frac{\partial \mathcal{L}}{\partial x_2} &= 0 & \lambda &= 0 \\
-2x_1 + 3\lambda &= 0 & -2x_2 + 4\lambda &= 0 & & \\
-2x_1 &= 0 & -2x_2 &= 0 & & \\
x_1 &= 0 & x_2 &= 0 & & \\
\text{check } 3(0) + 4(0) - 25 &\geq 0 & & & &
\end{aligned}$$

The only critical point we found failed the check. There is no maximizer where the constraint is nonbinding. Now consider the $g(\vec{x}) = 0$ case.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 0 & \frac{\partial \mathcal{L}}{\partial x_2} &= 0 & g(\vec{x}) &= 0 \\
-2x_1 + 3\lambda &= 0 & -2x_2 + 4\lambda &= 0 & 3x_1 + 4x_2 - 25 &= 0 \\
x_1 &= \frac{3}{2}\lambda & x_2 &= 2\lambda & & \\
& & & & \frac{9}{2}\lambda + 8\lambda - 25 &= 0 \\
& & & & \frac{25}{2}\lambda &= 25 \\
& & & & \lambda &= 2 \\
x_1 &= \frac{3}{2}(2) & x_2 &= 2(2) & & \\
x_1 &= 3 & x_2 &= 4 & & \\
\text{check } \lambda &\geq 0 & & & & \\
2 &\geq 0 & & & &
\end{aligned}$$

The point $(3, 4, 2)$ passes all of these conditions. We conclude that $(3, 4)$ is the only possible maximizer. Since this is not a sufficient condition, we cannot conclude $(3, 4)$ is a maximizer. There may be no maximizer at all.

Main Idea

To solve for the maximizer over an inequality constraint

1. Pick a case from complementary slackness
2. Set up and solve the system of equations
3. Check whether the solutions satisfy the inequalities
4. Repeat for the other case from complementary slackness

2.2.7 Section Summary

The important definitions and results from this section were

- The notation of an inequality constraint (Definition 2.12)
- The definition of an upper or lower level set (Definition 2.13)
- The sign of λ at a maximizer subject to an inequality constraint (Lemma 2.16)
- Complementary slackness (Definition 2.17)
- Conditions for a maximizer on an inequality constraint (Theorem 2.18)

2.3

The Kuhn-Tucker Conditions

Goals:

1. Solve for maximizers on multiple constraints using the Kuhn-Tucker conditions
2. Recognize what each case of the Kuhn-Tucker conditions is checking

2.3.1

Multiple Equality Constraints

In economics, it is easy to imagine a firm is constrained by more than one equation:

- A budget for capital and labor
- Availability of labor or other inputs
- A government regulation
- You can't purchase or produce a negative amount

What if we want to maximize an n-variable function $f(\vec{x})$ subject to more than one equality constraint?

$$g_1(\vec{x}) = g_2(\vec{x}) = \cdots = g_m(\vec{x}) = 0$$

The feasible set, is the intersection of the level sets $g_j(\vec{x}) = 0$. We have seen that a single level set is usually one dimension less than the space it lies in. The intersection of m level sets will usually be an $(n-m)$-dimensional shape.

Notation

The g_i are different functions, not partial derivatives of g. Whenever we have multiple g's around, we will need to write our partial derivatives using ∂ notation, for instance

$$\frac{\partial}{\partial x_2} g_3(\vec{x}) \text{ not } g_{3_2}(\vec{x})$$

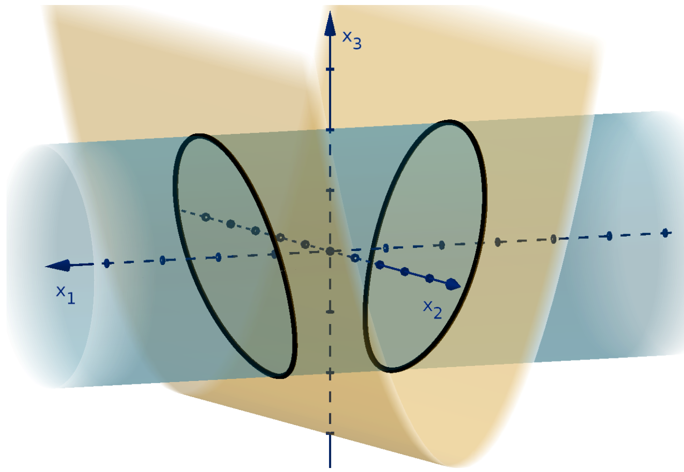


Figure 2.19: Two level sets intersecting at a pair of (one-dimensional) curves

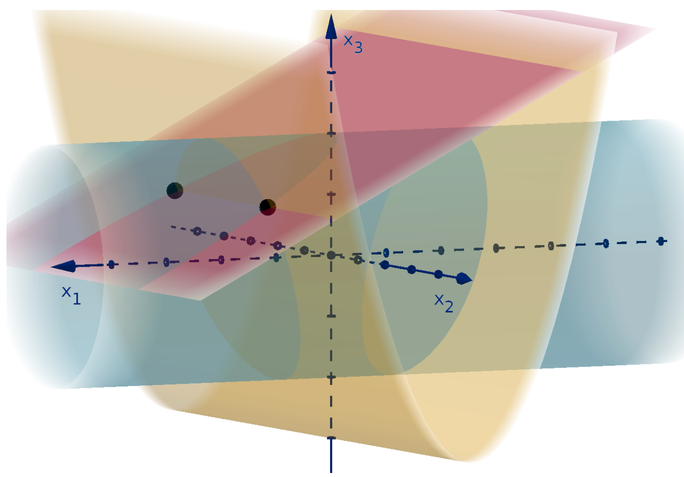


Figure 2.20: Three level sets intersecting at a pair of (zero-dimensional) points

Like with a single constraint, there are exceptions. Also like a single constraint, these exceptions rely on the inadequacy of the gradient vector(s). The following definition come from linear algebra. A corollary of the multivariable implicit function theorem characterizes the intersection of the level sets.

Definition 2.19

Vectors $\vec{v}_1$, $\vec{v}_2$, $\ldots \vec{v}_m$ are **linearly dependent** if there are constants k_j, not all 0, such that

$$\sum_{j=1}^{m} k_j \vec{v}_j = \vec{0}$$

If no such k_j exist then they are **linearly independent**

Equivalently, vectors are linearly dependent, if one of them is a linear combination of the others.

Corollary 2.20

Let $g_1(\vec{x}), g_2(\vec{x}), \ldots g_m(\vec{x})$ be a continuously differentiable functions at $\vec{a}$. If $\vec{a}$ lies on the level sets $g_j(\vec{x}) = c_j$ and the $\nabla g_j(\vec{a})$ are linearly independent, then the intersection of the level sets $g_j(\vec{x}) = c_j$ is a $(n-m)$-dimensional shape in some neighborhood of $\vec{a}$. Specifically, it is the graph of m differentiable functions on $\mathbb{R}^{n-m}$.

Remark

Any set containing the zero vector is linearly dependent. Thus this corollary agrees with our criterion for a single constraint. If the vector $\nabla g(\vec{a}) = \vec{0}$, then it is linearly dependent.

Two vectors are linearly dependent if they are parallel. With parallel gradients, the intersection of m level sets can have fewer than $(n-m)$ dimensions.

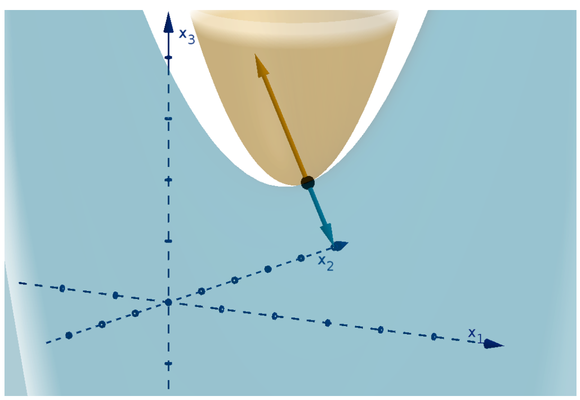


Figure 2.21: Two level sets with parallel gradients intersecting at a point

Parallel gradients can also occur when the level sets overlap, in this case the intersection can have more than $(n-m)$ dimensions.

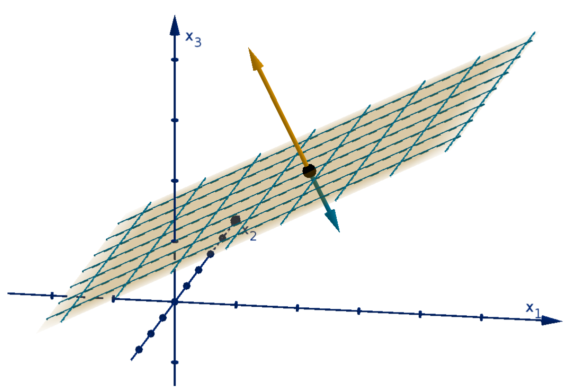


Figure 2.22: Two (overlapping) level sets whose intersection is a plane

2.3.1 Multiple Equality Constraints

Three vectors are linearly dependent if they are coplanar. Coplanar gradient vectors can also lead to intersections of unexpected dimension.

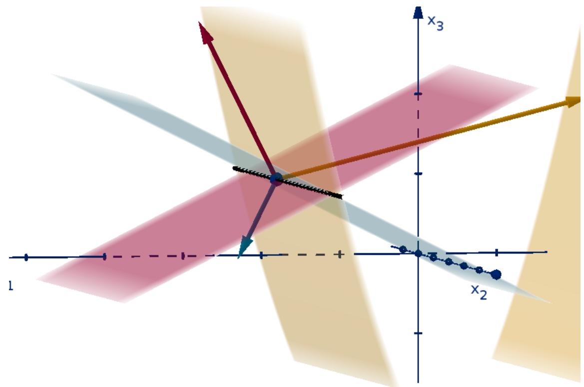


Figure 2.23: Three level sets with coplanar gradients intersecting on a line

Main Idea

Each additional equality constraint reduces the dimension of the feasible set by 1, unless its gradient vector is a linear combination of the existing gradient vectors.

2.3.2 The Lagrangian for Multiple Equality Constraints

For multiple constraints, we write a Lagrangian that uses all of the constraints. We need a different variable λ for each one. The first-order condition of this Lagrangian is a necessary condition for a local maximizer or local minimizer.

Definition 2.21

Given the problem of maximizing an n-variable objective function $f(\vec{x})$ subject to constraints $g_j(\vec{x}) = 0$ for $1 \leq j \leq m$, the **Lagrangian** is a function of the n components of $\vec{x}$ and m variables $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$.

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) + \sum_{j=1}^{m} \lambda_j g_j(\vec{x})$$

Theorem 2.22

If $\vec{a}$ is a local maximizer or local minimizer of $f(\vec{x})$ subject to constraints $g_j(\vec{x}) = 0$, all continuously differentiable at $\vec{a}$, then either

1. $(\vec{a}, \vec{\lambda})$ satisfies the first-order condition of $\mathcal{L}$ for some $\vec{\lambda}$ or
2. The $\nabla g_j(\vec{a})$ are linearly dependent

Notation

When developing the theory, we write our constraint functions and our Lagrange multipliers using a common letter and an index variable

$$g_1(\vec{x}), g_2(\vec{x}), \ldots \qquad \lambda_1, \lambda_2, \ldots$$

This makes it easier to generalize to any number of constraints, because can use Σ notation in the Lagrangian. For a small number of constraints, using an index is inconvenient. Economists often use different letters for each constraint and multiplier.

$$g(\vec{x}), h(\vec{x}), \ldots \qquad \lambda, \mu, \ldots$$

2.3.3 A Multi-Constraint Optimization

Find the maximizer of $f(x_1, x_2, x_3) = 3x_2$ on the constraints $x_1^2 + x_3^2 - 50 = 0$ and $x_1 + x_2 + x_3 = 0$.

Solution

The Lagrangian is

$$\mathcal{L}(x_1, x_2, x_3, \lambda_1, \lambda_2) = 3x_2 + \lambda_1(x_1^2 + x_3^2 - 50) + \lambda_2(x_1 + x_2 + x_3)$$

The first-order condition is

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_3} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0$$

$$2\lambda_1 x_1 + \lambda_2 = 0 \qquad 3 + \lambda_2 = 0 \qquad 2\lambda_1 x_3 + \lambda_2 = 0 \qquad x_1^2 + x_3^2 - 50 = 0 \qquad x_1 + x_2 + x_3 = 0$$

$$\lambda_2 = -3$$

$$x_1 = \frac{3}{2\lambda_1} \qquad x_3 = \frac{3}{2\lambda_1}$$

$$x_1 = x_3$$

$$x_1^2 + x_1^2 - 50 = 0$$

$$x_1 = \pm 5$$

$$\pm 5 = x_3$$

$$\pm 5 + x_2 \pm 5 = 0$$

$$x_2 = \mp 10$$

We conclude that $(\pm 5, \mp 10, \pm 5)$. The value $f(x_1, x_2, x_3) = 3x_2$ is larger when $x_2 = 10$. We conclude that no point besides $(-5, 10, 5)$ can be the maximizer of f on these constraints. Because this is not a sufficient condition, we cannot be sure that this is the maximizer. It may be that no maximizer exists at all.

We can visualize the maximizer of $f(x_1, x_2, x_3) = 3x_2$ by looking for the point in the level sets that is farthest in the x_2 direction. It appears that such a point exists at $(-5, 10, -5)$.

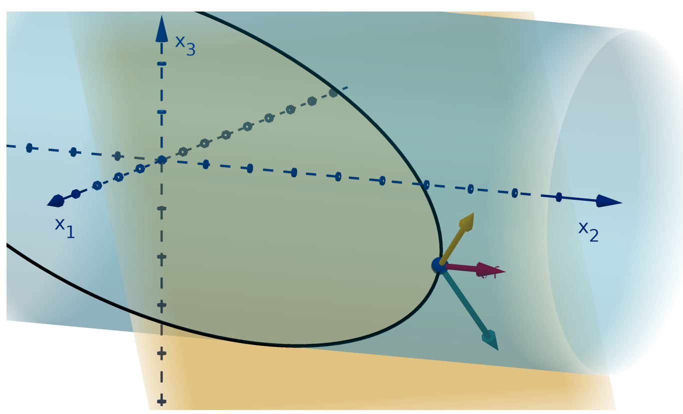


Figure 2.24: The gradients of $x_1^2 + x_3^2 - 50$, $x_1 + x_2 + x_3 = 0$, and $f(\vec{x}) = 3x_2$ at the maximizer

The partial derivatives of the Lagrangian are

$$\begin{aligned}\mathcal{L}_{x_i} =& \frac{\partial f}{\partial x_i}(\vec{x}) + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}(\vec{x}) \\ \mathcal{L}_{\lambda_j} =& g_j(\vec{x})\end{aligned}$$

Any solution $(\vec{a}, \vec{\lambda})$ to the first-order condition on $\mathcal{L}$ satisfies

- $f_{x_i}(\vec{a}) = -\sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}(\vec{a})$ for all i, so $\nabla f(\vec{a}) = -\sum \lambda_j \nabla g_j(\vec{a})$.
- $g_j(\vec{a}) = 0$ for all j so $\vec{a}$ lies on all the constraints.

We can examine the relationship between the gradients using the same procedure we used for a single constraint. We take a path $\vec{x}(t)$ through $\vec{a}$ at t_0. We assume that $\vec{x}(t)$ lies in the feasible set. This means it lies in all of the level sets $g_j(\vec{x}) = 0$.

If $\vec{a}$ is a local maximizer or minimizer of $f(\vec{x})$ on the feasible set then it must be a local maximizer or minimizer of the composition $f(\vec{x}(t))$ for all paths $\vec{x}(t)$ in the feasible set. This means the t-value corresponding to $\vec{a}$ (we will call it t_0) must satisfy the first-order condition of $f(\vec{x}(t))$. The derivative calculation should be familiar:

$$\begin{aligned}0 &= \frac{df}{dt}(t_0) \\ 0 &= \nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t_0) \\ 0 &= \nabla f(\vec{a}) \cdot \vec{x}'(t_0)\end{aligned}$$

Thus $\nabla f(\vec{a})$ is normal to the feasible set.

When we had a single constraint, we argued that there were two opposite normal directions to the feasible set, and $\nabla g(\vec{a})$ pointed in one of them. This relied on the fact that our feasible set was a level set and had dimension $(n-1)$. If we assume that the $\nabla g_j(\vec{a})$ are linearly independent, then the feasible set has dimension $(n-m)$ at $\vec{a}$.

There is an entire m-dimensional space of vectors normal to the feasible set at $\vec{a}$. If $\vec{a}$ is a maximizer, $\nabla f(\vec{a})$ must be a vector in this space.

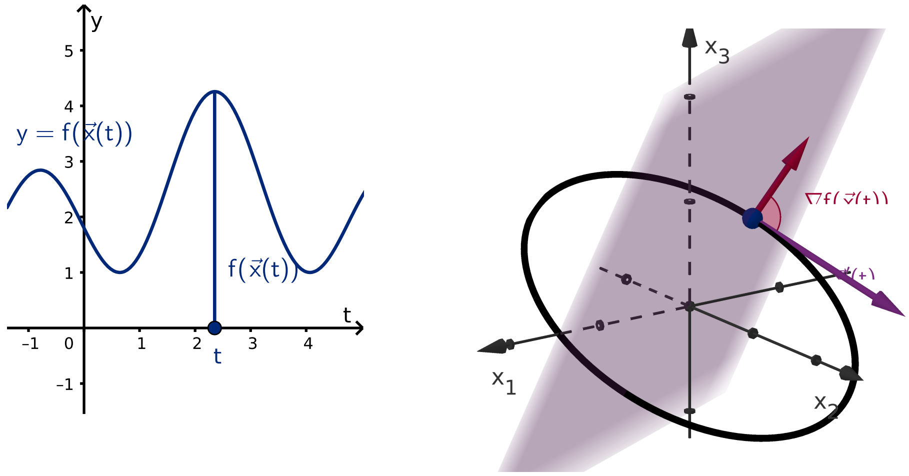


Figure 2.25: The gradient vector of the objective function and the normal plane of the feasible set at a local maximizer of the composition: $f(\vec{x}(t))$

The feasible set lies in each level set $g_j(\vec{x}) = 0$. The gradient of each g_j is normal its level set. We can put together what we know about the gradients and the shape of the feasible set.

- $f(\vec{a})$ is normal to the feasible set at a local maximizer a.
- Each $\nabla g_j(\vec{x})$ is normal to the feasible set at all points in the level set.
- If the $\nabla g_j(\vec{a})$ are linearly independent, then the normal space is m-dimensional.

m independent vectors in a m-dimensional space must span the space. We conclude that $\nabla f(\vec{a})$ is a linear combination of the $\nabla g_j(\vec{a})$. This is what the first-order condition of the Lagrangian requires.

Example

If $\vec{a}$ is a maximizer of a three-variable function subject to two equality constraints, then $\nabla f(\vec{a})$ must lie in the **normal plane**, which is spanned by $\nabla g_1(\vec{a})$ and $\nabla g_2(\vec{a})$.

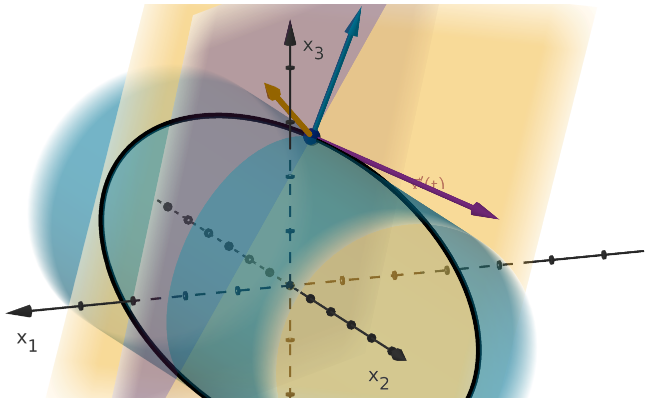


Figure 2.26: Two gradients spanning the normal plane of the feasible set

2.3.5 An Optimization with Dependent Gradients

When the gradients of the $g_j(\vec{x})$ are linearly dependent, the feasible set may not have the expected dimension of $(n-m)$. Even if it does, the gradient vectors need not span the normal space. In this case, calculus arguments cannot rule out a point from being a maximizer. Here is an example where insisting upon the first-order condition of the Lagrangian would incorrectly rule out a maximizer.

Consider the smooth function $f(x_1, x_2) = x_1^2 + x_2^2$ and the constraints $x_2 + 2 = 0$ and $x_2 - (x_1 - 3)^2 + 2 = 0$. If we attempt to apply the first-order condition we obtain the following.

- The constraints intersect only at $(3, -2)$.
- At $(3, -2)$ the gradients are

$$\nabla g_1(3,-2) = (0,1) \qquad \nabla g_2(3,-2) = (0,1) \qquad \nabla f(3,-2) = (6,-4).$$

  ∇f cannot be written as a linear combination of ∇g_1 and ∇g_2

Even though it doesn't satisfy the first-order condition, $(3, -2)$ must be the maximizer (and minimizer), since it is the only point that satisfies both constraints.

2.3.5 An Optimization with Dependent Gradients

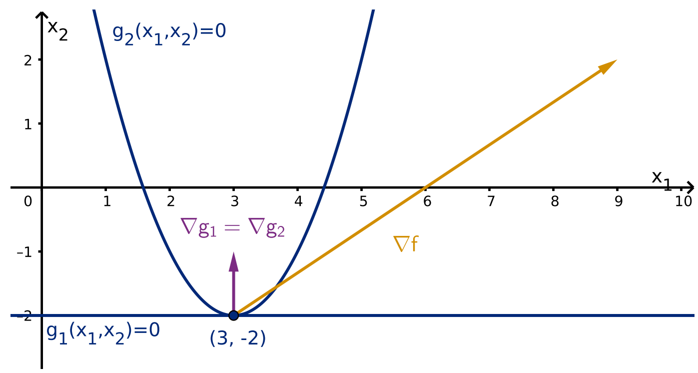


Figure 2.27: $(3, 2)$ is a maximizer, but ∇f cannot be written $\lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$.

If we count dimensions, the feasible set is a 0-dimensional subspace of $\mathbb{R}^n$. Its normal space is $2 - 0 = 2$-dimensional, which suggests that $\nabla f(3, -2)$ can be any vector in $\mathbb{R}^2$. This sounds bizarre but actually makes sense. $(3, -2)$ is the entire feasible set. It must be a maximizer, no matter what $\nabla f(3, -2)$ is. The exception for dependent gradients in Theorem 2.22 exists to avoid ruling out a maximizer in a situation like this.

2.3.6 The Kuhn-Tucker Conditions

The Kuhn-Tucker conditions are a robust set of necessary conditions for constrained optimizations with inequality constraints, potentially in addition to some number of equality constraints. They combine all of the ideas we have developed in this chapter. We will use the same Lagrangian that we would for equality constraints.

A Lagrangian For Multiple Equality and Inequality Constraints

Given an objective function $f(\vec{x})$ and constraints of the forms $g_j(\vec{x}) \geq 0$ or $g_j(\vec{x}) = 0$ the **Lagrangian** is

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) + \sum_{j=1}^{m} \lambda_j g_j(\vec{x}).$$

Like with a single inequality constraint, complementary slackness will remove the $\lambda_j g_j(\vec{x})$ when $g_j(\vec{x})$ does not bind.

Theorem 2.23 [The Kuhn-Tucker Conditions]

Given the objective function $f(\vec{x})$ and constraints of the forms $g_j(\vec{x}) \geq 0$ or $g_j(\vec{x}) = 0$, then at any local maximizer $\vec{a}$ one of the following must be true

1. There is some vector $\vec{\lambda}$ such that $(\vec{a}, \vec{\lambda})$ satisfies the **Kuhn-Tucker conditions**:
   - For each variable x_i,
     $$\mathcal{L}_{x_i}(\vec{a}, \vec{\lambda}) = 0$$
   - For each equality constraint function g_j,
     $$g_j(\vec{a}) = 0$$
   - For each inequality constraint function g_j,
     $$g_j(\vec{a}) \geq 0 \quad \text{and} \quad \lambda_j \geq 0 \quad \text{and} \quad \lambda_j g_j(\vec{a}) = 0$$
2. The binding $\nabla g_j(\vec{a})$ are linearly dependent

2.3.7 Solving the Kuhn-Tucker Conditions

According to Kuhn-Tucker, what points (x_1, x_2) could be maximizers of $f(x_1, x_2) = x_1 x_2^2$ given the constraints

$$12 - x_1 - 4x_2 \geq 0 \qquad\qquad 9 - x_1 - x_2 \geq 0$$

Solution

The Lagrangian is $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1 x_2^2 + \lambda_1(12 - x_1 - 4x_2) + \lambda_2(9 - x_1 - x_2)$. The Kuhn-Tucker conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} = x_2^2 - \lambda_1 - \lambda_2 &= 0 & \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 2x_1x_2 - 4\lambda_1 - \lambda_2 &= 0 \\ 12 - x_1 - 4x_2 &\geq 0 & 9 - x_1 - x_2 &\geq 0 \\ \lambda_1 &\geq 0 & \lambda_2 &\geq 0 \\ \lambda_1(12 - x_1 - 4x_2) &= 0 & \lambda_2(9 - x_1 - x_2) &= 0 \end{aligned}$$

The final equations carry our complementary slackness conditions. There are two ways to choose which factor is 0 for each, giving us $2 \times 2 = 4$ cases to check.

1 $\lambda_1 = 0$, $\lambda_2 = 0$

$$x_2^2 - \lambda_1 - \lambda_2 = 0 \qquad 2x_1x_2 - 4\lambda_1 - \lambda_2 = 0$$

$$x_2^2 = 0 \qquad 2x_1x_2 = 0$$

$$x_2 = 0$$

check $12 - x_1 - 4x_2 \geq 0$ $\qquad$ $6 - x_1 - x_2 \geq 0$

$$12 - x_1 \geq 0 \qquad 6 - x_1 \geq 0$$

$$x_1 \leq 12 \qquad x_1 \leq 9$$

So $(k, 0, 0, 0)$ satisfies the Kuhn-Tucker conditions for $k \leq 9$.

2 $\lambda_1 = 0$, $9 - x_1 - x_2 = 0$

$$x_2^2 - \lambda_1 - \lambda_2 = 0 \qquad 2x_1x_2 - 4\lambda_1 - \lambda_2 = 0 \qquad 9 - x_1 - x_2 = 0$$

$$x_2^2 - \lambda_2 = 0 \qquad 2x_1x_2 - \lambda_2 = 0$$

$$x_2^2 = 2x_1x_2$$

$$x_2^2 - 2x_1x_2 = 0$$

$$x_2(x_2 - 2x_1) = 0$$

$x_2 = 0$ covered in case 1

or $x_2 = 2x_1$

$$9 - x_1 - 2x_1 = 0$$

$$x_1 = 3$$

$$x_2 = 6$$

$$6^2 - \lambda_2 = 0$$

$$\lambda_2 = 36$$

check $12 - x_1 - 4x_2 \geq 0$ $\qquad$ $\lambda_2 \geq 0$

$$12 - 3 - 4(6) \geq 0 \qquad 36 \geq 0$$

$$-15 \geq 0$$

$(3, 6, 0, 36)$ does not satisfy the inequalities.

3 $12 - x_1 - 4x_2 = 0,\ \lambda_2 = 0$

$$x_2^2 - \lambda_1 - \lambda_2 = 0 \qquad 2x_1x_2 - 4\lambda_1 - \lambda_2 = 0 \qquad 12 - x_1 - 4x_2 = 0$$

$$x_2^2 - \lambda_1 = 0 \qquad 2x_1x_2 - 4\lambda_1 = 0$$

$$4x_2^2 = 4\lambda_1 \qquad 2x_1x_2 = 4\lambda_1$$

$$4x_2^2 = 2x_1x_2$$

$$4x_2^2 - 2x_1x_2 = 0$$

$$2x_2(2x_2 - x_1) = 0$$

$x_2 = 0$ covered in case 1

or $2x_2 = x_1$

$$12 - 2x_2 - 4x_2 = 0$$

$$x_2 = 2$$

$$x_1 = 4$$

$$2^2 - \lambda_1 = 0$$

$$\lambda_2 = 4$$

check $9 - x_1 - x_2 \geq 0$ $\qquad \lambda_1 \geq 0$

$$9 - 4 - 2 \geq 0 \qquad 4 \geq 0$$

$(4, 2, 4, 0)$ satisfies the Kuhn-Tucker conditions

4 $12 - x_1 - 4x_2 = 0,\ 9 - x_1 - x_2 = 0$

$$x_2^2 - \lambda_1 - \lambda_2 = 0 \qquad 2x_1x_2 - 4\lambda_1 - \lambda_2 = 0 \qquad 12 - x_1 - 4x_2 = 0 \qquad 9 - x_1 - x_2 = 0$$

$$12 - x_1 - 4x_2 = 0$$

$$-(9 - x_1 - x_2) = -0$$

$$3 - 3x_2 = 0$$

$$x_2 = 1$$

$$9 - x_1 - 1 = 0$$

$$x_1 = 8$$

$$1 - \lambda_1 - \lambda_2 = 0 \qquad 16 - 4\lambda_1 - \lambda_2 = 0$$

$$-(16 - 4\lambda_1 - \lambda_2) = -0$$

$$-15 + 3\lambda_1 = 0$$

$$\lambda_1 = 5$$

$$16 - 4(5) - \lambda_2 = 0$$

$$\lambda_2 = -4$$

check $\lambda_1 \geq 0$ $\qquad \lambda_2 \geq 0$

$$5 \geq 0 \qquad -4 \geq 0$$

$(8, 1, 5, -4)$ does not satisfy the Kuhn-Tucker conditions.

The Kuhn-Tucker conditions are necessary. No point except $(4, 2)$ or $(k, 0)$ for $k \leq 9$ can be a maximizer.

2.3.7 Solving the Kuhn-Tucker Conditions

By evaluating we see

$$f(4,2) = 16 \qquad\qquad f(k,0) = 0$$

So only $(4,2)$ could be a maximizer. We cannot conclude that it is a maximizer, until we learn a relevant sufficient condition.

Strategy

In general our method for solving the Kuhn Tucker conditions is

- Pick an equality from a factor of each complementary slackness condition.
- Solve the resulting system of equations.
- Discard any solutions that violate the remaining inequalities.
- Repeat for a different choice of equalities.

With m inequality constraints, we will need to repeat for all 2^m combinations of equalities.

2.3.8 Visualizing the Kuhn-Tucker Conditions

Each choice of equalities mutates the Kuhn-Tucker conditions into the necessary conditions from Theorem 2.22, where the binding $g_j(\vec{x}) \geq 0$ are treated as equality constraints. The following diagram shows how each choice corresponds to a different piece of the feasible set, where different constraints bind.

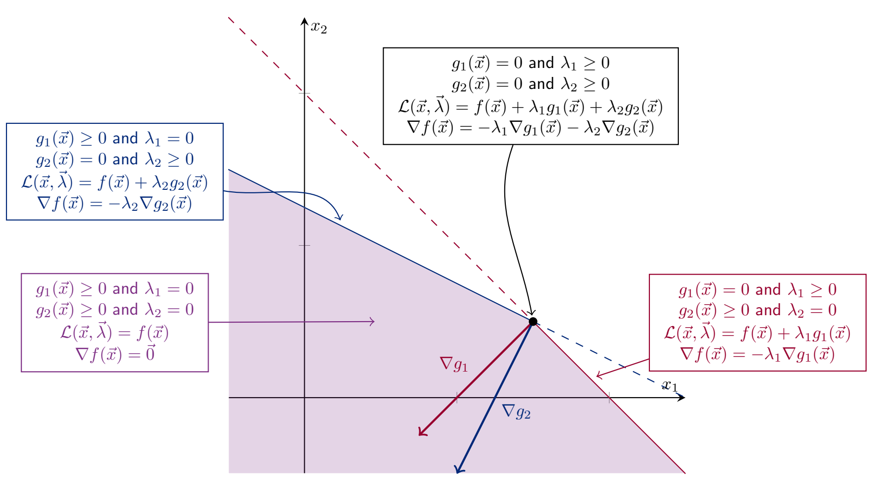


Figure 2.28: The four regions covered by two pairs of inequalities with complementary slackness

In the case of multiple binding inequality constraints, the condition that $\lambda_j \geq 0$ forces $\nabla f(\vec{a})$ to lie in the cone made by $-\nabla g_j(\vec{x})$.

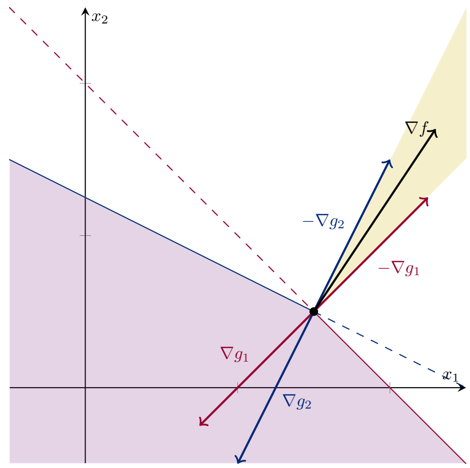


Figure 2.29: The cone where ∇f must lie if both constraints bind at a maximizer

2.3.9 Proving the Necessity of the Kuhn-Tucker Conditions

To justify the necessity of the Kuhn-Tucker conditions, we can list each condition and describe why it must apply.

1. A given $\vec{a}$ will satisfy some binding equations $g_j(\vec{x}) = 0$.
2. For the constraints that do not bind, setting $\lambda_j = 0$ turns $\mathcal{L}(\vec{x}, \vec{\lambda})$ into the Lagrangian for f with only the binding equality constraints. If $\vec{a}$ is a local maximizer, it must satisfy Theorem 2.22, specifically, $\frac{\partial \mathcal{L}}{\partial x_i}(\vec{a}) = 0$.
3. For each inequality constraint, a maximizer must satisfy $g_j(\vec{a}) \geq 0$ to be feasible.
4. Finally, if every path from $\vec{a}$ into the feasible region decreases f, then all $\lambda_j \geq 0$.

We have not given a convincing argument for the last statement yet. A formal proof is just below, but it may be more illuminating to convince yourself graphically. Try drawing a few ∇f that lie outside the cone between $-\nabla g_1$ and $-\nabla g_2$ in the previous figure. For each one, you should be able to identify a vector $\vec{x}'(t)$ that points into the feasible region but makes an acute angle with ∇f.

Here is a formal proof that $\lambda_j \geq 0$ at a maximizer.

Proof

Pick any inequality constraint $g_k(\vec{x}) \geq 0$. If $g_k(\vec{x}) \geq 0$ is not binding at $\vec{a}$, then $\lambda_k = 0$ and we are done.

We will therefore consider the case where $g_k(\vec{a}) \geq 0$ is binding and show $\lambda \geq 0$. Let S be the intersection of all the binding constraints **except** $g_k(\vec{x}) = 0$. Since the gradients of the binding constraints a linearly independent, we can conclude:

- The gradients of the binding constraints other than $\nabla g_k(\vec{a})$ span the normal space of S at $\vec{a}$
- $\nabla g_k(\vec{a})$ is not a linear combination of these gradients, so it does not lie in the normal space of S at $\vec{a}$

There must be a path $\vec{x}(t)$ in S through $\vec{a}$ such that $\vec{x}'(t_0)$ is not orthogonal to $\nabla g_k(\vec{a})$. We can pick such a path so that $\nabla g_k(\vec{a}) \cdot \vec{x}'(t_0) > 0$. If the first path we try produces a negative dot product, just traverse $\vec{x}(t)$ backwards instead. First we show that, in some neighborhood of t_0, $\vec{x}(t)$ lies in the feasible region for all $t > t_0$. We check that it satisfies each constraint.

- $\vec{a}$ lies in the interior of each upper level set $g_j(\vec{x}) \geq 0$ for each nonbinding g_j. $\vec{x}(t)$ will travel for some distance before leaving the upper level set.
- Since $\vec{x}(t)$ was chosen to lie in S it lies in the level set $g_j(\vec{x}) = 0$ for all binding g_j except g_k.
- Since $\nabla g_k(\vec{a}) \cdot \vec{x}'(t_0) > 0$, $\vec{x}(t)$ must travel into the upper level set $g_k(\vec{x}) \geq 0$.

Thus in some neighborhood of t_0, $\vec{x}(t)$ lies in the feasible set for $t > t_0$. Since $\vec{a}$ is a local maximizer, $f(\vec{x}(t_0)) \geq f(\vec{x}(t))$ for $t > t_0$ in this neighborhood. Thus $\frac{df}{dt}(t_0) \leq 0$. We use the chain rule to examine this inequality.

$$\begin{aligned}
\frac{df}{dt}(t_0) &\leq 0 \\
\nabla f(\vec{a}) \cdot \vec{x}'(t_0) &\leq 0 \\
-\sum_{j=1}^{m} \lambda_j \nabla g_j(\vec{a}) \cdot \vec{x}'(t_0) &\leq 0
\end{aligned}$$

We pause to examine the terms of this summation. Most are 0. Here is the reasoning.

- For nonbinding $g_j(\vec{x})$, $\lambda_j = 0$.
- For binding $g_j(\vec{x})$ except $j = m$, we have $\nabla g_j(\vec{a}) \cdot \vec{x}'(t_0) = 0$, since $\nabla g_j(\vec{a})$ is a normal vector of S and $\vec{x}(t)$ lies in S.
- Finally, $\nabla g_k(\vec{a}) \cdot \vec{x}'(t_0) > 0$ by our choice of $\vec{x}(t)$.

We apply these to our inequality.

$$\begin{aligned}
-\sum_{j=1}^{m} \lambda_j \nabla g_j(\vec{a}) \cdot \vec{x}'(t_0) &\leq 0 \\
-\lambda_k \nabla g_k(\vec{a}) \cdot \vec{x}'(t_0) &\leq 0 \\
\lambda_k &\geq 0
\end{aligned}$$

If we demand that $x_i \geq 0$ for each i, then we have added n new constraints to our Langrangian. We will use μ_i for the Lagrange multipliers of the x_i.

$$\mathcal{L}(\vec{x}, \vec{\lambda}, \vec{\mu}) = f(\vec{x}) + \sum_{j=1}^{m} \lambda_j g_j(\vec{x}) + \sum_{i=1}^{n} \mu_i x_i$$

This is unwieldy. If we are clever, we can do better.

For $1 \leq k \leq n$, the inequality conditions for μ_k are

$$\frac{\partial \mathcal{L}}{\partial \mu_k} = x_k \geq 0 \text{ and } \mu_k \geq 0$$

- If the constraint $x_k \geq 0$ is not binding, then $\mu_k = 0$. The $\mu_k x_k$ term is 0 in $\mathcal{L}$ and its partial derivatives. We can remove it, but we will still need to verify that $x_k \geq 0$ is satisfied.
- If the constraint is binding, then we can still remove the $\mu_k x_k$ term from L.
    - The term goes to 0 anyway in $\mathcal{L}_{x_i}$ for $i \neq k$.
    - In $\mathcal{L}_{x_k}$, there is a single $+\mu_k$ term. $\mathcal{L}_{x_k}$ is supposed to be 0, whereas $\mu_k \geq 0$. We can replicate this effect by removing $\mu_k x_k$ from $\mathcal{L}$, but requiring that the remaining terms of $\mathcal{L}_{x_k}$ have a sum less than or equal to 0.

For each k, we can remove the variable μ_k and the $\mu_k x_k$ term from our Lagrangian in exchange for some new conditions.

2.3.10 Kuhn-Tucker with Non-Negativity Constraints

Corollary 2.24 [The Kuhn-Tucker Conditions with Non-Negativity]

Given the objective function $f(\vec{x})$ and constraints of the forms $g_j(\vec{x}) \geq 0$ or $g_j(\vec{x}) = 0$, along with $x_i \geq 0$ for each i, then at any local maximizer $\vec{a}$ one of the following must be true.

1. There is some vector $\vec{\lambda}$ such that $(\vec{a}, \vec{\lambda})$ satisfies the **Kuhn-Tucker Conditions with Non-Negativity Constraints**:
   - For each i,
     $$a_i \geq 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial x_i}(\vec{a}, \vec{\lambda}) \leq 0 \quad \text{and} \quad a_i \frac{\partial \mathcal{L}}{\partial x_i}(\vec{a}, \vec{\lambda}) = 0$$
   - For each equality constraint function g_j,
     $$g_j(\vec{a}) = 0$$
   - For each inequality constraint function g_j,
     $$g_j(\vec{a}) \geq 0 \quad \text{and} \quad \lambda_j \geq 0 \quad \text{and} \quad \lambda_j g_j(\vec{a}) = 0$$
2. The binding constraints (including any $x_i = 0$) have linearly dependent gradients at $\vec{a}$.

2.3.11 Section Summary

The most important definitions and results from this section were

- The shape of an intersection of level sets (Corollary 2.20)
- The Lagrangian for multiple constraints (Definition 2.21)
- First-order condition for multiple equality constraints (Theorem 2.22)
- The Kuhn-Tucker conditions (Theorem 2.23)
- The Kuhn-Tucker conditions with non-negativity (Corollary 2.24)

Chapter 3

Comparative Statics

3.1 The Implicit Function Theorem

Goals:

1. Identify when one variable of an implicit equation can be written as a function of the other(s)
2. Compute the derivative of one variable with respect to another
3. Determine how the optimal choice responds to changes in an exogenous parameter

3.1.1 Comparative Statics

Comparative statics study situations with one or more **choice variables** x_i and one or more **exogenous parameters** α_j.

Example

A consumer maximizing her utility has the choice variable

- q: the quantity of a product they buy

and the exogenous parameter

- p: the price of that product

Example

From the point of view of a producer, the choice variables might be

- q: the quantity of a product they produce
- p: the price they sell for

the exogenous parameters may include

- w, r: the price of labor or capital
- t, s: a tax or subsidy imposed by the government
- some aspect of the demand function

Remark

The choice variables are the variables whose value is chosen by the agent who wants to maximize the function.

In economics, we assume that choosers are rational and well-informed. We assume they will learn the value of the parameters. After that, they will pick the value of the choice variables that maximizes their objective function. **Comparative statics** ask how the outcome changes as the value of the parameter changes. We will present the tools to compute two types of comparative statics

1. How does the optimal value of a choice variable change as a parameter changes?
2. How does the value of the objective function change as a parameter changes?

Notation

Given an objective function $f(x, \alpha)$, with choice variable x and parameter α, we expect that different values of α will lead the chooser to pick different values of x. The chooser's optimal choice is a function of α that we write $x^*(\alpha)$.

The first-order condition tells us that for each α, $x^*(\alpha)$ must satisfy the first-order condition:

$$f_x(x^*(\alpha), \alpha) = 0$$

Without an expression for $x^*(\alpha)$, trying to understand its rate of change raises a sequence of questions.

1. What is $\frac{dx^*(\alpha)}{d\alpha}$?

3.1.1 Comparative Statics

2. How do we know that $x^*(\alpha)$ is differentiable?
3. How do we know that $x^*(\alpha)$ is even a function?

Given a specific f, we can solve the first-order condition algebraically. If we can tell which solution is the maximizer (assuming one exists), then we can write an expression for x^* as a function of α and answer all these questions. In the case of a general or abstract f, this may not be possible. Fortunately, mathematics has the vocabulary to describe these questions in abstract. It also has a powerful tool to answer them.

3.1.2 Explicit Functions and Implicit Equations

Definition 3.1

An **explicit function** equates a function or dependent variable to an expression entirely in terms of the independent variables.

Example

The following equations describe explicit functions.

$$\begin{aligned} f(x) &= x^3 - \sqrt{x+7} \\ y &= \sin(x^2) \\ z &= 3xy + x^2 - 2 \\ g(x_1, x_2, x_3) &= e^{x_1} \cos(x_2 x_3) \end{aligned}$$

Since each input has at most one output, graphs of explicit functions pass the vertical line test. We also have a variety of tools for differentiating explicit functions (though not all are differentiable).

We do not have the luxury of always working with explicit functions. We use the following vocabulary when we wish to draw a contrast with explicit functions.

Definition 3.2

An **implicit equation** in two or more variables does not necessarily have any dependent variable which is equated to an expression in the others.

Sometimes we can solve an implicit equation to obtain an explicit function, but other times we cannot.

Example

$$2x + 3y = 12 \implies y = 4 - \frac{2}{3}x$$

$$x_1^2 + x_2^2 + x_3^2 = 25 \implies x_3 = \pm\sqrt{25 - x_1^2 - x_2^2} \quad \text{(multiple outputs for each input, not a function)}$$

$$13 - 3y^3 + xy^4 = y^5 \implies x = \frac{y^5 - 13 + 3y^3}{y^4} \quad \text{(can solve for } x \text{ but not for } y\text{)}$$

$$x^3 + y^3 = 6xy \implies y = \ ??? \quad \text{(requires the cubic formula, not a function)}$$

The solutions to any implicit equation form a level set of some function. We obtain the function by rewriting the equation in a form like:

$$F(x, y) = c \qquad \text{or} \qquad F(\vec{x}) = c$$

3.1.3 An Implicit Equation

Consider the implicit equation $x^3 + y^3 = 6xy$

a Write the equation in the form $F(x, y) = c$.

b Does this equation have a graph?

c Can we solve it to obtain an explicit function?

Solution

a We can write $x^3 + y^3 - 6xy = 0$

b Yes. The graph is the set of points (x, y) that satisfy the equation. Every equation has a graph, though some graphs are the empty set.

c We could look up the cubic formula and solve this. The cubic formula, like the quadratic formula, returns multiple values for the variable. The result is not a function. We can see this in the graph, which fails the vertical line test.

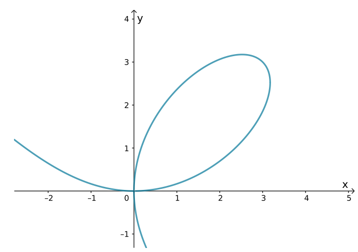


Figure 3.1: The graph of $x^3 + y^3 - 6xy = 0$

Sometimes we cannot write y as an explicit function of x in a way that describes the whole graph. If we are only interested in rates of change, we can restrict our attention to a small neighborhood of the graph. At most points, this neighborhood does look like the graph of some function $y = f(x)$.

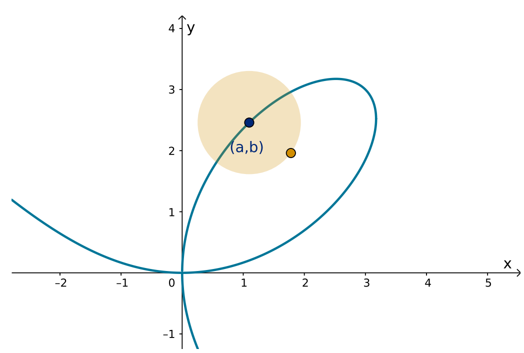


Figure 3.2: A neighborhood in which the graph $x^3 + y^3 - 6xy = 0$ is the graph of an explicit function

This is not always possible. Consider the part of $x^3 + y^3 - 6xy = 0$ near $(0, 0)$. In any neighborhood we choose, there are three branches of the graph that extend to the right. No matter how small a neighborhood we choose around $(0, 0)$, the graph will fail the vertical line test and cannot be written as $y = f(x)$.

The implicit function theorem tells us when a point on the graph of an implicit equation has a neighborhood that is identical to the graph of an explicit function. The basic version takes an implicit equation in two variables and writes a function that expresses one (the dependent variable) in terms of the other (the independent variable).

Theorem 3.3 [The Implicit Function Theorem]

Suppose $F(x,y)$ is a continuously differentiable function and (a,b) is a point on $F(x,y)=c$. If $F_y(a,b)\neq 0$, then there exists a differentiable function $f(x)$ such that $y=f(x)$ and $F(x,y)=c$ describe the same graph in some neighborhood of (a,b).

The theorem does not tell us what the function $f(x)$ is, only that it exists. Even so, we can express its derivatives in terms of F.

Corollary 3.4

The derivative of f with respect to x at a is given by

$$f'(a) = -\frac{F_x(a,b)}{F_y(a,b)}$$

The derivation of this formula is famous and not too difficult. To compute the derivative of f, we parameterize a path in the graph $y=f(x)$. Unlike our previous parametrizations, we will use x as the parameter. Differentiating x with respect to x is most palatable with Leibniz notation.

$$\begin{aligned} x &= x & \qquad \frac{dx}{dx} &= 1 \\ y &= f(x) & \qquad \frac{dy}{dx} &= f'(x) \end{aligned}$$

The points $(x,f(x))$ lie in $F(x,y)=c$ near (a,b). Thus the composition $F(x,f(x))$ is the constant function c. Differentiating $F(x,f(x))=c$ with respect to the parameter x to produces an equation that contains $f'(x)$. We solve this equation to obtain an expression for $f'(x)$.

$$
\begin{aligned}
F(x, f(x) &= c && (y = f(x) \text{ lies in } F(x,y) = c) \\
\frac{dF(x, f(x))}{dx} &= 0 && (\text{derivative of a constant is } 0) \\
F_x(x, f(x))\frac{dx}{dx} + F_y(x, f(x))\frac{df(x)}{dx} &= 0 && (\text{chain rule}) \\
F_x(x, f(x))(1) + F_y(x, f(x))f'(x) &= 0 && (\text{evaluate derivative of } x) \\
f'(x) &= -\frac{F_x(x, f(x))}{F_y(x, f(x))} && (\text{solve for } f'(x)) \\
f'(a) &= -\frac{F_x(a,b)}{F_y(a,b)} && (\text{evaluate at } x = a)
\end{aligned}
$$

The implicit function theorem guarantees that f exists and is differentiable in a neighborhood of a. Since we don't know how big this neighborhood is, $x = a$ is the only point at which we can be sure $f'(x)$ exists.

We often apply this formula before checking whether the implicit function theorem applies. Assuming F is continuously differentiable, the theorem will only fail when $F_y(a, b) = 0$. Conveniently, this formula will be undefined in that case.

We can also determine the derivative of f geometrically. Near (a, b), the graph $y = f(x)$ is the level set $F(x, y) = c$. We know the $\nabla F(a, b) = (F_x(a, b), F_y(a, b))$ is normal to the level set.

The gradient has a slope of $\frac{F_y(a,b)}{F_x(a,b)}$. The tangent line, which is perpendicular, has a negative reciprocal slope: $-\frac{F_x(a,b)}{F_y(a,b)}$. The slope of the tangent line is also the derivative $f'(a)$.

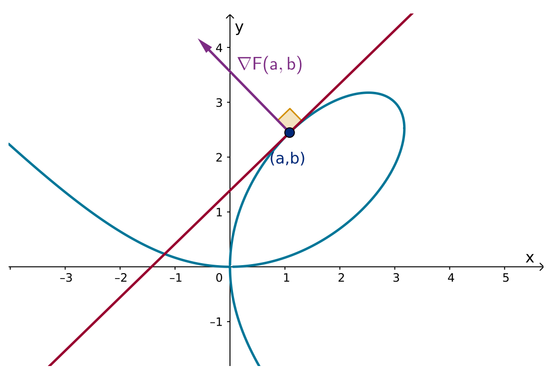


Figure 3.3: The gradient of F and the tangent line whose slope is $f'(a)$

If x and y satisfy $x^3 - 2xy^2 + 3y = -13$, show that y can be written as a function $f(x)$ of x near $(2,3)$ and compute $f'(2)$

Solution

We may first want to check that $(2,3)$ satisfies the implicit equation.

$$(2)^2 - 2(2)(3)^2 + 3(3) = -13$$

In order to apply the implicit function theorem, we need to know that $F(x,y) = x^3 - 2xy^2 + 3y$ has continuous partial derivatives. It is a polynomial, so it does. Finally, we need to show $F_y(2,3) \neq 0$.

$$\begin{aligned} F_y(x,y) &= -4xy + 3 \\ F_y(2,3) &= -4(2)(3) + 3 \\ F_y(2,3) &= -21 \neq 0 \end{aligned}$$

Therefore, by the implicit function theorem, there is a function $f(x)$ such that $y = f(x)$ and $x^3 - 2xy^2 + 3y = -13$ describe the same graph near $(2,3)$. The derivative of f at $x = 2$ is given by Corollary 3.4.

$$\begin{aligned} f'(2) &= -\frac{F_x(2,3)}{F_y(2,3)} \\ &= -\frac{3(2)^2 - 2(3)^2}{-4(2)(3) + 3} \\ &= -\frac{-6}{-21} \\ &= -\frac{2}{7} \end{aligned}$$

Figure 3.4: The graph of $x^3 - 2xy^2 + 3y = -13$ and its tangent line at $(2,3)$

3.1.6 The Derivative of the Optimal Choice

In comparative statics, we are interested in the function $x^*(\alpha)$, which is a solution to the equation $f_x(x, \alpha) = 0$. We apply the implicit function theorem where:

- α takes the role of the independent variable "x".
- x takes the role of the dependent variable "y".
- $f_x(x, \alpha)$ takes the role of the two-variable function "F".
- (b, a) is a point on the graph $f_x(x, \alpha) = 0$.
- The derivatives of F are second derivatives of f.

The implicit function theorem requires that

$$F_x(b, a) = f_{xx}(b, a) \neq 0.$$

It concludes there is a differentiable function $x^*(\alpha)$ such that $x = x^*(\alpha)$ matches the graph of $f_x(x, \alpha) = 0$ in a neighborhood of (b, a).

Corollary 3.5

Given a function $f(x, \alpha)$, suppose that

1. a is a value of α and $b = x^*(a)$
2. $x^*(\alpha)$ satisfies $f_x(x^*(\alpha), \alpha) = 0$ near (b, a)
3. $f(x, \alpha)$ has continuous second derivatives near (b, a)
4. $f_{xx}(b, a) \neq 0$

Then

$$\frac{dx^*(a)}{d\alpha} = -\frac{f_{x\alpha}(b, a)}{f_{xx}(b, a)}$$

This computes the derivative at a point. If, in some interval of α values, every point $(x^*(a), a)$ satisfies these conditions, then we can extend this to a derivative function for $x^*(\alpha)$.

$$\frac{dx^*(\alpha)}{d\alpha} = -\frac{f_{x\alpha}(x^*(\alpha), \alpha)}{f_{xx}(x^*(\alpha), \alpha)}$$

Remark

When working with comparative statics, we can reasonably assume the requirements of the implicit function theorem.

1. If an optimal choice exists, then it must satisfy the first-order condition.
2. It makes sense to work with smooth functions when modeling empirical data.
3. At a maximizer, f_{xx} is likely (though not required) to be negative. Alternately, assuming $f_{xx} < 0$ on the entire domain guarantees that $x^*(\alpha)$ is actually the global maximizer for each α.

3.1.7 The Construction of $f(x)$

Here we will give a construction of the function $y = f(x)$ that matches the graph of $F(x, y) = c$ near a point (a, b). The same argument works for more variables, but the pictures are harder to draw.

Constructing $f(x)$ requires the following tools:

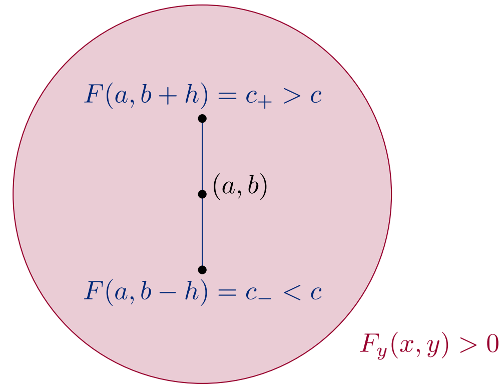
1. Lemma 1.11: If $f'(x) > 0$ then f is increasing.
2. Definition of continuity: The values of F can be kept arbitrarily close to $F(a, b)$ by restricting to points sufficiently close to (a, b).
3. The intermediate value theorem: If $f(x)$ is continuous and $f(a_-) < c < f(a_+)$ then there is a value k between a_- and a_+ such that $f(k) = c$.

The implicit function theorem requires that $F_y(a, b) \neq 0$. There are two cases to consider, but the arguments are analogous. We will consider the case $F_y(a, b) > 0$.

3.1.7 The Construction of $f(x)$

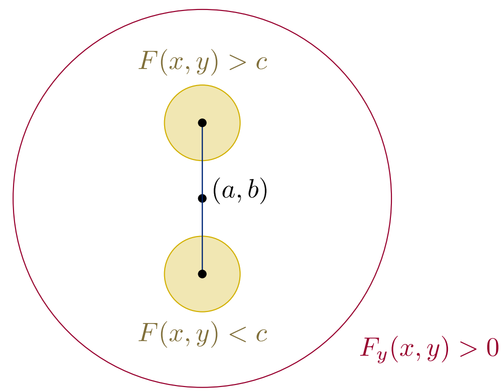
1. F is continuously differentiable, so there is a neighborhood of (a, b) where $F_y(x, y) > 0$.

2. Within this neighborhood, we can travel h units in the y-direction from (a, b). Let $F(a, b+h) = c_+$ and $F(a, b-h) = c_-$



3. Since $F_y(x, y) > 0$, we have $c_+ > c$ and $c_- < c$.

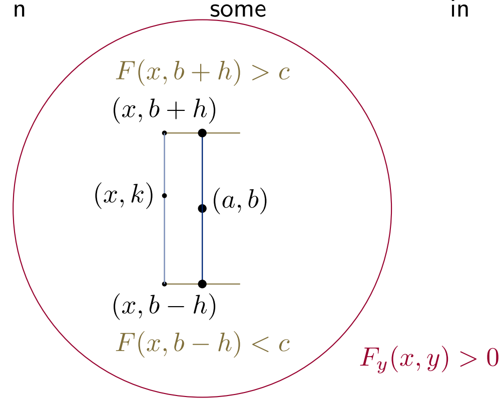
4. Since F is continuous, there is a neighborhood of $(a, b+h)$ where $F(x, y) > c$. There is a neighborhood of $(a, b-h)$ where $F(x, y) < c$.



n some in

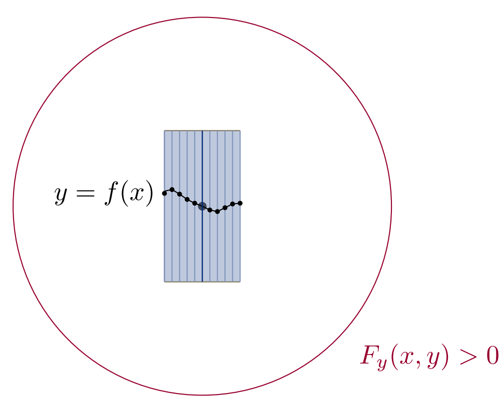
5. We consider segments from $(x, b+h)$ to $(x, b-h)$, with one endpoint in each neighborhood.

6. Apply the intermediate value theorem, since $F(x, b-h) < c < F(x, b+h)$, there is a k between $b-h$ and $b+h$ such that $F(x, k) = c$.



7. Since $F_y(x, y) > 0$, F is increasing along the segment, so it cannot take the value c more than once. The k in the previous step is unique.

8. Repeat this for every segment of the form $(x, b+h)$ to $(x, b-h)$, and define $f(x) = k$.



We can also apply the implicit function theorem to an implicit equation of $n > 2$ variables. In this case, one dependent variable can be expressed as a function of $n-1$ independent variables.

Theorem 3.6 [The Multivariable Implicit Function Theorem]

Suppose $F(\vec{x}, y)$ is a continuously differentiable function and $F(\vec{a}, b) = c$. If $F_y(\vec{a}, b) \neq 0$, then there exists a differentiable function $f(\vec{x})$ such that $y = f(\vec{x})$ and $F(\vec{x}, y) = c$ describe the same graph in some neighborhood of $(\vec{a}, b)$.

Since $f(\vec{x})$ is an $n-1$ variable function, the derivatives we can compute are the partial derivatives. The formula for these derivatives is analogous to the single-variable version.

Corollary 3.7

For each variable x_k, the partial derivative of f with respect to x_k at $\vec{a}$ is given by

$$f_{x_k}(\vec{a}) = -\frac{F_{x_k}(\vec{a}, b)}{F_y(\vec{a}, b)}$$

We can now justify our earlier characterization of level sets. At the time we made no mention of dependent and independent variables. This lack of distinction actually makes the implicit function theorem easier to apply.

Remark

There is nothing special about the letter y, nor the fact that it is the last variable of F. The variable "y" in the implicit function theorem can apply to any variable of an implicit equation, so long as the partial derivative with respect to that variable is not zero.

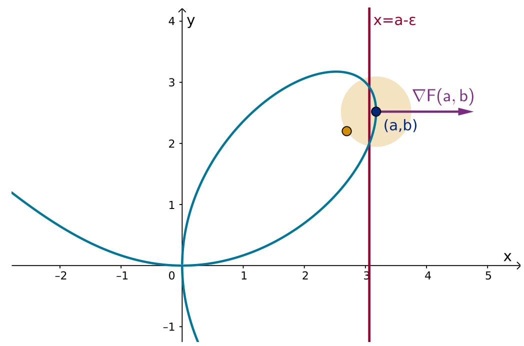


Figure 3.5: A point where $x^3 + y^3 - 6xy = 0$ cannot be written in the form $y = f(x)$ because it fails the vertical line test but can be rewritten as $x = f(y)$

If we are not picky about which variable is written as a function of the others, then the implicit function theorem only fails when all the partial derivatives are 0. As long as the gradient of the function is not the zero vector, one component can play the role of the dependent variable. This is exactly what our corollary requires.

Corollary 2.5

Let $g(\vec{x})$ be a continuously differentiable function at $\vec{a}$. If $\vec{a}$ lies on the level set $g(\vec{x}) = c$ and $\nabla g(\vec{a}) \neq \vec{0}$, then the level set $g(\vec{x}) = c$ is a $(n-1)$-dimensional shape in some neighborhood of $\vec{a}$. Specifically, it is the graph of a differentiable function of $n-1$ of the variables of $\mathbb{R}^n$.

The "differentiable function" is the function f produced by the implicit function theorem.

We can also apply this version of the implicit function theorem and its corollary to comparative statics. Consider a function of one choice variable and multiple parameters. We write this objective function as $f(x, \vec{\alpha})$. The implicit function theorem and its corollary can compute the partial derivatives of $x^*(\vec{\alpha})$.

Corollary 3.8

Given a function $f(x, \vec{\alpha})$, suppose that

1. $\vec{a}$ is a value of $\vec{\alpha}$ and $b = x^*(\vec{a})$.
2. $x^*(\vec{\alpha})$ satisfies $f_x(x^*(\vec{\alpha}), \vec{\alpha}) = 0$ near $(b, \vec{a})$
3. $f(x, \vec{\alpha})$ has continuous second derivatives near $(b, \vec{a})$
4. $f_{xx}(b, \vec{a}) \neq 0$

Then
$$\frac{\partial x^*(\vec{a})}{\partial \alpha_k} = -\frac{f_{x\alpha_k}(b, \vec{a})}{f_{xx}(b, \vec{a})}.$$

We can justify the multivariable implicit function and Corollary 3.7 using arguments similar to the single-variable versions. The construction of f is the same for both versions, except that $\vec{x}$ replaces x. The computation of the partial derivatives of f requires more adaptation.

Since $f_k(\vec{x})$ is a partial derivative, we treat x_k as a parameter. The other x_i are constants, held equal to the corresponding components of $\vec{a}$.

$$x_i = \begin{cases} a_i & \text{if } i \neq k \\ x_k & \text{if } i = k \end{cases} \qquad \frac{dx_i}{dx_k} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$$

$$y = f(\vec{x}) \qquad \frac{dy}{dx_k} = f_{x_k}(\vec{x})$$

The strategy is the same. We differentiate $F(\vec{x}, f(\vec{x}))$ with respect to x_k, solve for $f_{x_k}(\vec{x})$, and evaluate at $x_k = a_k$.

$$\begin{aligned}
F(\vec{x}, f(\vec{x})) &= c && ((\vec{x}, f(\vec{x})) \text{ lies in } F(\vec{x}, y) = c) \\
\frac{dF(\vec{x}, f(\vec{x}))}{dx_k} &= 0 && (\text{derivative of a constant is } 0) \\
\sum_{i=1}^{n} F_{x_i}(\vec{x}, f(\vec{x}))\frac{dx_i}{dx_k} + F_y(\vec{x}, f(\vec{x}))f_{x_k}(\vec{x}) &= 0 && (\text{chain rule}) \\
F_{x_k}(\vec{x}, f(\vec{x}))\frac{dx_k}{dx_k} + F_y(\vec{x}, f(\vec{x}))f_{x_k}(\vec{x}) &= 0 && (\frac{dx_i}{dx_k} = 0 \text{ for } i \neq k) \\
F_{x_k}(\vec{x}, f(\vec{x}))(1) + F_y(\vec{x}, f(\vec{x}))f_{x_k}(\vec{x}) &= 0 && (\text{evaluate derivative of } x_k) \\
f_{x_k}(\vec{x}) &= -\frac{F_{x_k}(\vec{x}, f(\vec{x}))}{F_y(\vec{x}, f(\vec{x}))} && (\text{solve for } f_{x_k}(\vec{x})) \\
f_{x_k}(\vec{a}) &= -\frac{F_{x_k}(\vec{a}, b)}{F_y(\vec{a}, b)} && (\text{evaluate at } \vec{x} = \vec{a})
\end{aligned}$$

3.1.9

The Implicit Function Theorem for Multiple Equations

We have seen two instances previously where the solution to multiple implicit equations was relevant.

1. The feasible set of multiple equality constraints is the intersection of multiple level sets.
2. Critical points of a multivariable function satisfy $f_{x_i}(\vec{x}) = 0$ for each i.

A graph of the form $y = f(\vec{x})$ in $\mathbb{R}^{n+1}$ will have dimension n. In general, each equation we wish to satisfy lowers the dimension of our space of solutions by 1. If we want to express an intersection of level sets, $y = f(\vec{x})$ will not have the right dimension.

The way to handle this loss of dimension in an explicit function is to increase the number of dependent variables. Specifically, if we have n-variables $\vec{x}$ and m-variables $\vec{y}$ then the graph of a family of functions $y_j = f_j(\vec{x})$ will have dimension n in $\mathbb{R}^{n+m}$

The most general version of the implicit function theorem states when a family of implicit equations can be expressed as a family of explicit functions instead.

Notation

Given a family of functions

$$\mathcal{F}(\vec{x},\vec{y}) = (F_1(\vec{x},\vec{y}), F_2(\vec{x},\vec{y}), \ldots, F_m(\vec{x},\vec{y}))$$

the derivative of $\mathcal{F}$ with respect to y_j is the family of functions

$$\frac{\partial \mathcal{F}}{\partial y_j}(\vec{x},\vec{y}) = \left(\frac{\partial F_1}{\partial y_j}(\vec{x},\vec{y}), \frac{\partial F_2}{\partial y_j}(\vec{x},\vec{y}), \ldots, \frac{\partial F_n}{\partial y_j}(\vec{x},\vec{y})\right)$$

Note that we are using subscripts here to indicate different components of the vector $\mathcal{F}$, not as partial derivatives.

Theorem 3.9 [The Implicit Function Theorem for Multiple Dependent Variables]

Suppose $\vec{y}$ is an m-vector and $\mathcal{F}(\vec{x},\vec{y})$ is a family of m continuously differentiable functions such that $\mathcal{F}(\vec{a},\vec{b}) = (c_1,\ldots,c_m)$. If the vectors $\frac{\partial \mathcal{F}}{\partial y_j}(\vec{a},\vec{b})$ are linearly independent, then there exists a family of differentiable functions $(f_j(\vec{x}))$ such that the equations $y_j = f_j(\vec{x})$ describe the same graph as $\mathcal{F}(\vec{x},\vec{y}) = (c_1,\ldots,c_m)$ in some neighborhood of $(\vec{a},\vec{b})$.

We can use the chain rule to solve for partial derivatives $\frac{\partial f_j(\vec{a},\vec{b})}{\partial x_k}$, but the derivative of any equation $F_j(\vec{x},\vec{y}) = c_j$ with respect to x_k will contain the derivatives of all the f_j. To compute the derivative we want, we need to differentiate all of the implicit equations and solve a system of equations. Here is the simplest example.

Example

Consider two implicit equations $F_1(x, y_1, y_2) = c_1$ and $F_2(x, y_1, y_2) = c_2$. Assuming $\mathcal{F}_{y_1}$ and $\mathcal{F}_{y_2}$ are linearly independent, the implicit function guarantees differentiable explicit functions $y_1 = f_1(x)$ and $y_2 = f_2(x)$. Differentiating the original implicit equations with respect to x gives

$$\frac{\partial F_1}{\partial x}\frac{dx}{dx} + \frac{\partial F_1}{\partial y_1}f_1'(x) + \frac{\partial F_1}{\partial y_2}f_2'(x) = 0$$
$$\frac{\partial F_2}{\partial x}\frac{dx}{dx} + \frac{\partial F_2}{\partial y_1}f_1'(x) + \frac{\partial F_2}{\partial y_2}f_2'(x) = 0$$

We can use this system to solve for $f_1'(x)$ and $f_2'(x)$.

3.1.10 The Derivative of Optimal Choice with Multiple Choice Variables

Suppose your utility is a function of two choice variables and one exogenous parameter.

$$u(x_1, x_2, \alpha)$$

Your optimal choices $x_1^*(\alpha)$ and $x_2^*(\alpha)$ satisfy the following implicit equations:

$$u_1(x_1^*(\alpha), x_2^*(\alpha), \alpha) = 0$$
$$u_2(x_1^*(\alpha), x_2^*(\alpha), \alpha) = 0.$$

By the implicit function theorem, we can write $x_1^*(\alpha)$ and $x_2^*(\alpha)$ as differentiable explicit functions of α if their derivatives with respect to x_1 and x_2 are linearly independent. Here are those derivatives.

$$\frac{\partial(u_1, u_2)}{\partial x_1}(x_1^*(\alpha), x_2^*(\alpha), \alpha) = (u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha), u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha))$$
$$\frac{\partial(u_1, u_2)}{\partial x_2}(x_1^*(\alpha), x_2^*(\alpha), \alpha) = (u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha), u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha))$$

These are the columns of the second upper left square minor of the Hessian of u. Columns of a matrix are independent, if the matrix has a nonzero determinant.

$$|Hu(x_1^*(\alpha), x_2^*(\alpha), \alpha)_2| = \begin{vmatrix} u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{vmatrix} \neq 0$$

Assuming this holds, we can use the chain rule to differentiate both implicit equations with respect to α. We obtain

$$u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha)\frac{dx_1^*(\alpha)}{d\alpha} + u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha)\frac{dx_2^*(\alpha)}{d\alpha} + u_{1\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha)\frac{d\alpha}{d\alpha} = 0$$
$$u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha)\frac{dx_1^*(\alpha)}{d\alpha} + u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha)\frac{dx_2^*(\alpha)}{d\alpha} + u_{2\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha)\frac{d\alpha}{d\alpha} = 0$$

3.1.10 The Derivative of Optimal Choice with Multiple Choice Variables

In a concrete example we could solve for $\frac{dx_1^*(\alpha)}{d\alpha}$ and $\frac{dx_2^*(\alpha)}{d\alpha}$ directly using algebra. If that seems difficult or the problem is abstract, we can borrow an approach from linear algebra. We write this system of equations as a matrix product.

$$\begin{bmatrix} u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{bmatrix} \begin{bmatrix} \dfrac{dx_1^*(\alpha)}{d\alpha} \\ \dfrac{dx_2^*(\alpha)}{d\alpha} \end{bmatrix} = \begin{bmatrix} -u_{1\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ -u_{2\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{bmatrix}$$

Cramer's rule writes the solution to a matrix equation as a ratio of determinants.

$$\frac{dx_1^*(\alpha)}{d\alpha} = \frac{\begin{vmatrix} -u_{1\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ -u_{2\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{vmatrix}}{\begin{vmatrix} u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{vmatrix}}$$

$$\frac{dx_2^*(\alpha)}{d\alpha} = \frac{\begin{vmatrix} u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & -u_{1\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & -u_{2\alpha}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{vmatrix}}{\begin{vmatrix} u_{11}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{12}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \\ u_{21}(x_1^*(\alpha), x_2^*(\alpha), \alpha) & u_{22}(x_1^*(\alpha), x_2^*(\alpha), \alpha) \end{vmatrix}}$$

Notice the denominator in each formula is $|Hu(x_1^*(\alpha), x_2^*(\alpha), \alpha)_2|$. We replace the corresponding column with the right side of the equation to obtain the numerator.

Cramer's rule can apply to any number of variables, so we can extend this procedure to more than two choice variables. We can also handle multiple parameters. This method would compute partial derivatives of $x_i^*(\vec{\alpha})$ with respect to some α_j.

Main Ideas

- The optimal choice of multiple variables satisfies the multivariable first-order condition.
- The implicit function theorem applies to the optimal choice variables, if the Hessian minor corresponding to the choice variables has a nonzero determinant. If the optimal choice satisfies the second-order condition, this is automatic.
- We can differentiate the multivariable first-order condition with respect to a parameter and obtain a linear system of equations in the derivatives of the choice variables.
- Cramer's rule is a reliable way to solve such systems.

3.1.11 Section Summary

The key definitions and results from this section were

- Explicit functions and implicit equations (Definitions 3.1 and 3.2)
- The implicit function theorem (Theorem 3.3)
- The derivative of the function guaranteed by the implicit function theorem (Corollary 3.4)
- The derivative of the function $x^*(\alpha)$ (Corollary 3.5)

3.2 The Envelope Theorem

Goals:

1. Use the envelope theorem to compute the derivative of value functions.

3.2.1 Value Functions

Definition 3.10

Suppose we have

- $f(x, \alpha)$, a differentiable function.
- x is a choice variable
- α is a parameter
- $x^*(\alpha)$ is the x that maximizes $f(x, \alpha)$ for a given α.

Then the outcome that will occur for each α is the **value function**

$$V(\alpha) = f(x^*(\alpha), \alpha)$$

Vocabulary

In the case that f has an economic meaning, we sometimes use the term **indirect**, along with the $*$ notation, to refer to its value function of f.

- If $u(x, \alpha)$ is a utility function then $u^*(\alpha) = u(x^*(\alpha), \alpha)$ is the **indirect utility function**.
- If $\pi(x, \alpha)$ is a profit function then $\pi^*(\alpha) = \pi(x^*(\alpha), \alpha)$ is the **indirect profit function**.

$x^*(\alpha)$ is a solution to $f_x(x, \alpha) = 0$. Assuming f is continuously differentiable, the implicit function guarantees that $x^*(\alpha)$ is a differentiable function. This means that V is a differentiable function. If we want to understand how a change in α affects the value of V, we want to compute $V'(\alpha)$.

$V'(\alpha)$ computes how a change in a parameter affects the an outcome, assuming the agents involved make the optimal choices. This derivative can answer a variety of questions in economics.

Example

- How will increasing a tax impact company profits?
- Will increasing a subsidy increase consumer well-being?

In the case that we have an expression for f, computing $V'(\alpha)$ is not difficult:

1. Solve for $x^*(\alpha)$
2. Substitute $x^*(\alpha)$ into $f(x, \alpha)$ to obtain an expression for $V(\alpha)$
3. Differentiate $V(\alpha)$

But is there a better way?

3.2.2 The Envelope Theorem

The envelope theorem gives us an alternative method. We compute $V'(\alpha)$ by parametrizing the values of α and x^* in a neighborhood where $x^*(\alpha)$ is differentiable. We use α as the parameter.

$$\alpha = \alpha \qquad\qquad \frac{d\alpha}{d\alpha} = 1$$

$$x = x^*(\alpha)$$

We can apply the chain rule to $V(\alpha) = f(x^*(\alpha), \alpha)$. We use the fact that $x^*(\alpha)$ satisfies the first-order condition for all α. Specifically, have $f_x(x^*(\alpha), \alpha) = 0$.

$$\begin{aligned}
V'(\alpha) &= \frac{df(x, \alpha)}{d\alpha} \\
&= f_x(x^*(\alpha), \alpha)\frac{dx^*(\alpha)}{d\alpha} + f_\alpha(x^*(\alpha), \alpha)\frac{d\alpha}{d\alpha} && \text{(chain rule)} \\
&= (0)\frac{dx^*(\alpha)}{d\alpha} + f_\alpha(x^*(\alpha), \alpha)(1) && \text{(FOC)} \\
&= f_\alpha(x^*(\alpha), \alpha)
\end{aligned}$$

Remark

This computation requires us to know that $\vec{x}^*(\alpha)$ is a differentiable function. Here are two ways to verify this.

1. For a concrete function, compute $\vec{x}^*(\alpha)$. Verify directly that it is differentiable.
2. In more abstract settings, apply the implicit function theorem. Check that
   - f has continuous second derivatives
   - $f_{xx}(x^*(\alpha), \alpha) \neq 0$ for all α.

.

Theorem 3.11 [The Envelope Theorem, Single-Variable]

Suppose

- $f(x, \alpha)$ is a differentiable function, and
- $x^*(\alpha)$ that maximizes f for each α is a differentiable function.

The following two derivatives are equal:

$$\underbrace{V'(\alpha)}_{\substack{\text{derivative of}\\ \text{value function}}} = \underbrace{f_\alpha(x^*(\alpha), \alpha)}_{\substack{\text{partial derivative}\\ \text{of original function}}}$$

The envelope theorem allows us to compute a partial derivative of f instead of the total derivative of V. In some sense, the envelope theorem is saying that the change in x^* does not matter. This makes sense, because at a maximizer, we cannot increase the value of the function by changing x.

This is an interesting insight into the behavior of value functions, but all we have done is traded one derivative for another. We still need to compute $x^*(\alpha)$ to evaluate the partial derivative. It is natural to ask: does the envelope theorem save us any work in practice?

Compare the following methods of computing $V'(\alpha)$:

Without the envelope theorem

1. Compute $x^*(\alpha)$
2. Substitute into $f(x, \alpha)$ to get $V(\alpha)$
3. Differentiate $V(\alpha)$

With the envelope theorem

1. Compute $x^*(\alpha)$
2. Partially differentiate $f(x, \alpha)$
3. Substitute $x^*(\alpha)$

Remark

In concrete situations, the first method gives us a more complicated function to differentiate. In abstract situations, the first method may be impossible or give us an answer in a less useful form.

The envelope theorem can also be justified visually. If we pick a specific a, we can compare two functions

1. The value function: $V(\alpha) = f(x^*(\alpha), \alpha)$, which uses the best x for each α
2. The **stubborn function**: $V_0(\alpha) = f(x^*(a), \alpha)$, which sticks with the best x for a, even if α changes.

The stubborn function has the following properties

- Since its x-coordinate is constant, the derivative of V_0 is equal to the partial derivative f_α.
- $V_0(a) = V(a)$
- For any other α, $x^*(a)$ cannot be a better choice than $x^*(\alpha)$. Thus $V_0(\alpha) \leq V(\alpha)$.

The graph $y = V_0(\alpha)$ meets $y = V(\alpha)$ at a but does not go above it. They must be tangent, which means their derivatives are equal.

$$V'(a) = V_0'(a) = f_\alpha(x^*(a), a)$$

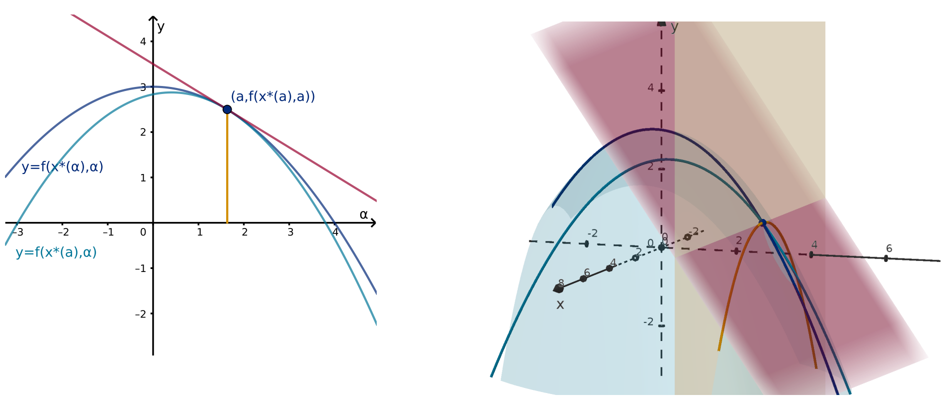


Figure 3.6: The graphs of the value function and stubborn function and their tangent line

The envelope theorem gets its name from the fact that $y = V(\alpha)$ envelopes all of the stubborn functions $y = V_0(\alpha)$, generated by choosing different values of a.

There are several ways to generalize the envelope theorem. First we can consider a function of more choice variables or more parameters.

1. If f is a function of n choice variables, then each choice variable has an optimal value that depends of α. We obtain a family of functions $x_i^*(\alpha)$. We can write them as a vector $\vec{x}^*(\alpha)$. The value function is
$$V(\alpha) = f(\vec{x}^*(\alpha), \alpha)$$
2. If f is a function of m parameters α_j, we can express the parameters as a vector $\vec{\alpha}$. The optimal choice of each choice variable is a function of all the α_j. The value function also a function of all the α_j.
$$V(\vec{\alpha}) = f(\vec{x}^*(\vec{\alpha}), \vec{\alpha})$$

Since the $x_i^*(\vec{\alpha})$ satisfy the family of equations $f_{x_i}(\vec{x}, \vec{\alpha}) = 0$, we need the multi-equation version of the implicit function theorem to justify their existence and differentiability.

The value function is a function of multiple parameters, so the envelope theorem computes its partial derivatives.

Theorem 3.12 [The Envelope Theorem, Multivariable]

Suppose

- $f(\vec{x}, \vec{\alpha})$ is a differentiable function
- $\vec{x}^*(\vec{\alpha})$ that maximizes f for each $\vec{\alpha}$ is a differentiable function

For any region of $\mathbb{R}^m$ where $\vec{x}^*(\vec{\alpha})$ is differentiable and any coordinate α_k of $\vec{\alpha}$ we have:

$$V_{\alpha_k}(\vec{\alpha}) = f_{\alpha_k}(\vec{x}^*(\vec{\alpha}), \vec{\alpha})$$

We can again show this with a parametrization. Since we need the partial derivative, we treat α_k as a parameter. The other α_j are constants, held equal to the corresponding components of some fixed $\vec{a}$. This parametrization will not cover the entire domain of $\vec{\alpha}$, only those that lie in the α_k-direction from $\vec{a}$.

$$\alpha_j = \begin{cases} a_j & j \neq k \\ \alpha_k & j = k \end{cases} \qquad \frac{d\alpha_j}{d\alpha_k} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$
$$\vec{x} = \vec{x}^*(\vec{\alpha})$$

In general, x_i^* is a multivariable function of $\vec{\alpha}$. In this parametrization, x_i is a function only of the parameter α_k. By the first-order condition, $\vec{x}^*(\vec{\alpha})$ satisfies $f_{x_i}(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) = 0$ for each x_i. We can apply the chain rule to $V(\vec{\alpha}) = f(\vec{x}^*(\vec{\alpha}), \vec{\alpha})$ to compute $V_{\alpha_k}(\vec{\alpha})$.

$$\begin{aligned}
V_{\alpha_k}(\vec{\alpha}) &= \frac{df(\vec{x}^*(\vec{\alpha}), \vec{\alpha})}{d\alpha_k} \\
&= \sum_{i=1}^{n} f_{x_i}(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) \frac{dx_i^*(\vec{\alpha})}{d\alpha_k} + \sum_{j=1}^{m} f_{\alpha_j}(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) \frac{d\alpha_j}{d\alpha_k} \\
&= \sum_{i=1}^{n} (0) \frac{dx_i^*(\vec{\alpha})}{d\alpha_k} + \sum_{j \neq k} f_{\alpha_j}(\vec{x}^*(\vec{\alpha}), \vec{\alpha})(0) + f_{\alpha_k}(\vec{x}^*(\vec{\alpha}), \vec{\alpha})(1) \\
&= f_{\alpha_k}(\vec{x}^*(\vec{\alpha}), \vec{\alpha})
\end{aligned}$$

This is valid for any $\vec{\alpha}$ that lies on our path through $\vec{a}$ in the α_k-direction. We can apply this reasoning to any $\vec{a}$ though, so the computation holds for the entire domain of $\vec{x}^*(\vec{\alpha})$.

Our final generalization of the envelope theorem assumes that $\vec{x}^*(\vec{\alpha})$ is the optimal choice given an equality constraint, $g(\vec{x}, \vec{\alpha}) = 0$.

Theorem 3.13 [The Envelope Theorem, Constrained]

Suppose

- $f(\vec{x}, \vec{\alpha})$ is a differentiable objective function
- $g(\vec{x}, \vec{\alpha})$ is a differentiable constraint function
- $\vec{x}^*(\vec{\alpha})$ that maximizes f subject to $g(\vec{x}, \vec{\alpha}) = 0$ for each $\vec{\alpha}$ is a differentiable function
- $\lambda^*(\vec{\alpha})$ is the value of λ that solves the first-order conditions of $\mathcal{L}$ along with $\vec{x}^*(\vec{\alpha})$ and $\vec{\alpha}$.

For any region of $\mathbb{R}^m$ where $\vec{x}^*(\vec{\alpha})$ and $\lambda^*(\vec{\alpha})$ are differentiable and any coordinate α_k of $\vec{\alpha}$ we have:

$$V_{\alpha_k}(\vec{\alpha}) = \mathcal{L}_{\alpha_k}(\vec{x}^*(\vec{\alpha}), \vec{\alpha}, \lambda^*(\vec{\alpha}))$$

Proving this version requires us to use the fact that $g(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) = 0$ for all $\vec{\alpha}$. This means that

$$\begin{aligned}
\mathcal{L}(\vec{x}^*(\vec{\alpha}), \vec{\alpha}, \lambda^*(\vec{\alpha})) &= f(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) + \lambda^*(\vec{\alpha}) g(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) \\
&= f(\vec{x}^*(\vec{\alpha}), \vec{\alpha}) + \lambda^*(\vec{\alpha})(0) \\
&= V(\vec{\alpha})
\end{aligned}$$

$\vec{x}^*(\vec{\alpha})$ does not necessarily satisfy the first-order condition of f, but for a fixed α it does satisfy the first order condition of $\mathcal{L}(\vec{x}^*(\vec{\alpha}), \lambda^*(\alpha))$. Specifically:

$$\begin{aligned}
\mathcal{L}_{x_i}(\vec{x}^*(\vec{\alpha}), \lambda^*(\alpha)) &= 0 \\
\mathcal{L}_{\lambda}(\vec{x}^*(\vec{\alpha}), \lambda^*(\alpha)) &= 0
\end{aligned}$$

We can use the same parametrization as the unconstrained case, with the understanding that $\vec{x}^*(\vec{\alpha})$ now describes the maximizer of the constrained optimization and with $\lambda^*(\vec{\alpha})$ the corresponding λ value.

$$\begin{aligned}
V_{\alpha_k}(\vec{\alpha}) &= \frac{df(\vec{x}^*(\vec{\alpha}),\vec{\alpha})}{d\alpha_k} \\
&= \frac{d\mathcal{L}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))}{d\alpha_k} \\
&= \sum_{i=1}^{n}\mathcal{L}_{x_i}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))\frac{dx_i^*(\vec{\alpha})}{d\alpha_k} + \sum_{j=1}^{m}\mathcal{L}_{\alpha_j}(\vec{x}^*(\vec{\alpha},\vec{\alpha},\lambda^*(\vec{\alpha}))\frac{d\alpha_j}{d\alpha_k} + \mathcal{L}_{\lambda}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))\frac{d\lambda^*(\vec{\alpha})}{d\alpha_k} \\
&= \sum_{i=1}^{n}(0)\frac{dx_i^*(\vec{\alpha})}{d\alpha_k} + \sum_{j\neq k}\mathcal{L}_{\alpha_j}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))(0) + \mathcal{L}_{\alpha_k}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))(1) + (0)\frac{d\lambda^*(\vec{\alpha})}{d\alpha_k} \\
&= \mathcal{L}_{\alpha_k}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))
\end{aligned}$$

Further Generalizations

We can extend this reasoning to multiple equality constraints without any trouble.

We can also also extend the envelope theorem to inequality constraints, where we set $\lambda^*(\vec{\alpha}) = 0$ for any $\vec{\alpha}$ such that the constraint is not binding at the maximizer $\vec{x}^*(\vec{\alpha})$. If $\vec{x}^*$ and λ^* are differentiable functions of α in this setting, we can still conclude that

$$V_{\alpha_k}(\vec{\alpha}) = \mathcal{L}_{\alpha_k}(\vec{x}^*(\vec{\alpha}),\vec{\alpha},\lambda^*(\vec{\alpha}))$$

However, we expect $\vec{x}^*(\vec{\alpha})$ will not be differentiable at the transition between the binding and nonbinding cases.

Proving the inequality case is identical to the equality case, except

$$\underbrace{\mathcal{L}_{\lambda}(\vec{x}(t),\vec{\alpha}(t),\lambda(t))}_{=0\text{ if binding}} \quad \underbrace{\lambda'(t)}_{\substack{=0\text{ if}\\ \text{nonbinding}}} \quad = 0$$

Remark

The generalizations of the envelope theorem require that $\vec{x}^*(\vec{\alpha})$ is a differentiable function. Like in the single-variable case, it usually makes sense to solve for $\vec{x}^*(\vec{\alpha})$ and directly verify that it is differentiable. If we need to use the implicit function theorem instead, we need the multivariable version. The requirement for that is

$$|Hf_{\vec{x}}(\vec{x},\vec{\alpha})| \neq 0 \text{ or } |H\mathcal{L}_{\vec{x},\lambda}(\vec{x},\vec{\alpha})| \neq 0$$

depending on whether there is a constraint.

3.2.4

Section Summary

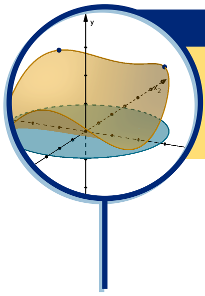
The most important definitions and results from this section were

- The definition of a value function (Definition 3.10)
- Three versions of the envelope theorem (Theorems 3.11, 3.12 and 3.13)

3.2.4 Section Summary

Chapter 4

Sufficient Conditions



4.1 The Extreme Value Theorem

Goals:

1. Recognize when the extreme value theorem applies.
2. Apply the extreme value theorem to identify a maximizer.

4.1.1 The Extreme Value Theorem

In previous examples, we often found that only one point satisfied the necessary conditions for a maximizer. Still we did not conclude that this point was a maximizer. We would have found it very useful to know that a maximizer existed in these circumstances. We then could have identified our point as that maximizer. Even if we only narrowed our search down to two or three potential maximizers, the information that one of them is in fact the maximizer would have been helpful.

This is what the extreme value theorem does for us.

Theorem 4.1 [The Extreme Value Theorem]

If $f(\vec{x})$ is a continuous function and S is a closed and bounded subset of the domain of f, then there exists an $\vec{x}^*$ that maximizes $f(\vec{x})$ subject to $\vec{x} \in S$.

In order to apply this theorem, we need to be able to identify when a region S is closed and bounded. Here are the definitions of those terms.

Definition 4.2

Let S be a subset of $\mathbb{R}^n$.

- S is **closed** if it contains all of the points on its boundary.
- S is **bounded** if there is some upper limit to how far its points get from the origin (or any other fixed point). If there are points of S arbitrarily far from the origin, then S is **unbounded**.

For one-variable functions, we can use the fact that a union of finitely many closed intervals (or isolated points) is closed. If the intervals are each finite length, then the union is also bounded.

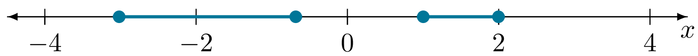


Figure 4.1: A finite union of closed intervals

The same holds for curves. A curve must include its endpoints, or have no endpoints to be closed.

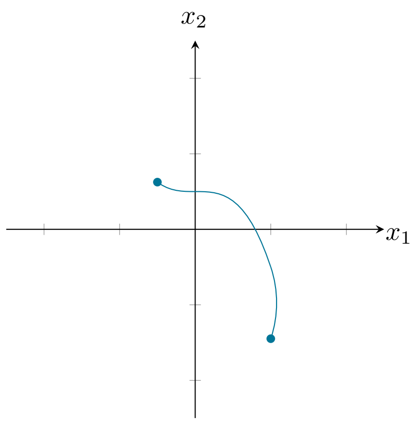


Figure 4.2: A closed curve

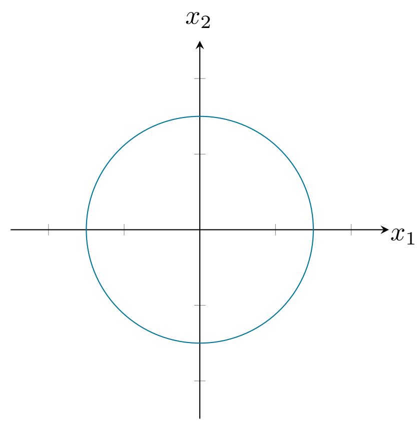


Figure 4.3: The graph $x_1^2 + x_2^2 = 9$ is closed.

Generally, a region defined by a strict inequality will not contain its boundary points and thus will not be closed.

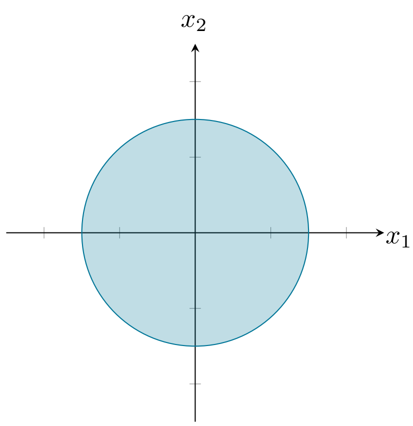


Figure 4.4: $x_1^2 + x_2^2 \leq 9$ is closed.

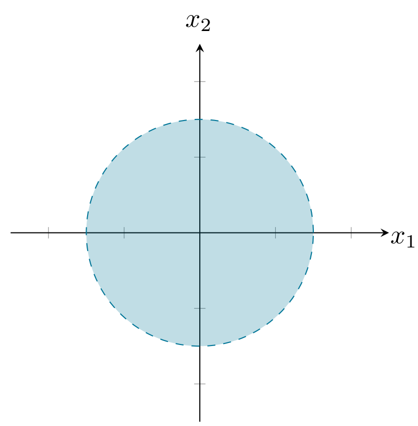


Figure 4.5: $x_1^2 + x_2^2 < 9$ is not closed.

If multiple inequalities are involved and relevant, they must all be nonstrict in order to avoid removing boundary points. An interesting case is the removal of a single interior point. If we exclude that point from S, then that point becomes a boundary point. Any neighborhood of it contains points in S and a point not in S.

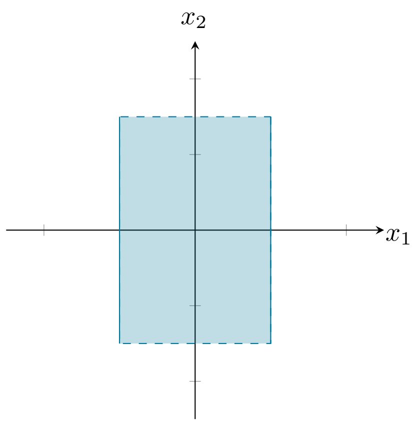


Figure 4.6: $-2 \leq x_1 \leq 2$ and $-3 < x_2 < 3$ is not closed.

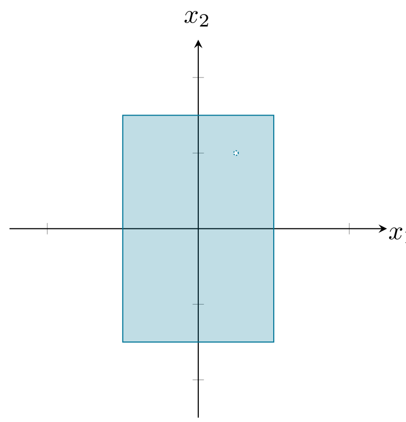


Figure 4.7: $-2 \leq x_1 \leq 2$ and $-3 \leq x_2 \leq 3$ and $(x_1, x_2) \neq (1, 2)$ is not closed.

Boundedness is a simpler concept and easy to check. If you can draw a circle around S, it is bounded. If no circle is big enough, it is unbounded.

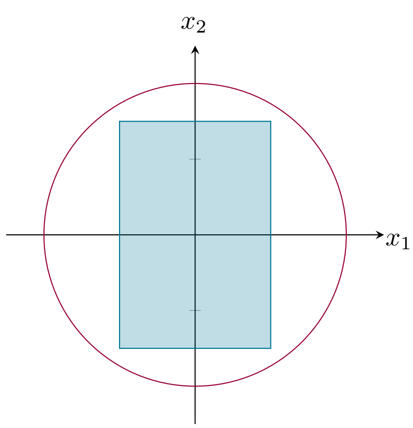


Figure 4.8: $-2 \leq x_1 \leq 2$ and $-3 \leq x_2 \leq 3$ is bounded.

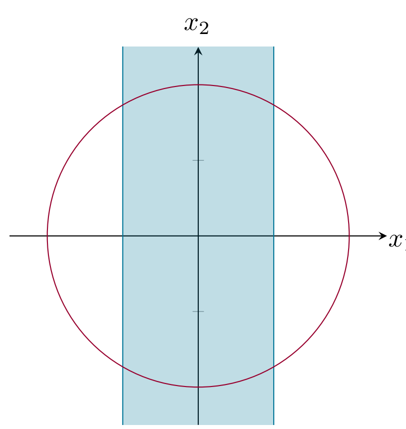


Figure 4.9: $-2 \leq x_1 \leq 2$ is unbounded.

The examples above are a good way to visually recognize a closed and bounded set. What if we have an equation or inequality instead of a graph? The following theorems answer some of these questions.

Theorem 4.3

If $f(\vec{x})$ is a continuous function, then any level set or upper level set of f is closed.

Theorem 4.4

1. The intersection of any number of closed sets is closed.
2. The union of any finite number of closed sets is closed.

The second theorem is specifically useful for feasible sets defined by multiple constraints. Under our formulation of constrained optimization, the feasible set is an intersection of level sets and upper level sets. As long as the constraint functions are continuous, the feasible set will be closed.

The extreme value theorem is a standard result in analysis. While we will not prove it, we can at least demonstrate that each hypothesis is necessary.

Example

Consider $f(x) = x^2$ on the region $S = \{x \ : \ x \geq 0\}$. $f(x)$ is continuous and S is closed but not bounded. $f(x)$ grows without bound in S and has no maximum.

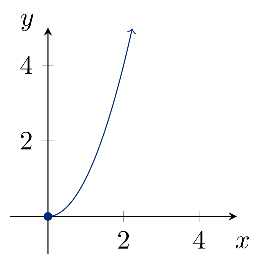


Figure 4.10: The graph of $y = x^2$ over $[0, \infty)$

Example

Consider

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

on the region $S = \{x \ : \ 0 \leq x \leq 2\}$. S is closed and bounded, but $f(x)$ is not continuous. $f(x)$ approaches a value of 4 but never reaches it. There is no maximizer a. For any $a < 2$, there is a b closer to 2 with $f(b) > f(a)$.

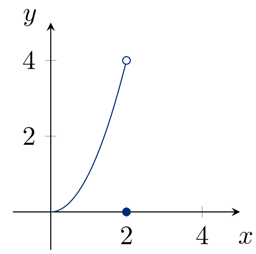


Figure 4.11: The graph of $y = f(x)$ over $[0, 2]$

Example

Consider $f(x) = x^2$ on the region $S = \{x \ : \ 0 \leq x < 2\}$. $f(x)$ is continuous and S is bounded but not closed. Again $f(x)$ approaches a value of 4 but never reaches it. There is no maximizer a. For any a in S, there is a b closer to 2 with $f(b) > f(a)$.

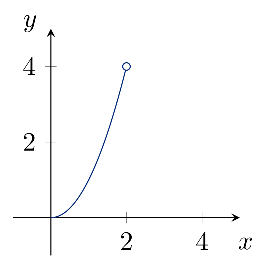


Figure 4.12: The graph of $y = x^2$ over $[0, 2)$

Find the maximizer(s) of $f(x_1, x_2) = x_1^2 + x_1^2 x_2 + 2x_2^2$ subject to $8 - x_1^2 - x_2^2 \geq 0$.

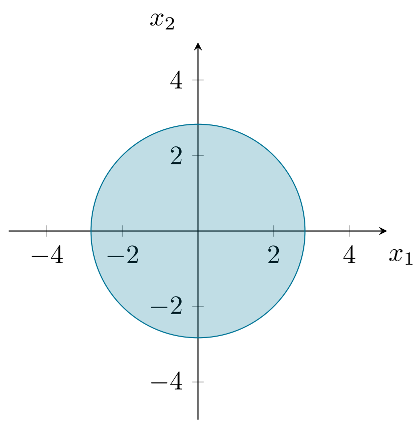


Figure 4.13: The feasible set

Solution

Apply the necessary conditions for maximizers subject to an inequality. The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_1^2 x_2 + 2x_2^2 + \lambda(8 - x_1^2 - x_2^2)$$

The conditions are

$$L_{x_1}(x_1, x_2, \lambda) = 2x_1 + 2x_1 x_2 - 2\lambda x_1 = 0$$
$$L_{x_2}(x_1, x_2, \lambda) = x_1^2 + 4x_2 - 2\lambda x_2 = 0$$
$$8 - x_1^2 - x_2^2 \geq 0, \quad \lambda \geq 0 \quad \text{and} \quad \lambda(8 - x_1^2 - x_2^2) = 0$$

We solve each case of the complementary slackness.

1 Set $\lambda = 0$

$$2x_1 + 2x_1 x_2 = 0 \qquad x_1^2 + 4x_2 = 0$$
$$2x_1(1 + x_2) = 0$$
$$\text{if } x_1 = 0 \qquad 0^2 + 4x_2 = 0$$
$$x_2 = 0$$
$$\text{if } x_2 = -1 \qquad x_1^2 + 4(-1) = 0$$
$$x_1 = \pm 2$$

Check that $(0, 0, 0)$ and $(\pm 2, -1, 0)$ satisfy $8 - x_1^2 - x_2^2 \geq 0$. They do.

2 Set $8 - x_1^2 - x_2^2 = 0$. We need to solve

$$2x_1 + 2x_1 x_2 - 2\lambda x_1 = 0 \qquad x_1^2 + 4x_2 - 2\lambda x_2 = 0 \qquad 8 - x_1^2 - x_2^2 = 0$$

One good approach is to factor

$$\begin{aligned} 2x_1 + 2x_1x_2 - 2\lambda x_1 &= 0 \\ 2x_1(1 + x_2 - \lambda) &= 0 \\ x_1 = 0 &\text{ or } \lambda = 1 + x_2 \end{aligned}$$

We treat these two cases separately.

a If $x_1 = 0$,

$$8 - (0)^2 - x_2^2 = 0$$

$$x_2 = \pm 2\sqrt{2} \qquad (0)^2 + 4(\pm 2\sqrt{2}) - 2\lambda(\pm 2\sqrt{2}) = 0$$

$$\begin{aligned} \pm 8\sqrt{2} &= \pm 4\sqrt{2}\lambda \\ 2 &= \lambda \end{aligned}$$

b If $\lambda = 1 + x_2$,

$$x_1^2 + 4x_2 - 2(1 + x_2)x_2 = 0$$

$$x_1^2 = 2x_2^2 - 2x^2$$

$$\begin{aligned} 8 - (2x_2^2 - 2x_2) - x_2^2 &= 0 \\ 3x_2^2 - 2x_2 - 8 &= 0 \\ -(3x_2 + 4)(x_2 - 2) &= 0 \end{aligned}$$

if $x_2 = 2$

$$\lambda = 1 + 2 \qquad x_1^2 = 8 - (2)^2$$

$$\lambda = 3 \qquad x_1 \pm 2$$

if $x_2 = -\dfrac{4}{3}$

$$\lambda = 1 - \frac{4}{3} \qquad x_1^2 = 8 - \left(\frac{4}{3}\right)^2$$

$$\lambda = -\frac{1}{3} \qquad x_1 = \pm\frac{2\sqrt{14}}{3}$$

We verify that $(0, \pm 2\sqrt{2}, 2)$, $(\pm 2, 2, 3)$ and $\left(\pm\frac{2\sqrt{14}}{3}, \frac{4}{3}, -\frac{1}{3}\right)$ satisfy $\lambda \geq 0$. The third one does not.

Now we can apply the extreme value theorem. The function $f(x_1, x_2)$ is continuous. The feasible set is the upper level set of a continuous function: $10 - x_1^2 - x_2^2 \geq 0$, so it is closed. The feasible set is a disk, so it is bounded. The extreme value theorem tells us a maximizer must exist.

Only a point that satisfies our necessary condition can be that maximizer. To determine which one, we evaluate $f(x_1, x_2)$ at each.

$$f(0,0) = (0)^2 + (0)^2(0) + 2(0)^2 = 0$$
$$f(\pm 2, -1) = (\pm 2)^2 + (\pm 2)^2(-1) + 2(-1)^2 = 2$$
$$f(0, \pm 2\sqrt{2}) = (0)^2 + (0)^2(\pm 2\sqrt{2}) + 2(\pm 2\sqrt{2})^2 = 16$$
$$f(\pm 2, 2) = (\pm 2)^2 + (\pm 2)^2(2) + 2(2)^2 = 20$$

Because they produce the greatest values among the candidates, the maximizers are $(2,2)$ and $(-2,2)$.

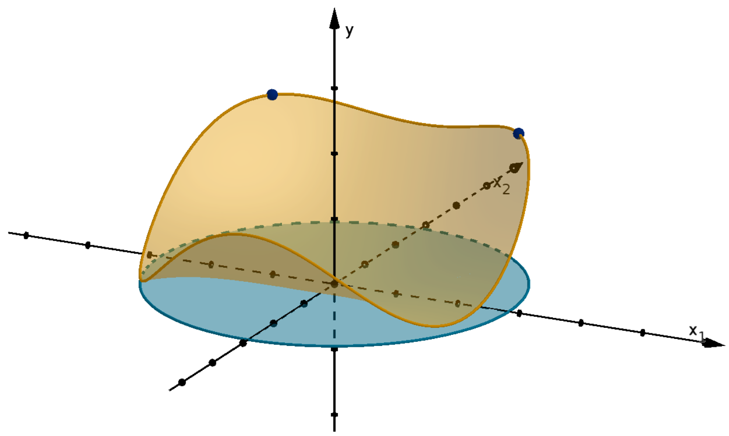


Figure 4.14: The maximizers of $y = x_1^2 + x_1^2 x_2 + 2x_2^2$ subject to $8 - x_1^2 - x_2^2 \geq 0$

Main Ideas

- The algebraic expressions tell us when the objective function is continuous and the feasible set is closed.
- Draw the feasible set to decide whether it is bounded.
- If the EVT applies, we can evaluate f at all of the points that passed our necessary conditions. The one that attains the greatest value is the maximizer.

4.1.3 Section Summary

The most important definitions and results from this section were

- The extreme value theorem (Theorem 4.1)
- The meaning of closed and bounded (Definition 4.2)
- Level sets, upper level sets, and their intersections are closed (Theorems 4.3 and 4.4)

4.2

The Bordered Hessian

Goals:

1. Use the bordered Hessian to identify local maximizers of a function subject to a constraint.

4.2.1

The Bordered Hessian

We used the Hessian matrix to recognize maximizers and minimizers in unconstrained optimization. This worked because the Hessian computed the second derivative over a straight line $\vec{a}+t\vec{v}$. An equality constraint does not usually contain straight lines. The test for the second derivative will need to take account of the shape of the level set $g(\vec{x})=0$.

Definition 4.5

Given a constrained optimization problem with Hessian $\mathcal{L}$, the matrix $H\mathcal{L}(\lambda,\vec{x})$ is called the **bordered Hessian** of the constrained optimization problem.

Example

The 2-variable bordered Hessian has the form

$$H\mathcal{L}(\lambda,x_1,x_2)=\begin{bmatrix}\mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda 1} & \mathcal{L}_{\lambda 2}\\ \mathcal{L}_{\lambda 1} & \mathcal{L}_{11} & \mathcal{L}_{12}\\ \mathcal{L}_{\lambda 1} & \mathcal{L}_{12} & \mathcal{L}_{22}\end{bmatrix}=\begin{bmatrix}0 & g_1 & g_2\\ g_1 & f_{11}+\lambda g_{11} & f_{12}+\lambda g_{12}\\ g_2 & f_{21}+\lambda g_{21} & f_{22}+\lambda g_{22}\end{bmatrix}$$

Why is this called "bordered?"

- The bottom right 2×2 minor looks like a Hessian.
- It is bordered to the left and above by ∇g.

Notice that we have switched the order of variables in our Lagrangian. This is common when writing the bordered Hessian. Placing the border on the top allows us to write our condition for a local maximizer in a familiar way. If we instead prioritized consistency, we could keep the λ last and modify our condition.

Theorem 4.6

Let $f(\vec{x})$ be an n-variable function and $g(\vec{x}) = 0$ be a constraint. If $(\ell, \vec{a})$ satisfies the first-order condition of the Lagrangian and the upper left square minors of $H\mathcal{L}(\ell, \vec{a})$ satisfy

$$(-1)^i |M_i| < 0 \qquad 2 \le i \le n+1,$$

then $\vec{a}$ is a strict local maximizer of f among points on the constraint.

Remark

- We do not test $|M_1|$, since $M_1 = [0]$.
- We generally do not need to worry about M_2 either, since

$$|M_2| = \begin{vmatrix} 0 & g_1 \\ g_1 & f_{11} + \lambda g_{11} \end{vmatrix} = -(g_1)^2$$

- Notice the inequality is reversed from the unconstrained second-order condition.

Variant of Theorem 4.6

Let $f(\vec{x})$ be an n-variable function and $g(\vec{x}) = 0$ be a constraint. If $(\ell, \vec{a})$ satisfies the first-order condition of the Lagrangian and the upper left square minors of $H\mathcal{L}(\ell, \vec{a})$ satisfy

$$|M_i| < 0 \qquad 2 \le i \le n+1,$$

then $\vec{a}$ is a strict local minimizer of f among points on the constraint.

Remark

We might hope to quickly extend this to a global condition, but unfortunately, $H\mathcal{L}$ only tells us about the second derivatives at a critical point. Deriving a condition that works for all points on the constraint is possible but more complicated.

Let $f(x_1, x_2) = x_1^2 + x_2^2$ on the domain

$$D = \{(x_1, x_2) \ : \ x_1 > 0, x_2 > 0\}$$

Find the critical point of f on the constraint $x_1^4 + x_2^4 = 2$. Classify it as a local maximizer or local maximizer of f (or neither) on the constraint.

Solution

The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(2 - x_1^4 - x_2^4)$$

Here are the first-order conditions. We can solve them, using the fact that $x_1 > 0$ and $x_2 > 0$.

$$\begin{aligned} 2x_1 - 4\lambda x_1^3 &= 0 & 2x_2 - 4\lambda x_2^3 &= 0 & 2 - x_1^4 - x_2^4 &= 0 \\ x_1^2 &= \frac{1}{2\lambda} & x_2^2 &= \frac{1}{2\lambda} & & \\ x_1^2 &= x_2^2 & & & & \\ x_1 &= x_2 & & & & \\ & & & & 2 - x_1^4 - x_1^4 &= 0 \\ & & & & 2 &= 2x_1^4 \\ & & & & 1 &= x_1 \\ 1 &= x_2 & & & & \\ 1 &= \frac{1}{2\lambda} & & & & \\ \lambda &= \frac{1}{2} & & & & \end{aligned}$$

The critical point is $\left(1, 1, \frac{1}{2}\right)$. Switching the order of the variables, we compute

$$HL(\lambda, x_1, x_2) = \begin{bmatrix} 0 & -4x_1^3 & -4x_2^3 \\ -4x_1^3 & 2 - 12\lambda x_1^2 & 0 \\ -4x_2^3 & 0 & 2 - 12\lambda x_2^2 \end{bmatrix}$$

$$HL\left(\frac{1}{2}, 1, 1\right) = \begin{bmatrix} 0 & -4 & -4 \\ -4 & -4 & 0 \\ -4 & 0 & -4 \end{bmatrix}$$

The determinants of the upper left square minors are

$$|M_2| = \begin{vmatrix} 0 & -4 \\ -4 & -4 \end{vmatrix} = -16$$

$$|M_3| = \begin{vmatrix} 0 & -4 & -4 \\ -4 & -4 & 0 \\ -4 & 0 & -4 \end{vmatrix} = 0 + 4\begin{vmatrix} -4 & 0 \\ -4 & -4 \end{vmatrix} - 4\begin{vmatrix} -4 & -4 \\ -4 & 0 \end{vmatrix} = 128$$

$(-1)^2(-16) < 0$ and $(-1)^3(128) < 0$. According to Theorem 4.6, $(1,1)$ is a strict local maximizer.

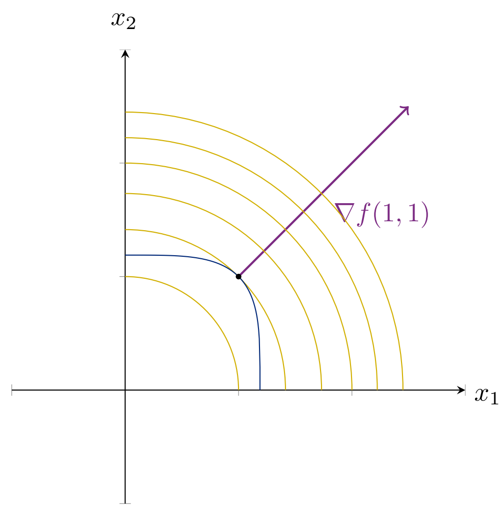


Figure 4.15: The level sets of $x_1^2 + x_2^2$ and the constraint $x_1^4 + x_2^4 = 2$

4.2.3 The Multi-Constraint Bordered Hessian

For more variables, the test requires more determinants.

Theorem 4.7

Let $f(\vec{x})$ be an n-variable function and $\{g_j(\vec{x}) = 0\}$ be a set of m constraint equations. If $(\vec{\ell}, \vec{a})$ satisfies the first-order condition of the Lagrangian and the upper left square minors of $H\mathcal{L}(\vec{\ell}, \vec{a})$ satisfy

$$(-1)^i |M_i| < 0 \qquad 2m \leq i \leq n + m,$$

then $\vec{a}$ is a strict local maximizer of $f(\vec{x})$ among the feasible points.

4.2.4 The Bordered Hessian and Inequality Constraints

If we combine the determinant of the bordered Hessian with the requirement that $\ell > 0$, then we can guarantee that $f(\vec{x}) < f(\vec{a})$ on the equality constraint and in the direction of $\nabla g(\vec{a})$ from the equality constraint. This guarantees a neighborhood of some size in which $\vec{a}$ is a maximizer among feasible points.

4.2.5 Section Summary

The most important definitions and results from this section were

- The definition of the bordered Hessian (Definition 4.5)
- The bordered Hessian determinant test for a maximizer (Theorems 4.6)

4.3

Separation

Goals:

1. Use normal vectors to describe half-spaces of a hyperplane.
2. Apply optimization by separation arguments.

4.3.1

Upper Level Sets and Optimization

Recall our necessary conditions for a maximizer of $f(\vec{x})$ subject to $g(\vec{x}) \geq 0$. If the constraint is binding, we learned to check that $\nabla f(\vec{a})$ is parallel to $\nabla g(\vec{a})$, meaning the level sets are parallel. We also check that $\lambda \geq 0$, meaning that $\nabla f(\vec{a})$ points away from the feasible set. However, these checks are not sufficient. We will construct an example that passes these conditions but isn't a maximizer.

Recall the following definition

Definition 2.13

The **upper level sets** of a function $f(\vec{x})$ with domain D are the sets

$$\{\vec{x} \in D \mid f(\vec{x}) \geq c\} \text{ for some number } c$$

The **lower level sets** are

$$\{\vec{x} \in D \mid f(\vec{x}) \leq c\} \text{ for some number } c$$

The following characterization will be important to our arguments.

Lemma 2.14

Suppose $\nabla f(\vec{x}) \neq \vec{0}$, and $\vec{x}(t)$ is a path that passes through $\vec{a}$ in the level set $f(\vec{x}) = c$ at t_0

- If $\vec{x}'(t_0)$ makes an acute angle with $\nabla f(\vec{a})$, then $\vec{x}(t)$ travels into the upper level set $f(\vec{x}) \geq c$.
- If $\vec{x}'(t_0)$ makes an obtuse angle with $\nabla f(\vec{a})$, then $\vec{x}(t)$ travels into the lower level set $f(\vec{x}) \leq c$.

Consider the following diagram of a feasible set and an upper level set. The point $\vec{a}$ satisfies $\nabla f(\vec{a}) = -\lambda \nabla g(\vec{a})$ and $\lambda \geq 0$, but we can see that there are also feasible points in the interior of the upper level set, like $\vec{b}$. $f(\vec{b})$ may be greater than $f(\vec{a})$.

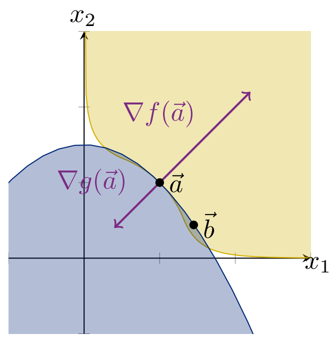


Figure 4.16: An upper level set and feasible set for which $\vec{a}$ is not a maximizer

We would like a condition to rule out this behavior. Suppose that a line separates the upper level set $f(x) \geq c$ from the feasible set. This prevents any higher values of f from appearing in the feasible set.

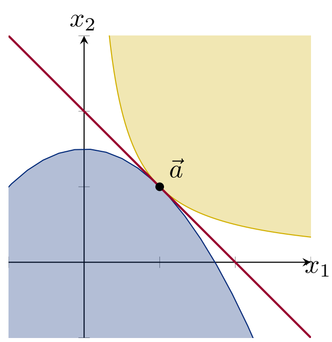


Figure 4.17: The upper level set of f, a feasible set, and a separating line

4.3.2 Hyperplanes and Half-Spaces

To formalize this reasoning, we first present the notation for a separating line and generalize it to higher dimensions. Every line h in $\mathbb{R}^2$ has a **normal vector** $\vec{v}$. h divides $\mathbb{R}^2$ into two **half-planes**:

- h^+ on the side of $\vec{v}$
- h^- on the other side.

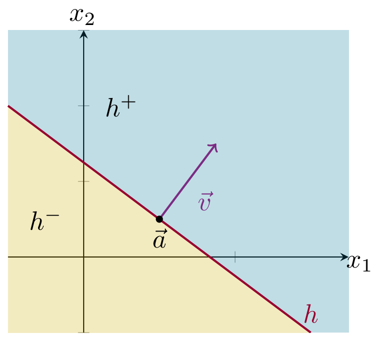


Figure 4.18: A line, its normal vector $\vec{v}$, and its half spaces

For any point $\vec{x}$, the vector that points from $\vec{a}$ to $\vec{x}$ is

$$\vec{x} - \vec{a}.$$

The angle of this vector with $\vec{v}$ tells us which half-plane contains $\vec{x}$. The dot product tells us whether this angle is acute, obtuse or right.

Lemma 4.8

Suppose h is a line in $\mathbb{R}^2$ with normal vector $\vec{v}$, and $\vec{a}$ is a point on h. For any point $\vec{x}$

$$\vec{v} \cdot (\vec{x} - \vec{a}) \begin{cases} > 0 & \text{if } \vec{x} \text{ lies in } h^+ \\ = 0 & \text{if } \vec{x} \text{ lies on } h \\ < 0 & \text{if } \vec{x} \text{ lies in } h^- \end{cases}$$

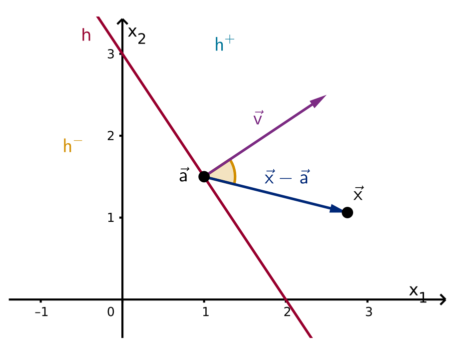


Figure 4.19: The angle between a vector and the normal vector of a line

The analogous object in $\mathbb{R}^3$ is a plane h. It has a normal vector and divides $\mathbb{R}^3$ into two **half-spaces** h^+ and h^-. The sign of $\vec{v} \cdot (\vec{x} - \vec{a})$ tests which half-space $\vec{x}$ lies in. Specifically, $\vec{x}$ lies on the plane, if $\vec{v} \cdot (\vec{x} - \vec{a}) = 0$.

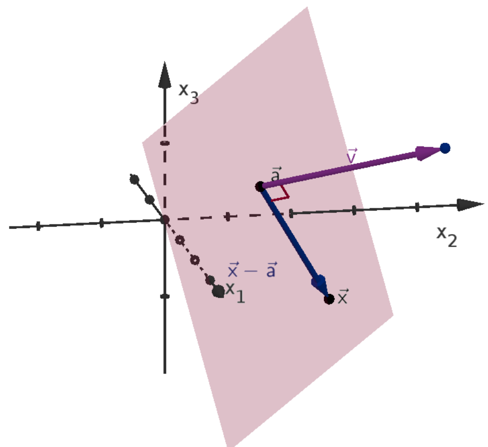


Figure 4.20: A plane, its normal vector $\vec{v}$, and a vector on it

This reasoning works in any dimension to define a set of points whose displacement from a known point $\vec{a}$ is orthogonal to a normal vector $\vec{v}$.

Definition 4.9

- In $\mathbb{R}^2$, $\vec{v} \cdot (\vec{x} - \vec{a}) = 0$ defines a line.
- In $\mathbb{R}^3$, $\vec{v} \cdot (\vec{x} - \vec{a}) = 0$ defines a plane.
- In $\mathbb{R}^n$, $\vec{v} \cdot (\vec{x} - \vec{a}) = 0$ defines a **hyperplane**, a linear $(n-1)$-dimensional subspace.

When dimension is general or ambiguous, we use the term hyperplane as a catch-all term. We can rewrite our dot product lemma to reflect the n-dimensional case.

Variant of Lemma 4.8

Suppose h is a hyperplane with normal vector $\vec{v}$ and $\vec{a}$ is a point on h. For any point $\vec{x}$

$$\vec{v} \cdot (\vec{x} - \vec{a}) \begin{cases} > 0 & \text{if } \vec{x} \text{ lies in } h^+ \\ = 0 & \text{if } \vec{x} \text{ lies on } h \\ < 0 & \text{if } \vec{x} \text{ lies in } h^- \end{cases}$$

4.3.2 Hyperplanes and Half-Spaces

For a more concise test, we can let $k = \vec{v} \cdot \vec{a}$. We can rewrite our dot product

$$\vec{v} \cdot (\vec{x} - \vec{a}) = \vec{v} \cdot \vec{x} - \vec{v} \cdot \vec{a} = \vec{v} \cdot \vec{x} - k$$

and the lemma

$$\vec{v} \cdot \vec{x} \begin{cases} > k & \text{if } \vec{x} \text{ lies in } h^+ \\ = k & \text{if } \vec{x} \text{ lies on } h \\ < k & \text{if } \vec{x} \text{ lies in } h^- \end{cases}$$

4.3.3 Optimization by Separation

In Figure 4.17, a line separates the upper level set of f and the feasible set. The point where they meet appears to be a maximizer of f. We now have the notation to formalize this argument.

Theorem 4.10

Suppose we have a continuous objective function $f(\vec{x})$ and some constraint(s). Suppose $\vec{x}^*$ is in the feasible set and let $f(\vec{x}^*) = c$. If there is a hyperplane h such that

1. the upper level set $f(\vec{x}) \geq c$ has no points in h^-
2. the feasible set has no points in h^+

then $\vec{x}^*$ is a maximizer of $f(\vec{x})$ subject to the constraint(s).

This method is called **optimization by separation** and h is a **separating hyperplane**. It is a sufficient condition, but it requires us to know the proper hyperplane h.

Notice both the upper level set and the feasible set may contain have points on h. In fact, $\vec{x}^*$ is one of them. What if they share an additional point $\vec{a}$ in h? $\vec{a}$ must lie on the boundary of the upper level set, arbitrarily close to points for which $f(\vec{x}) < c$. Since $f(\vec{x})$ is continuous, $f(\vec{a}) = c$. We can still conclude that $\vec{x}^*$ is a maximizer, but not a unique maximizer.

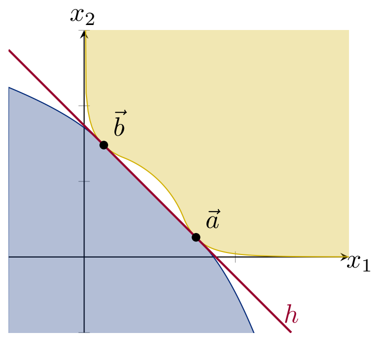


Figure 4.21: An upper level set of f and the feasible set intersecting multiple times along their separating line

We can modify this theorem by stipulating that $\vec{x}^*$ is either

1. the only point of the upper level set on h or
2. the only feasible point on h.

In this case we can conclude that $\vec{x}^*$ is the unique maximizer.

If we want to avoid talking directly about the hyperplane h, we can think of $\vec{v} \cdot \vec{x}$ as a function of $\vec{x}$. Its level sets are the hyperplanes $\vec{v} \cdot \vec{x} = k$. The value of $\vec{v} \cdot \vec{x}$ increases as we travel in the direction of $\vec{v}$.

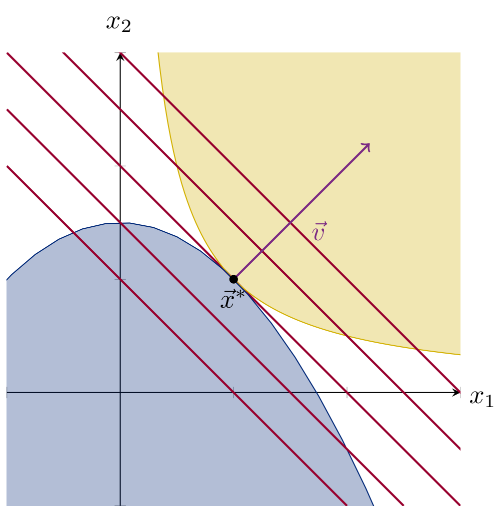


Figure 4.22: Some level sets of $\vec{v} \cdot \vec{x}$ that intersect an upper level set of f and a feasible set

Our separation argument can be rephrased to require that the upper level set meets only higher values of $\vec{v} \cdot \vec{x}$ while the feasible set meets only lower values of $\vec{v} \cdot \vec{x}$. We can verify this algebraically.

Suppose $\vec{x}$ be a point in the upper level set of $f(\vec{x})$. Lemma 4.8 states that $\vec{x}$ does not lie in h^-, if

$$\begin{aligned} \vec{v} \cdot (\vec{x} - \vec{x}^*) &\geq 0 \\ \vec{v} \cdot \vec{x} &\geq \vec{v} \cdot \vec{x}^* \end{aligned}$$

This inequality indicates that the value of $\vec{v}\cdot\vec{x}$ at $\vec{x}^*$ is less than or equal to the value at any other point in the upper level set. In other words, $\vec{x}^*$ minimizes the function $\vec{v}\cdot\vec{x}$ subject to the constraint $f(\vec{x}) \geq c$.

To show that the feasible set has no points in h^+ we can check that for each feasible $\vec{x}$:

$$\vec{v}\cdot(\vec{x}-\vec{x}^*) \leq 0$$
$$\vec{v}\cdot\vec{x} \leq \vec{v}\cdot\vec{x}^*$$

This means that the value of $\vec{v}\cdot\vec{x}$ at $\vec{x}^*$ is greater than or equal to the value at any other $\vec{x}$ in the feasible set. In other words, $\vec{x}^*$ maximizes the function $\vec{v}\cdot\vec{x}$ subject to the constraints of the original optimization problem. We can restate Theorem 4.10 in terms of this new vocabulary.

Alternate Formulation of Theorem 4.10

Suppose we have a continuous objective function f and some constraints. If $f(\vec{x}^*) = c$ and for some $\vec{v} \neq \vec{0}$ we have

1. $\vec{x}^*$ minimizes $\vec{v}\cdot\vec{x}$ subject to $f(\vec{x}) \geq c$
2. $\vec{x}^*$ maximizes $\vec{v}\cdot\vec{x}$ subject to the constraints

Then $\vec{x}^*$ maximizes $f(\vec{x})$ given the constraints.

We can see that this is a condition for a maximizer without knowing much about hyperplanes. If the feasible points all have values of $\vec{v}\cdot\vec{x}$ less than or equal to k and the upper level sets all have values greater than or equal to k, then the feasible set and the upper level can only meet where $\vec{v}\cdot\vec{x} = k$.

4.3.4 The Tangent Hyperplane

An observant reader will have noticed that our separating lines have always been tangent to the level curve $f(\vec{x}) = c$. This is not a coincidence. It occurs in higher dimensions as well, where the tangent lines become tangent hyperplanes.

Definition 4.11

Suppose a point $\vec{a}$ lies on level set $f(\vec{x}) = c$, and $\nabla f(\vec{a}) \neq \vec{0}$. The **tangent hyperplane** to $f(\vec{x}) = c$ at $\vec{a}$ is the hyperplane containing of all the tangent lines to $f(\vec{x}) = c$ at $\vec{a}$. Its normal vector is $\nabla f(\vec{a})$. Its equation is

$$\nabla f(\vec{a})\cdot(\vec{x}-\vec{a}) = 0$$

If $\nabla f(\vec{a}) \neq 0$, the only candidate for a separating hyperplane is the tangent hyperplane to $f(\vec{x}) = c$. Any other hyperplane would contain some vector $\vec{w}$ such that $\nabla f(\vec{a}) \cdot \vec{w} > 0$, meaning it would cut into the upper level set of f at $\vec{a}$.

On the other hand, we have seen examples where a tangent line fails to keep the upper level set out of h^-.

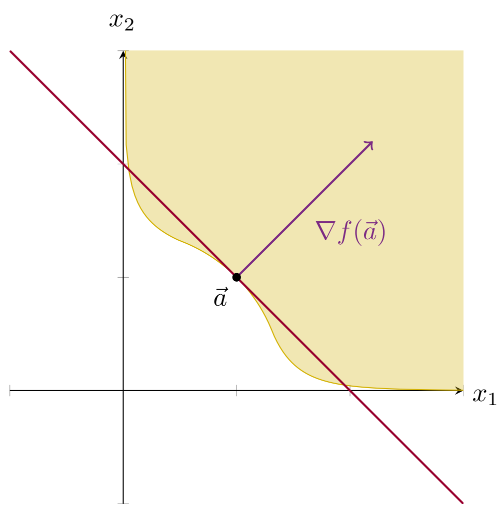


Figure 4.23: An upper level set that crosses its own tangent line

For this reason, optimization by separation is only realistic for functions with nicely shaped level sets. We will discover a class of such functions in the next section.

4.3.5 Section Summary

The most important definitions and results from this section were

- Hyperplanes and the dot product test (Definition 4.9 and Lemma 4.8)
- Optimization by separation, both formulations (Theorem 4.10 and its variant)
- Equation of a tangent hyperplane to a level set (Definition 4.11)

4.4 Concave Programming

Goals:

1. Understand the geometric consequences of concavity on constrained optimizations.
2. Apply sufficient conditions for maximizers of concave functions with concave constraints.

4.4.1 Kuhn-Tucker Sufficiency

When is a separation argument possible? When does the tangent hyperplane separate the upper level set from the feasible set? Concavity is one way to guarantee this. There is an a rich theory and toolset for optimization that is specific to concave functions. This is the field of **concave programming**. Here is our main result.

Theorem 4.12

Given an objective function $f(\vec{x})$ and constraints $g_j(\vec{x}) \geq 0$, suppose $(\vec{x}^*, \vec{\lambda}^*)$ satisfies the Kuhn-Tucker conditions:

- $\mathcal{L}_{x_i}(\vec{x}^*, \vec{\lambda}^*) = 0$ for all i
- $g_j(\vec{x}^*) \geq 0$ and $\lambda_j^* \geq 0$ with complementary slackness for each j

If $f(\vec{x})$ and the $g_j(\vec{x})$ are all concave functions, then $\vec{x}^*$ maximizes $f(\vec{x})$, subject to the constraints.

It is worth understanding the full argument of this theorem that follows, but the essential ideas are:

1. The upper level sets of a concave function are convex sets. This applies not only to $f(\vec{x}) \geq c$ but also to the feasible set, which is an intersection of the upper level sets: $g_j(\vec{x}) \geq 0$.
2. Using this convexity, we can show that the tangent hyperplane to $f(\vec{x}) = c$ separates the upper level set $f(\vec{x}) \geq c$ from the feasible set.

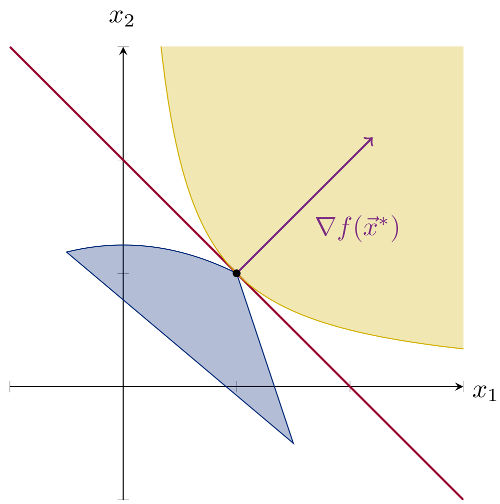


Figure 4.24: A feasible region defined by multiple concave inequalities separated from the upper level set of f by a tangent line

Remark

This sufficient condition is especially powerful because once we find a point that satisfies Kuhn-Tucker, we can stop looking. If, by luck or cleverness, we find a solution in the first case of complementary slackness we try, then we have found the maximizer. We need not examine the other cases at all.

4.4.2 Applying the Kuhn-Tucker Sufficient Condition

Find the maximum value of $f(x_1, x_2) = 4x_1 + x_2$ on the region

$$S = \{(x_1, x_2)\ :\ 25 - 7x_1 - x_2 \geq 0, x_2 \geq 0, x_2 + x_1^2 - 5 \geq 0, \text{ and } 25 - x_1^2 - x_2^2 \geq 0\}$$

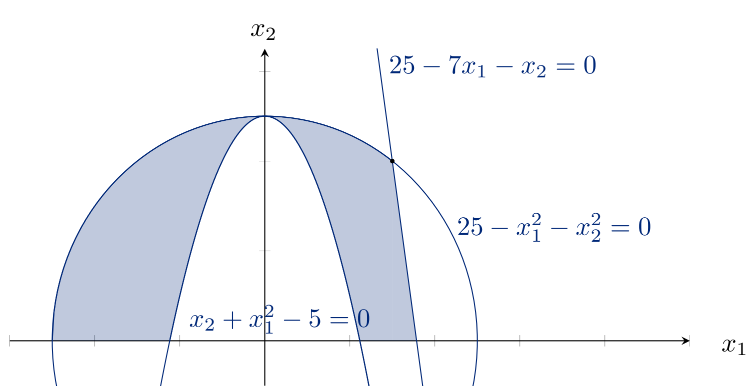


Figure 4.25: The feasible set S

Solution

Increasing x_1 seems to be the most important factor in increasing f, but larger x_2 helps too. We should draw and examine the region S. The set $x_2 + x_1^2 - 5 \geq 0$ appears to be nonconvex. On the other hand, based on our diagram, that inequality appears not to bind. We will instead maximize f over the region

$$T = \{(x_1, x_2) \ : \ 25 - 7x_1 - x_2 \geq 0, x_2 \geq 0, \text{ and } 25 - x_1^2 - x_2^2 \geq 0\}$$

S is a subset of T so a maximizer in T that lies in S is a maximizer in S. This modified problem has the following Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = 4x_1 + x_2 + \lambda_1(25 - 7x_1 - x_2) + \lambda_2 x_2 + \lambda_3(25 - x_1^2 - x_2^2).$$

The Kuhn-Tucker conditions are

$$\begin{aligned}
\mathcal{L}_{x_1} &= 4 - 7\lambda_1 - 2\lambda 3 x_1 = 0 \\
\mathcal{L}_{x_2} &= 1 - \lambda_1 + \lambda_2 - 2\lambda_3 x_2 \\
25 - 7x_1 - x_2 \geq 0 &\qquad \lambda_1 \geq 0 \qquad \lambda_1(25 - 7x_1 - x_2) = 0 \\
x_2 \geq 0 &\qquad \lambda_2 \geq 0 \qquad \lambda_2 x_2 = 0 \\
25 - x_1^2 - x_2^2 \geq 0 &\qquad \lambda_3 \geq 0 \qquad \lambda_3(25 - x_1^2 - x_2^2) = 0
\end{aligned}$$

We might guess that $25 - 7x_1 - x_2 = 0$ and $25 - x_1^2 - x_2^2 = 0$ are binding at the maximizer and $x_2 \geq 0$ is not. Based on that guess, we first consider the case where $\lambda_2 = 0$. Next we use the binding constraints to solve for x_1 and x_2.

$$\begin{aligned}
25 - 7x_1 - x_2 &= 0 \\
25 - 7x_1 &= x_2
\end{aligned}$$

$$25 - x_1^2 - x_2^2 = 0$$

$$\begin{aligned}
25 - x_1^2 - (25 - 7x_1)^2 &= 0 \\
25 - x_1^2 - 49x_1^2 + 350x_1 - 625 &= 0 \\
-50x_1^2 + 350x_1 - 600 &= 0 \\
-50(x_1 - 3)(x_1 - 4) &= 0 \\
x_1 &= 3 \text{ or } 4
\end{aligned}$$

$$\begin{aligned}
25 - 7(3) = 4 &= x_2 \text{ or} \\
25 - 7(4) = -3 &= x_2
\end{aligned}$$

We can look ahead and see that $x_2 = -3$ will not satisfy our inequalities, so we set $(x_1, x_2) = (3, 4)$. We use the remaining equations to solve for the λ_1 and λ_3.

$$\begin{aligned}
4 - 7\lambda_1 - 2\lambda_3 x_1 &= 0 \\
4 - 7\lambda_1 - 6\lambda_3 &= 0
\end{aligned}$$

$$\begin{aligned}
1 - \lambda_1 - 2\lambda_3 x_2 &= 0 \\
1 - \lambda_1 - 8\lambda_3 &= 0 \\
1 - 8\lambda_3 &= \lambda_1
\end{aligned}$$

$$\begin{aligned}
4 - 7 + 56\lambda_3 - 6\lambda_3 &= 0 \\
50\lambda_3 &= 3 \\
\lambda_3 &= \frac{3}{50}
\end{aligned}$$

$$\begin{aligned}
1 - \frac{24}{50} &= \lambda_1 \\
\frac{13}{25} &= \lambda_1
\end{aligned}$$

Our solution is $\left(3, 4, \frac{13}{25}, 0, \frac{3}{50}\right)$. We check, and it satisfies the remaining inequalities.

$$\lambda_1 = \frac{13}{25} \geq 0$$
$$x_2 = 4 \geq 0$$
$$\lambda_3 = \frac{3}{50} \geq 0$$

There are 7 other cases of complementary slackness to check, but we can avoid them. Our sufficiency theorem applies to $(3, 4)$. We only need to check for concavity of the relevant functions.

- $4x_1 + x_2$ is concave because it is linear.
- $25 - 7x_1 - x_2$ is concave because it is linear.
- x_2 is concave because it is linear.
- $25 - x_1^2 - x_2^2$ is strictly concave because its hessian is $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, which is negative definite for all (x_1, x_2).
- We ignored the constraint $x_2 + x_1^2 - 5 \geq 0$.

Based on these checks, the theorem applies, and $(3, 4)$ must be a maximizer of f on T. Since $(3, 4)$ satisfies

$$x_2 + x_1^2 - 5 = 4 + 9 - 5 \geq 0$$

it lies in S as well. Since $S \subseteq T$, we conclude $(3, 4)$ is also a maximizer of f in S. The maximum value is $f(3, 4) = 4(3) + (4) = 16$.

Main Idea

The most common methods to check concavity are

- linear functions are concave
- functions with negative definite Hessians are strictly concave

If the functions are concave, and we guess the right combination of binding constraints, then we only need to check the Kuhn-Tucker conditions for that case.

4.4.2 Applying the Kuhn-Tucker Sufficient Condition

Remark

The underlying separation argument was between the convex upper level set $f(x_1, x_2) \geq 16$ and the convex set

$$T = \{(x_1, x_2) \ : \ 25 - 7x_1 - x_2 \geq 0, x_2 \geq 0 \text{ and } 25 - x_1^2 - x_2^2 \geq 0\}$$

Since $(3, 4)$ is a maximizer on T, it is a maximizer on S.

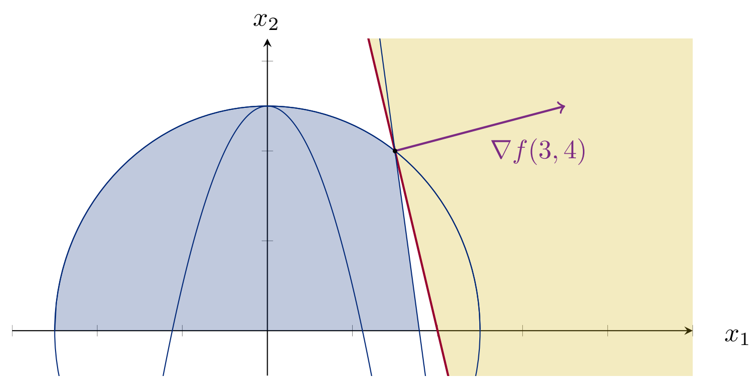


Figure 4.26: A convex set T containing S, an upper level set, and a separating line

The reasoning of this example also suggests the following variant.

Variant of Theorem 4.12

Given an objective function $f(\vec{x})$ and constraints $g_j(\vec{x}) \geq 0$, suppose $(\vec{x}^*, \vec{\lambda}^*)$ satisfies the Kuhn-Tucker conditions.

If $f(\vec{x})$ and the **binding** $g_j(\vec{x})$ are all concave, then $\vec{x}^*$ maximizes $f(\vec{x})$, subject to the constraints.

4.4.3 Proving Kuhn-Tucker Sufficiency

This will be an extensive argument with many parts, but there are two reasons to give it our attention.

1. Some of the lemmas along the way are useful or interesting in their own right.
2. There are many ways to modify this argument for different circumstances. If you understand the original argument, you can understand or even generate these variants.

Our first step is to understand the upper level sets of concave functions.

Lemma 4.13

If $f(\vec{x})$ is a concave function, then the upper level set $f(\vec{x}) \geq c$ is a convex set.

We can argue this directly from the definition of an upper level set, the definition of a convex set, and the following inequality for concave functions:

$$f((1-t)\vec{a} + t\vec{b}) \geq (1-t)f(\vec{a}) + tf(\vec{b}).$$

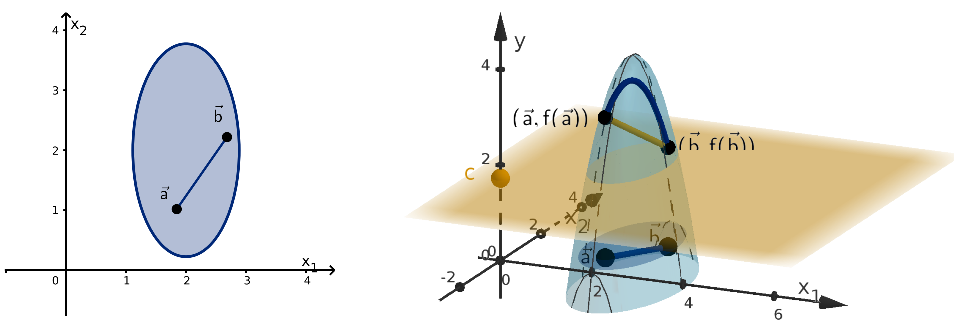


Figure 4.27: The segment from $\vec{a}$ to $\vec{b}$ in the upper level set of $f(\vec{x})$

Proof

Let $\vec{a}$ and $\vec{b}$ be points in the upper level set $f(\vec{x}) \geq c$. We will show that the segment between $\vec{a}$ and $\vec{b}$ also lies in this upper level set. The points on the segment from $\vec{a}$ to $\vec{b}$ are parametrized by

$$(1-t)\vec{a} + t\vec{b} \qquad 0 \leq t \leq 1$$

If we evaluate f along these points we get

$$\begin{aligned} f((1-t)\vec{a} + t\vec{b}) &\geq (1-t)f(\vec{a}) + tf(\vec{b}) && (f \text{ is concave}) \\ &\geq (1-t)c + tc && (\vec{a} \text{ and } \vec{b} \text{ lie in the upper level set}) \\ &\geq c \end{aligned}$$

Since $f((1-t)\vec{a} + t\vec{b}) \geq c$, $(1-t)\vec{a} + t\vec{b}$ lies in the upper level set. Since this is true for every t between 0 and 1, the entire segment from $\vec{a}$ to $\vec{b}$ lies in the upper level set. Since this is true for all $\vec{a}$ and $\vec{b}$ in the upper level set, we conclude that the upper level set is convex.

We now know that if f is concave, then its upper level sets are convex. For a separation argument, we also would like to know they do not cross their tangent hyperplane. Fortunately, this is the case.

Lemma 4.14

Let

- $f(\vec{x})$ be a continuously differentiable function
- $\vec{a}$ be a point such that $f(\vec{a}) = c$ and $\nabla f(\vec{a}) \neq \vec{0}$
- h be the tangent hyperplane to $f(\vec{x}) = c$ at $\vec{a}$.

If upper level set $f(\vec{x}) \geq c$ is convex, then it does not intersect h^-.

Proof

Suppose $\vec{b}$ is any point in the upper level set $f(\vec{x}) \geq c$. We want to show that $\vec{b}$ does not lie in h^-. Since the upper level set is convex, the entire segment

$$\begin{aligned} \vec{x}(t) &= (1-t)\vec{a} + t\vec{b} & 0 \leq t \leq 1 \\ &= \vec{a} + t(\vec{b} - \vec{a}) \end{aligned}$$

must lie within it. This specifically requires that the direction vector $\vec{x}'(0) = \vec{b} - \vec{a}$ points into the upper level set at $\vec{a}$. Lemma 2.14 tells us that if $\nabla f(\vec{a}) \cdot \vec{x}'(0) < 0$, then $\vec{x}(t)$ travels into the lower level set. Thus $\nabla f \cdot (\vec{b} - \vec{a}) \geq 0$. By Lemma 4.8, $\vec{b}$ does not lie in h^-.

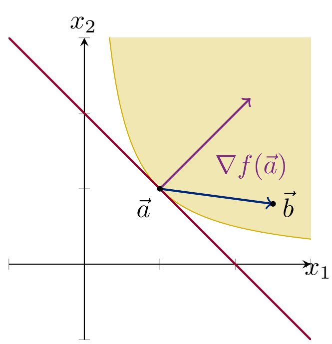


Figure 4.28: The vector $\vec{b} - \vec{a}$ in a convex upper level set making an acute angle with $\nabla f(\vec{a})$

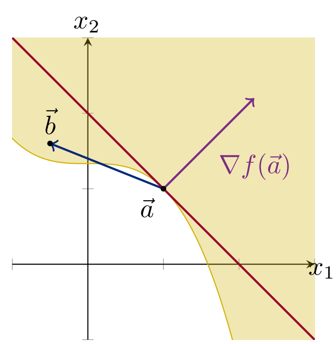


Figure 4.29: The vector $\vec{b} - \vec{a}$ leaving an upper level set and making an obtuse angle with $\nabla f(\vec{a})$

Now we turn our attention to the constraint functions. Lemma 4.13 applies as well to each g_j as it does to f. If each function $g_j(\vec{x})$ is concave, then each upper level set $g_j(\vec{x}) \geq 0$ is convex. With some very familiar looking conditions on $\nabla g_j(\vec{a})$, we can ensure that the feasible set stays on one side of the tangent hyperplane to $f(\vec{x}) = c$.

Corollary 4.15

Let $f(\vec{a}) = c$ and $\nabla f(\vec{a}) \neq \vec{0}$. Let h be the tangent hyperplane to $f(\vec{x}) = c$ at $\vec{a}$. If

1. The upper level sets $g_j(\vec{x}) \geq 0$ are convex
2. $\nabla f(\vec{a}) = -\sum_j \lambda_j \nabla g_j(\vec{a})$ for some numbers λ_j
3. For each j, $\lambda_j \geq 0$ if $g_j(\vec{a}) = 0$ and $\lambda_j = 0$ otherwise

then the intersection of the upper level sets $g_j(\vec{x}) \geq 0$ does not intersect h^+.

The geometry is easiest to visualize if we consider a single constraint $g(\vec{x}) \geq 0$. In this case, $\nabla f(\vec{a}) = -\lambda \nabla g(\vec{a})$. For any $\vec{b}$ in $g(\vec{x}) \geq 0$, the vector $\vec{b} - \vec{a}$ makes an acute angle with $\nabla g(\vec{a})$. Thus it makes an obtuse angle with $\nabla f(\vec{a})$, meaning $\vec{b}$ lies in h^-.

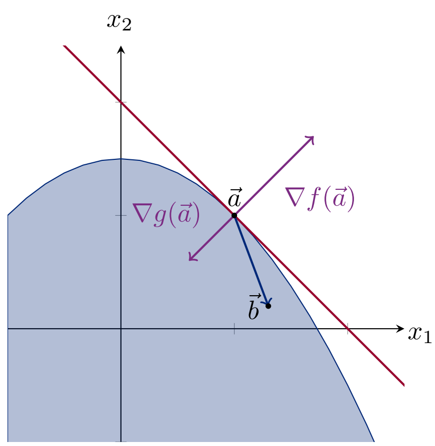


Figure 4.30: The vector $\vec{b} - \vec{a}$ in a convex feasible set making an obtuse angle with $\nabla f(\vec{a})$

For multiple constraints, angles are too difficult to discern. The dot product provides a cleaner argument.

Proof

Let $\vec{b}$ be a point in the intersection of the upper level sets. For each $g_j(\vec{x})$ that is binding at $\vec{a}$, we can apply Lemma 4.14 to the convex upper level set $g_j(\vec{x}) \geq 0$. As that proof argued, we can conclude that $\nabla g_j \cdot (\vec{b} - \vec{a}) \geq 0$.

By Lemma 4.8, the sign of $\nabla f(\vec{a}) \cdot (\vec{b} - \vec{a})$ will tell us whether $\vec{b}$ lies in h^+. We can substitute as follows:

$$\begin{aligned}\nabla f(\vec{a}) \cdot (\vec{b} - \vec{a}) &= -\left(\sum_j \lambda_j \nabla g_j(\vec{a})\right) \cdot (\vec{b} - \vec{a}) \\ &= -\sum_j \lambda_j \nabla g_j(\vec{a}) \cdot (\vec{b} - \vec{a})\end{aligned}$$

We can determine the sign of each term of the summation.

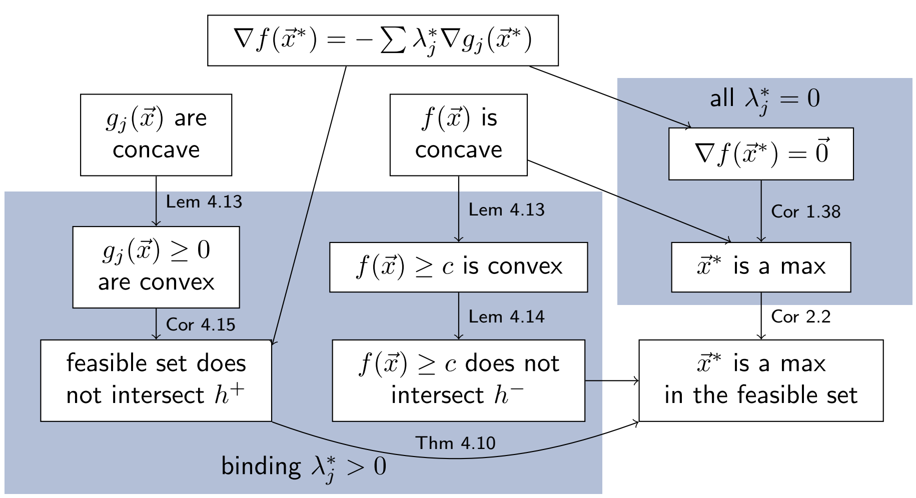
1. For nonbinding g_j, $\lambda_j = 0$.
2. For binding g_j, $\lambda_j \geq 0$ and $\nabla g_j(\vec{a}) \cdot (\vec{b} - \vec{a}) \geq 0$.

We conclude that $\nabla f(\vec{a}) \cdot (\vec{b} - \vec{a}) \leq 0$. Thus $\vec{b}$ does not lie in h^+.

We now have the ingredients to prove Theorem 4.12. There are two cases of complementary slackness to consider when proving this theorem.

1. If all $\lambda_j^* = 0$ then $\nabla f(\vec{x}^*) = 0$. Since f is concave, a variant of Corollary 1.23 tells us that $\vec{x}^*$ is an unconstrained maximizer of f. Corollary 2.2 tells us that since it lies in the feasible set, is also a maximizer there.
2. If some $\lambda_j^* > 0$ then we can put together the results of this section to conclude the upper level set of f and the feasible set lie on opposite sides of h. This is the requirement for optimization by separation established in Theorem 4.10, so $\vec{x}^*$ is a maximizer.

Here is a diagram containing the reasoning for **each case**.



In a separation argument, the upper level set and the feasible set may meet at many points in h. For example, we could have an entire line segment of intersection, and every point on that segment would satisfy Kuhn-Tucker.

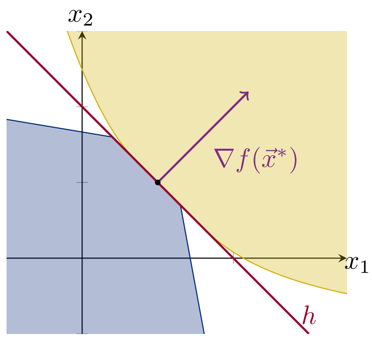


Figure 4.31: An upper level set of f and the feasible set intersecting along a separating line

This pathology only exists if both sets contain multiple points on the separating hyperplane. If one of the sets is strictly convex, this will not happen. We can achieve this with strict concavity. Each lemmas we used has a variant for strict concavity.

Variant of Lemma 4.13

If $f(\vec{x})$ is a strictly concave function then the upper level set $f(\vec{x}) \geq c$ is strictly convex.

Variant of Lemma 4.14

Suppose we have

- a continuously differentiable function $f(\vec{x})$
- a point $\vec{a}$ such that $f(\vec{a}) = c$ and $\nabla f(\vec{a}) \neq \vec{0}$
- the tangent hyperplane to $f(\vec{x}) = c$ at $\vec{a}$, denoted h.

If upper level set $f(\vec{x}) \geq c$ is strictly convex, then it lies entirely within h^+, except for the point $\vec{a}$.

Variant of Corollary 4.15

Let $f(\vec{a}) = c$ and $\nabla f(\vec{a}) \neq \vec{0}$. Let h be the tangent hyperplane to $f(\vec{x}) = c$ at $\vec{a}$. If

1. The upper level sets $g_j(\vec{x}) \geq 0$ are strictly convex
2. $\nabla f(\vec{a}) = -\sum_j \lambda_j \nabla g_j(\vec{a})$ for some numbers λ_j
3. For each j, $\lambda_j \geq 0$ if $g_j(\vec{a}) = 0$ and $\lambda_j = 0$ otherwise

then the intersection of the upper level sets $g_j(\vec{x}) \geq 0$ lies entirely within h^- except the point $\vec{a}$.

We can use these lemmas to guarantee a unique maximizer. We can either keep the upper level set or the feasible set from having multiple points on h.

Theorem 4.12 for a Unique Maximizer

Given an objective function $f(\vec{x})$ and constraints $g_j(\vec{x}) \geq 0$, suppose $(\vec{x}^*, \vec{\lambda}^*)$ satisfies the Kuhn-Tucker conditions. If $f(\vec{x})$ and the binding $g_j(\vec{x})$ are concave, and additionally either

1. $f(\vec{x})$ is strictly concave, or
2. at least one constraint binds and the binding $g_j(\vec{x})$ are strictly concave,

then $\vec{x}^*$ is the unique maximizer of $f(\vec{x})$, subject to the constraints.

Another avenue of modification is to include equality constraints. One method is to treat the equality constraint as an inequality constraint. The level set is a subset of the upper level set. By Corollary 2.2, a maximizer over an inequality constraint that happens to bind is also a maximizer over the equality constraint. This requires the constraint function to be concave and its λ_j to be positive.

Alternately, if the equality constraint is linear, then its level set is a hyperplane, which is convex. Thus the feasible set is still convex. Corollary 4.15 still holds regardless of the sign of λ_j, because $\nabla g_j(\vec{a}) \cdot (\vec{b} - \vec{a}) = 0$. We can formalize this possibility with the following variant.

Theorem 4.12 with Equality Constraints

Given an objective function $f(\vec{x})$ and constraints of the forms $g_j(\vec{x}) \geq 0$ and $g_j(\vec{x}) = 0$, suppose $(\vec{x}^*, \vec{\lambda}^*)$ satisfies the Kuhn-Tucker conditions.

If $f(\vec{x})$ and the binding $g_j(\vec{x})$ are all concave, and the equality constraints are linear, then $\vec{x}^*$ maximizes $f(\vec{x})$, given the constraints.

This section contains just a few possibilities. There are other ways to modify our sufficiency theorems that allow them to apply in more situations or to draw stronger conclusions.

4.4.5 Slater's Condition

Concavity has produced a convenient sufficient condition for constrained optimization. It can also help simplify our necessary conditions. Recall that the Kuhn-Tucker conditions are a necessary condition, but they have an exception. Theorem 2.23 states that a maximizer $\vec{a}$ must either satisfy the Kuhn-Tucker conditions or have binding $\nabla g_j(\vec{a})$ that are linearly dependent.

Checking for linearly dependent gradient vectors is difficult, especially if we do not know where to look for them. Slater's condition allows us to skip that check in some situations.

Slater's condition requires that the objective and constraint functions are concave. It also requires the feasible set to have an interior, rather than collapsing to a lower dimensional set. Formally, it demands the existence of a point $\vec{b}$ in the interior. This point is not special. If the feasible region has an interior, then you can identify infinitely many points inside it by looking at a diagram.

Theorem 4.16 [Slater's Condition]

Suppose that the functions $f(\vec{x})$ and $g_j(\vec{x})$ satisfy **Slater's condition**:

1. $f(\vec{x})$ and the $g_j(\vec{x})$ are all concave functions
2. there is at least one point $\vec{b}$ in the **interior** of the feasible set, meaning

$$g_j(\vec{b}) > 0 \text{ for all } j$$

If $\vec{a}$ is a local maximizer of $f(\vec{x})$ subject to $g_j(\vec{x}) \geq 0$, then $\vec{a}$ satisfies the Kuhn-Tucker conditions.

Remark

Slater's condition checks for concavity, just like Theorem 4.12 (our sufficient condition). This means that if $f(\vec{x})$ and $g_j(\vec{x})$ satisfy Slater's condition, then the Kuhn-Tucker conditions are both necessary and sufficient for a maximizer.

Slater's condition doesn't prevent the gradient vectors of the binding g_j from being linearly dependent. Instead it argues that even if the gradient vectors are linearly dependent at $\vec{a}$, the Kuhn-Tucker conditions must still be satisfied in order for $\vec{a}$ to be a maximizer.

Slater's condition can be strengthened to handle equality constraints in the case that those are linear.

4.4.5 Slater's Condition

Variant of Theorem 4.16

Suppose that we are maximizing $f(\vec{x})$ subject to constraints of the form $g_j(\vec{x}) \geq 0$ or $g_j(\vec{x}) = 0$. Suppose further that

1. $f(\vec{x})$ and the inequality constraint functions $g_j(\vec{x})$ are concave
2. the equality constraint functions $g_j(\vec{x})$ are linear
3. there is at least one point $\vec{b}$ in the **relative interior** of the feasible set, meaning

$$g_j(\vec{b}) > 0 \text{ for all inequality constraints}$$

$$g_j(\vec{b}) = 0 \text{ for all equality constraints}$$

If $\vec{a}$ is a local maximizer, then it satisfies the Kuhn-Tucker conditions.

4.4.6 The Separating Hyperplane Theorem

Throughout this section, we have used tangent hyperplanes to separate convex sets. It is possible to make these arguments without separation, however. Here is a short alternative proof of Theorem 4.12.

Proof

$\vec{x}^*$ satisfies the first-order condition of a **stubborn lagrangian**, in which $\vec{\lambda}$ is held constant at $\vec{\lambda}^*$.

$$\mathcal{L}^*(\vec{x}) = f(\vec{x}) + \sum_j \lambda_j^* g_j(\vec{x})$$

Since $\lambda_j^* \geq 0$, $\mathcal{L}^*$ is a sum of concave functions. By Theorem 1.20, $\mathcal{L}^*$ is concave. Thus $\vec{x}^*$ is a maximizer of $\mathcal{L}^*$. $\vec{x}^*$ also satisfies $\lambda_j^* g_j(\vec{x}^*) = 0$ by complementary slackness. If we compare $\vec{x}^*$ and any other feasible point $\vec{a}$, we have

$$\mathcal{L}^*(\vec{x}^*) \geq \mathcal{L}^*(\vec{a})$$

$$f(\vec{x}^*) + \sum_j \lambda_j^* g_j(\vec{x}^*) \geq f(\vec{a}) + \sum_j \lambda_j^* g_j(\vec{a})$$

$$f(\vec{x}^*) + 0 \geq f(\vec{a}) + \sum_j \underbrace{\lambda_j^*}_{\geq 0} \underbrace{g_j(\vec{a})}_{\geq 0}$$

$$f(\vec{x}^*) \geq f(\vec{a})$$

Since this holds for any feasible $\vec{a}$, we can conclude that $\vec{x}^*$ is a maximizer of $f(\vec{x})$ subject to $g_j(\vec{x}) \geq 0$.

This argument does not establish a separating hyperplane, even though we know one exists from our longer proof. It turns out that in any successful maximization argument for a concave function over a convex feasible set, the hyperplane is there, whether we use it or not. The following famous theorem guarantees that.

Theorem 4.17 [The Separating Hyperplane Theorem]

If S and U are two convex sets, at least one has a non-empty interior, and they share no interior points in common, then there is a vector $\vec{v}$ and a number k such that

1. For all $\vec{x}$ in U, $\vec{v} \cdot \vec{x} \geq k$
2. For all $\vec{x}$ in S, $\vec{v} \cdot \vec{x} \leq k$

Remark

This theorem tells us that if a convex upper level set U and a convex feasible set S meet only at their boundaries, then the hyperplane $\vec{v} \cdot \vec{x} = k$ separates them. It does not tell us what $\vec{v}$ is or how to construct it.

As you might infer from this remark, the applications of the separating hyperplane theorem tend to be abstract. The proof requires ideas from analysis.

4.4.7 Section Summary

The most important definitions and results from this section were

- Upper level sets of concave functions are convex (Lemma 4.13)
- Convex upper level sets lie in one half space of their tangent hyperplanes (Lemma 4.14)
- Kuhn-Tucker sufficiency for concave functions (Theorem 4.12)
- Slater's condition (Theorem 4.16)

This is also a good time to summarize the sufficient conditions we have covered. They differ in when they can be applied and what conclusion they draw.

4.4.7 Section Summary

Sufficient Conditions for Constrained Optimization

Condition	Limited to	Conclusion
EVT	bounded sets	one of the critical points is a maximizer
Bordered Hessian	binding constraint	this critical point is a local maximizer
KT Sufficiency	concave functions	this critical point is a maximizer

4.5

Quasiconcavity

Goals:

1. Recognize a quasiconcave function from its graph or level sets
2. Apply quasiconcavity in sufficient conditions for a maximizer
3. Use concavity and nondecreasing compositions to test for quasiconcavity
4. Use a determinant test to verify quasiconcavity

4.5.1

Limitations of Concave Programming

Concave programming provided sufficient conditions for a maximizer on one or more constraints. Our argument excluded functions with badly behaved level sets in order to guarantee separation, but did it exclude too many?

The function $f(x) = \frac{3}{x^2-2x+2}$ has convex upper level sets, but it is not a concave function. Could we make a separation argument for it?

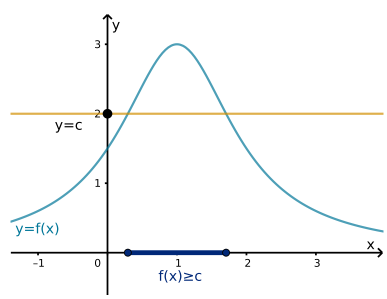


Figure 4.32: The graph of $y = \frac{3}{x^2-2x+2}$ and a convex upper level set

Another example: $f(x_1, x_2) = x_1 x_2$ (restricted to $x_1, x_2 > 0$) has strictly convex upper level sets. It is not a concave function, because

$$Hf(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4.5.1 Limitations of Concave Programming

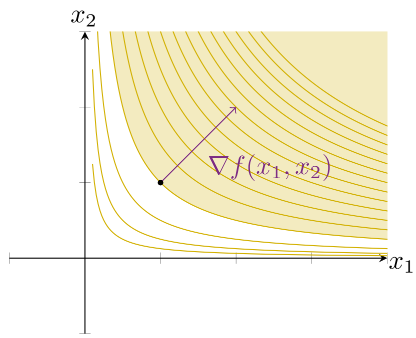


Figure 4.33: Several level sets and one upper level set of $f(x_1, x_2) = x_1 x_2$

4.5.2 Quasiconcave Functions

Theorem 4.12 requires that the objective function and constraint functions are concave, but the only part of the proof that used their concavity was the first step: that concave functions have convex upper level sets. We have now seen examples of other functions that have this property. These examples belong to a broader class of functions, on which the main ideas of concave programming still apply. We call them quasiconcave.

Definition 4.18

A function $f(\vec{x})$ is **quasiconcave** if for any points $\vec{a}$ and $\vec{b}$ in the domain of $f(\vec{x})$,

$$f((1-t)\vec{a} + t\vec{b}) \geq \min\{f(\vec{a}), f(\vec{b})\} \text{ for all } 0 \leq t \leq 1$$

It is **quasiconvex** if

$$f((1-t)\vec{a} + t\vec{b}) \leq \max\{f(\vec{a}), f(\vec{b})\} \text{ for all } 0 \leq t \leq 1$$

Quasiconcavity is a statement about the height of the graph over a path that goes across and then up.

$$f(\underbrace{(1-t)\vec{a} + t\vec{b}}_{\text{line from } \vec{a} \text{ to } \vec{b}}) \geq \underbrace{\min\{f(\vec{a}), f(\vec{b})\}}_{\text{horizontal line}} \text{ for all } 0 \leq t \leq 1$$

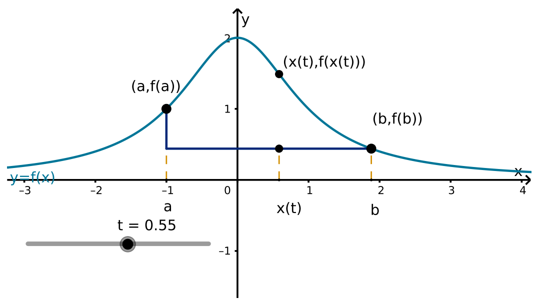


Figure 4.34: The graph of a quasiconcave function and the across and up path

Remark

"Across and up" is designed so that the path has a height that is the minimum of $f(\vec{a})$ and $f(\vec{b})$ for each $\vec{x}$ between them. If it lies below the graph then

$$f((1-t)\vec{a}+t\vec{b}) \geq \min\{f(\vec{a}), f(\vec{b})\} \text{ for all } 0 \leq t \leq 1.$$

Do not make an "across and down" or "up and across" path by mistake. Those will test a different inequality.

The name quasiconcavity suggests that these functions should be similar to concave functions. In fact, concavity is a stronger condition than quasiconcavity. A visual argument is best here. Theorem 1.19 showed that a function is concave, if and only if its graph lies above its secants.

The across and up path lies below the secant from $(\vec{a}, f(\vec{a}))$ to $(\vec{b}, f(\vec{b}))$. If $y = f(\vec{x})$ lies above the secant, then it also lies above the across and up path.

Theorem 4.19

If $f(\vec{x})$ is a concave function, then it is also a quasiconcave function.

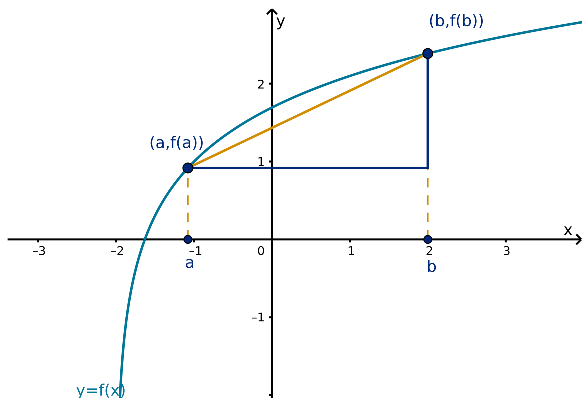


Figure 4.35: The secant and the across and up path below $y = f(x)$

Quasiconcavity is easiest to recognize in single-variable functions. We can verify that across and up paths stay below the graph $y = f(x)$ for functions with the following shapes.

Lemma 4.20

Suppose $f(x)$ is a one-variable function. $f(x)$ is quasiconcave, if one of the following is true about $f(x)$:

1. $f(x)$ is non-decreasing.
2. $f(x)$ is non-increasing.
3. For some a, $f(x)$ is non-decreasing before a and nonincreasing after a.

Functions that satisfy 1 or 2 are called **monotone**.

Remark

These conclusions only make sense for single-variable functions. Increasing and decreasing are not words that describe multivariable functions.

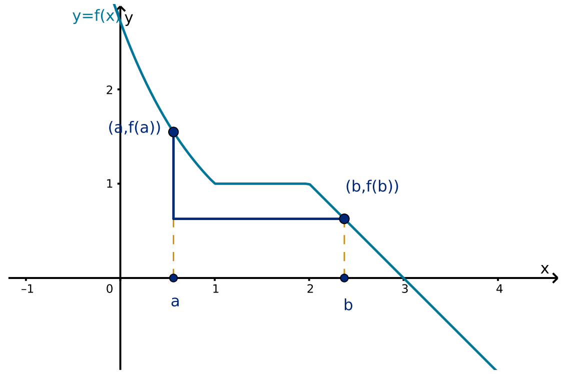


Figure 4.36: The across and up path below a non-increasing $y = f(x)$

Our goal in defining quasiconcavity was to produce a large class of functions whose upper level sets allowed for separation arguments. In fact, the quasiconcave functions are the broadest possible such class. Every function with convex upper level sets is quasiconcave, and every quasiconcave function has convex upper level sets.

Lemma 4.21

$f(\vec{x})$ is a quasiconcave function, if and only if every upper level set $f(\vec{x}) \geq c$ is a convex set.

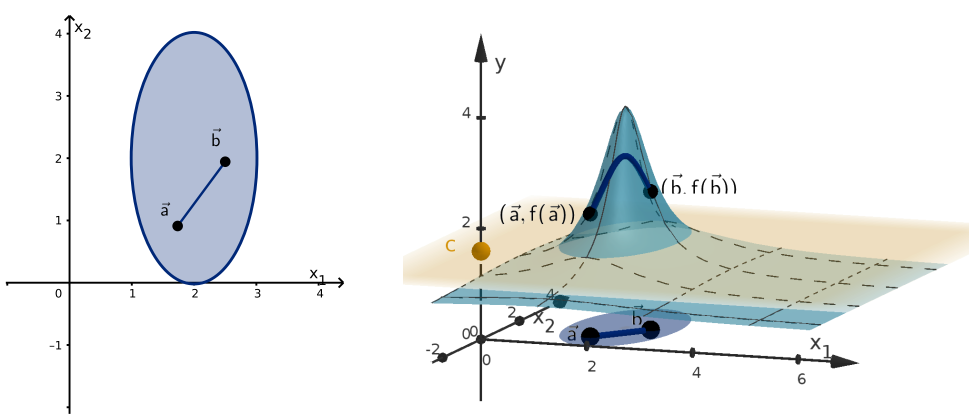


Figure 4.37: The segment from $\vec{a}$ to $\vec{b}$ in the upper level set of $f(\vec{x})$

The proof relies on combining the definitions of quasiconcavity, upper level sets, and convex sets.

Proof

There are two directions of argument that we need to make.

1. Suppose that $f(\vec{x})$ is quasiconcave. Let $\vec{a}$ and $\vec{b}$ be in the upper level set $f(\vec{x}) \geq c$. By definition, we know that $f(\vec{a}) \geq c$ and $f(\vec{b}) \geq c$. The points on the segment from $\vec{a}$ to $\vec{b}$ are parametrized by

$$(1-t)\vec{a} + t\vec{b} \qquad 0 \leq t \leq 1$$

If we evaluate f along these points we get

$$\begin{aligned} f((1-t)\vec{a} + t\vec{b}) &\geq \min\{f(\vec{a}), f(\vec{b})\} && (f \text{ is quasiconcave}) \\ &\geq c && (\vec{a} \text{ and } \vec{b} \text{ lie in the upper level set}) \end{aligned}$$

Thus the segment lies in the upper level set $f(\vec{x}) \geq c$. Since this holds for any $\vec{a}$ and $\vec{b}$ in any upper level set, we conclude that every upper level set of $f(\vec{x})$ is convex.

2. Now suppose that every upper level set of $(\vec{x})$ is convex. Let $\vec{a}$ and $\vec{b}$ be any points in the domain. Let $c = \min\{f(\vec{a}), f(\vec{b})\}$, so $\vec{a}$ and $\vec{b}$ both lie in the upper level set $f(\vec{x}) \geq c$. Since the upper level set is convex, the segment between them

$$(1-t)\vec{a} + t\vec{b} \qquad 0 \leq t \leq 1$$

lies in this set. Thus for $0 \leq t \leq 1$,

$$f((1-t)\vec{a} + t\vec{b}) \geq c = \min\{f(\vec{a}), f(\vec{b})\}$$

Since this holds for all $\vec{a}$ and $\vec{b}$, we conclude $f(\vec{x})$ is quasiconcave.

4.5.3 Quasiconcave Programming

Quasiconcave programming describes the methods we use to solve constrained optimization when the objective functions and constraint functions are quasiconcave. Most methods are analogs of results for concavity, but because quasiconcavity is a weaker condition, they often need additional conditions or draw weaker conclusions. We begin with a result for unconstrained optimization.

Theorem 4.22

If $\vec{x}^*$ is a strict local maximizer of a quasiconcave function $f(\vec{x})$, then $\vec{x}^*$ is the unique global maximizer of $f(\vec{x})$.

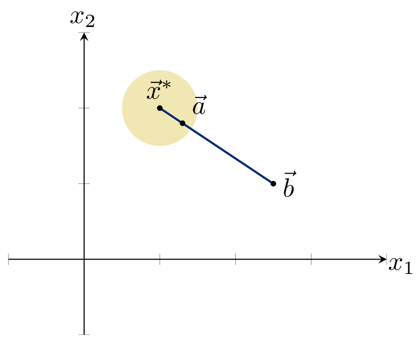


Figure 4.38: A segment, a neighborhood in which $\vec{x}^*$ is a maximizer, and a point $\vec{a}$ that lies in both

Proof

Let $\vec{b}$ be any other point in the domain of $f(\vec{x})$. $\vec{x}^*$ is the unique maximizer in some neighborhood, and the segment from $\vec{x}^*$ to $\vec{b}$ travels through this neighborhood. Let $\vec{a} \neq \vec{x}^*$ be a point that lies both on the segment and in the neighborhood.

$$f(\vec{a}) \geq \min\{f(\vec{x}^*), f(\vec{b})\} \qquad (f(\vec{x}) \text{ is quasiconcave})$$

$$f(\vec{x}^*) > \min\{f(\vec{x}^*), f(\vec{b})\} \qquad (f(\vec{x}^*) > f(\vec{a}))$$

$$f(\vec{x}^*) > f(\vec{b}) \qquad (f(\vec{x}^*) \text{ is not less than itself})$$

Since this holds for any $\vec{b}$, we conclude that $\vec{x}^*$ is the unique maximizer of $f(\vec{x})$,

Remark

For a quasiconcave function, knowing $\vec{a}$ is a critical point is not enough to conclude that $\vec{a}$ is a maximizer. For instance $f(x) = x^3$ is increasing, so it is quasiconcave. 0 is a critical point but not a maximizer.

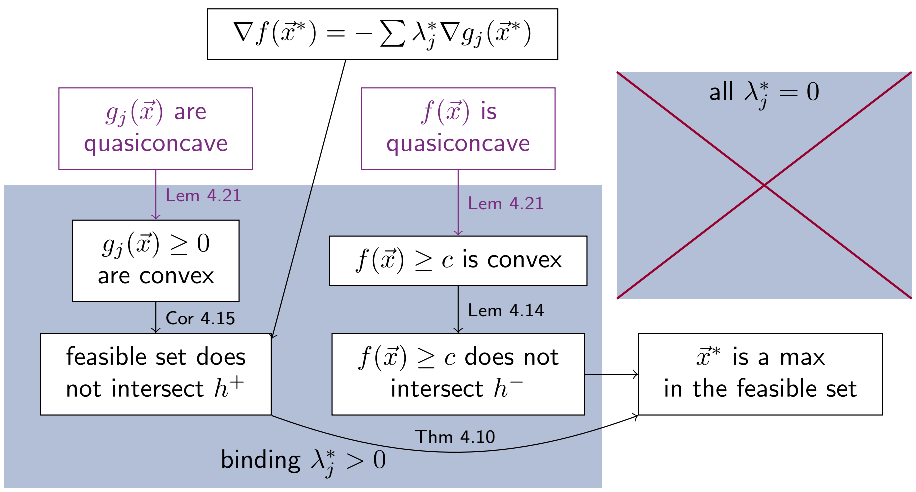
This means that our Kuhn-Tucker sufficiency theorem cannot translate directly to quasiconcavity, because there is no way to cover the $\nabla f(\vec{x}) = \vec{0}$ case. The simplest workaround is to write a theorem that only applies when $\nabla f(\vec{x}^*) \neq \vec{0}$.

4.5.3 Quasiconcave Programming

Theorem 4.23

Given an objective function $f(\vec{x})$ and constraints $g_j(\vec{x}) \geq 0$, suppose $\nabla f(\vec{x}^*) \neq \vec{0}$ and $(\vec{x}^*, \vec{\lambda}^*)$ satisfies the Kuhn-Tucker conditions. If $f(\vec{x})$ and the $g_j(\vec{x})$ are all quasiconcave, then $\vec{x}^*$ maximizes $f(\vec{x})$, subject to the constraints.

We can use almost the same argument that we used for the $\lambda_j > 0$ case with concave functions. Only the first steps need to change. Once we establish that the upper level sets are convex, the rest of the proof is identical to Theorem 4.12.



4.5.4 Positive Transformations and Quasiconcavity

The condition of quasiconcavity is **ordinal** rather than **cardinal** in nature. We care that certain points attain greater values than others, but not how much greater. Contrast this to concavity where $f((1-t)\vec{a} + t\vec{b})$ has to be at least $t(f(\vec{b}) - f(\vec{a}))$ greater than $f(\vec{a})$.

For this reason, quasiconcavity is a relevant property of utility functions. The values of a utility function reflect no inherent information beyond relative preferences. We can rescale a quasiconcave function $f(\vec{x})$ without affecting whether $f(\vec{a}) > f(\vec{b})$. The following definition and theorem formalizes this flexibility.

Definition 4.24

A function $f(\vec{x})$ is a **positive transformation** of a function $g(\vec{x})$ if there is an increasing function $p(x)$ such that $g(\vec{x}) = p(f(\vec{x}))$

Example

Consider the function $f(x_1, x_2) = x_1 x_2$ on the domain

$$\mathbb{R}^2_+ = \{(x_1, x_2) \ : \ x_1 > 0, x_2 > 0\}.$$

$p(x) = \ln x$ is an increasing function so

$$\begin{aligned} g(x_1, x_2) &= p(f(x_1, x_2)) \\ &= \ln x_1 + \ln x_2 \end{aligned}$$

is a positive transformation of f.

Positive transformation is a symmetric relation. If $p(x)$ is increasing, then so is $p^{-1}(x)$. That means that if $g(\vec{x}) = p(f(\vec{x}))$ is a positive transformation of $f(\vec{x})$, then $f(\vec{x}) = p^{-1}(g(\vec{x}))$ is also a positive transformation of $g(\vec{x})$.

Theorem 4.25

Let $f(\vec{x})$ be a function, and let $g(\vec{x})$ be a positive transformation of $f(\vec{x})$. $f(\vec{x})$ is quasiconcave if and only if $g(\vec{x})$ is quasiconcave.

Proof

Since $p(x)$ is an increasing function, larger values of $f(\vec{x})$ correspond to larger values of $g(\vec{x})$. This means that

$$f((1-t)\vec{a} + t\vec{b}) \geq f(\vec{a}), \text{ if and only if } g((1-t)\vec{a} + t\vec{b}) \geq g(\vec{a}),$$

$$\text{and } f((1-t)\vec{a} + t\vec{b}) \geq f(\vec{b}), \text{ if and only if } g((1-t)\vec{a} + t\vec{b}) \geq g(\vec{b}).$$

Putting these together gives

$$f((1-t)\vec{a} + t\vec{b}) \geq \min\{f(\vec{a}), f(\vec{b})\}, \text{ if and only if } g((1-t)\vec{a} + t\vec{b}) \geq \min\{g(\vec{a}), g(\vec{b})\}.$$

If $f(\vec{x})$ is quasiconcave, then this inequality is satisfied for all $\vec{a}$ and $\vec{b}$ in the domain and t in $[0, 1]$. This means that $g(\vec{x})$ is also quasiconcave. If $f(\vec{x})$ is not quasiconcave, then this inequality is not satisfied for some $\vec{a}$, $\vec{b}$ and t. That means that $g(\vec{x})$ is also not quasiconcave.

We will use this theorem to verify that positive transformations of concave functions are quasiconcave. Here is an example.

Example

Consider the function $f(x_1, x_2) = x_1 x_2$ on the domain

$$\mathbb{R}^2_+ = \{(x_1, x_2) \ : \ x_1 > 0, x_2 > 0\}.$$

We can reason that $f(x_1, x_2)$ is quasiconcave as follows:

1. $g(x_1, x_2) = \ln x_1 + \ln x_2$ is a positive transformation of $f(x_1, x_2)$.
2. Apply Theorem 1.41. $g(x_1, x_2)$ is concave, because its Hessian is negative definite.

$$Hg(x_1, x_2) = \begin{bmatrix} -\frac{1}{x_1^2} & 0 \\ 0 & -\frac{1}{x_2^2} \end{bmatrix}$$

3. Apply Theorem 4.19. Since $g(x_1, x_2)$ is concave, it is also quasiconcave.
4. Apply Theorem 4.25. Since $g(x_1, x_2)$ is quasiconcave, $f(x_1, x_2)$ is quasiconcave.

Remark

Notice this reasoning does not apply to concavity.

- $g(x_1, x_2) = \ln x_1 + \ln x_2$ is concave
- $f(x_1, x_2) = x_1 x_2$ is not concave

To use this method, we must find a function $g(\vec{x})$ that is a positive transformation of $f(\vec{x})$ and is also concave. If we do not have an obvious candidate, random guessing is not practical. We would like a more straightforward procedure.

4.5.5 The Bordered Hessian and Quasiconcavity

We can verify quasiconcavity by direct computation. We will derive that method here.

Recall Lemma 4.14. It showed that convex upper level sets lie in the positive half-space of their tangent hyperplanes. Another way to say this is that $\vec{a}$ is the maximizer of $f(\vec{x})$ among points on the tangent hyperplane to its level set.

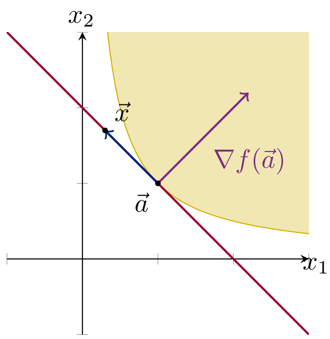


Figure 4.39: A maximizer of $f(\vec{x})$ on the tangent line to its level set

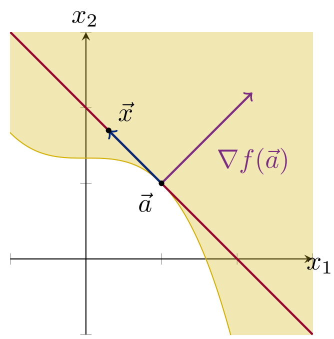


Figure 4.40: A local minimizer of $f(\vec{x})$ on the tangent line to its level set

We can prove a reverse version of this lemma with some modifications.

1. We generalize the tangent hyperplane to "points that satisfy $\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$."
2. We require that each point is a **strict local** maximizer subject to that constraint.

If this holds for all $\vec{a}$, we can conclude that $f(\vec{x})$ is quasiconcave.

Lemma 4.26

Suppose $f(\vec{x})$ is a continuously differentiable function on a convex domain. If every $\vec{a}$ in the domain of $f(\vec{x})$ is a strict local maximizer of $f(\vec{x})$ subject to $\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$, then $f(\vec{x})$ is quasiconcave.

The proof shows that $f(\vec{x})$ satisfies the inequality that defines quasiconcavity.

Proof

Let $f(\vec{x})$ be a function and $g(\vec{x})$ be a positive transformation of $f(\vec{x})$. Let $\vec{a}$ and $\vec{b}$ be points in the domain of $f(\vec{x})$. In order to show $f(\vec{x})$ is quasiconcave, we need to evaluate $f(\vec{x})$ along the segment

$$\vec{x}(t) = (1-t)\vec{a} + t\vec{b} \quad 0 \leq t \leq 1$$

Consider the composition $f(\vec{x}(t))$. By the extreme value theorem, this has a minimizer on $[0, 1]$. Let $0 < t_0 < 1$. We claim that t_0 is not a minimizer of $f(\vec{x}(t))$, there are two cases:

1. If $\frac{df(\vec{x}(t_0))}{dt} \neq 0$ then t_0 does not satisfy the first-order condition, so it is not a minimizer.
2. If $\frac{df(\vec{x}(t_0))}{dt} = 0$ then by the chain rule

$$\nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t_0) = 0$$

$$\nabla f(\vec{x}(t_0)) \cdot (\vec{b} - \vec{a}) = 0$$

For any t, the vector $\vec{x}(t) - \vec{x}(t_0)$ is parallel to $\vec{b} - \vec{a}$. Thus every $\vec{x}(t)$ satisfies $\nabla f(\vec{x}(t_0)) \cdot (\vec{x}(t) - \vec{x}(t_0)) = 0$. By the hypothesis of this lemma, $\vec{x}(t_0)$ is a strict local maximizer subject to $\nabla f(\vec{x}(t_0)) \cdot (\vec{x} - \vec{x}(t_0)) = 0$. We can conclude that t_0 is not a minimizer of $f(\vec{x}(t))$.

Since there is a minimizer of $f(\vec{x}(t))$ on $[0, 1]$, and it cannot occur for $0 < t < 1$, the minimizer must be $t = 0$ or $t = 1$. That means that

$$f((1-t)\vec{a} + t\vec{b}) \geq \min\{f(\vec{a}), f(\vec{b})\} \text{ for all } 0 \leq t \leq 1$$

Since this holds for any $\vec{a}$ and $\vec{b}$, we conclude $f(\vec{x})$ is quasiconcave.

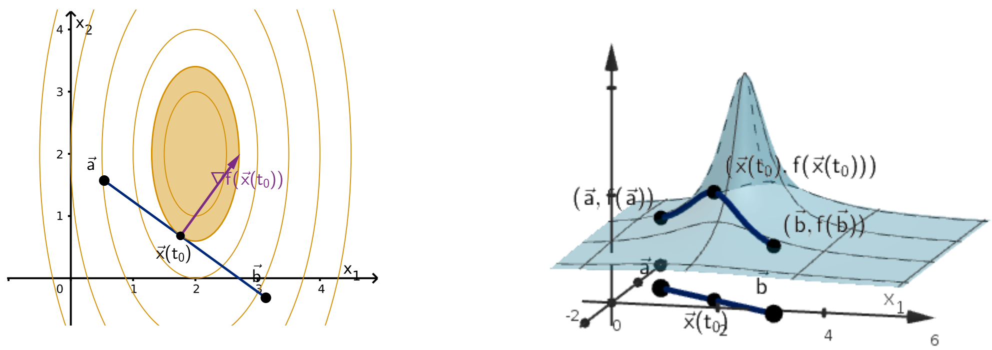


Figure 4.41: A critical point of the composition $f(\vec{x}(t))$, its gradient, and its upper level set

According to thus lemma, we now have a new way to verify quasiconcavity. We must check that every point $\vec{a}$ is a strict local maximizer of $f(\vec{x})$ subject to $\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$. We use different methods at different points $\vec{a}$. Which method we use depends on $\nabla f(\vec{a})$.

1. If $\nabla f(\vec{a}) = \vec{0}$ then every point $\vec{x}$ satisfies the constraint, making it meaningless. We can check that $\vec{a}$ is an unconstrained strict local maximizer by checking the determinants of the minors of $Hf(\vec{a})$ (Theorem 1.31).

2. If $\nabla f(\vec{a}) \neq \vec{0}$ then the Lagrangian of $f(\vec{x})$ subject to $\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$ is

$$\begin{aligned}\mathcal{L}(\lambda, \vec{x}) &= f(\vec{x}) + \lambda(\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})) \\ &= f(\vec{x}) + \lambda \sum_{i=1}^{n} f_i(\vec{a})(x_i - a_i)\end{aligned}$$

   We can check the determinants of the minors of $H\mathcal{L}(\lambda, \vec{a})$ (Theorem 4.6).

The Hessian of

$$\mathcal{L}(\lambda, \vec{x}) = f(\vec{x}) + \lambda \sum_{i=1}^{n} f_i(\vec{a})(x_i - a_i)$$

has a memorable form.

Example

For $n = 2$, the bordered Hessian is

$$HL(\lambda, \vec{x}) = \begin{bmatrix} 0 & f_1(\vec{a}) & f_2(\vec{a}) \\ f_1(\vec{a}) & f_{11}(\vec{x}) & f_{12}(\vec{x}) \\ f_2(\vec{a}) & f_{21}(\vec{x}) & f_{22}(\vec{x}) \end{bmatrix}.$$

Evaluating at $\vec{x} = \vec{a}$ gives

$$HL(\lambda, \vec{a}) = \begin{bmatrix} 0 & f_1(\vec{a}) & f_2(\vec{a}) \\ f_1(\vec{a}) & f_{11}(\vec{a}) & f_{12}(\vec{a}) \\ f_2(\vec{a}) & f_{21}(\vec{a}) & f_{22}(\vec{a}) \end{bmatrix}.$$

Due to its importance in this test, we call $H\mathcal{L}(\lambda, \vec{x})$ the **bordered Hessian of f**. We denote it $BHf(\vec{x})$, since it does not depend on λ.

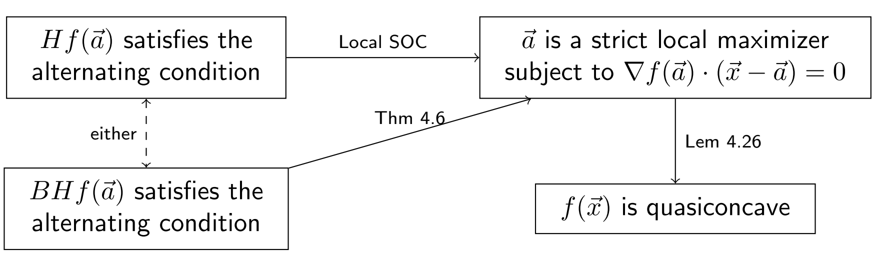
Theorem 4.27

Suppose $f(\vec{x})$ is a continuously differentiable function on a convex domain. If for each $\vec{x}$ in the domain either

1. $Hf(\vec{x})$ satisfies $(-1)^i|M_i| > 0$ for $1 \leq i \leq n$, or
2. $BHf(\vec{x})$ satisfies $(-1)^i|M_i| < 0$ for $2 \leq i \leq n+1$,

then $f(\vec{x})$ is a quasiconcave function.

Here is a diagram of the steps of the proof.



The requirement that $\vec{a}$ be a strict local maximizer of $f(\vec{x})$ subject to $\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$ is actually stronger than we need. We could make a valid argument only requiring a nonstrict local maximizer. This would add some extra complexity to the proof of Lemma 4.26. In exchange we would gain the ability to use the test for a negative semidefinite matrix instead of negative definite (Theorem 1.42). That test is unpleasant to perform, so such a result may not be worth the effort.

4.5.6 Verifying Qusiconcavity Using the Bordered Hessian

Let $f(x_1, x_2) = x_1 x_2$ on the domain

$$\mathbb{R}^2_+ = \{(x_1, x_2) \ : \ x_1 > 0, x_2 > 0\}.$$

Show that $f(x_1, x_2)$ is quasiconcave.

Solution

We need to check the Hessian where $\nabla f(x_1, x_2) = (x_2, x_1) = (0, 0)$. This only occurs at $(0, 0)$, which is not in the domain. Everywhere else we can check the bordered Hessian.

$$BHf(\vec{x}) = \begin{bmatrix} 0 & f_1(\vec{x}) & f_2(\vec{x}) \\ f_1(\vec{x}) & f_{11}(\vec{x}) & f_{12}(\vec{x}) \\ f_2(\vec{x}) & f_{21}(\vec{x}) & f_{22}(\vec{x}) \end{bmatrix} = \begin{bmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{bmatrix}$$

The minor determinants we need to check are

$$|M_2| = \begin{vmatrix} 0 & x_2 \\ x_2 & 0 \end{vmatrix} = -x_2^2 < 0$$

$$\begin{aligned} |M_3| &= \begin{vmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{vmatrix} \\ &= 0 - x^2 \begin{vmatrix} x_2 & 1 \\ x_1 & 0 \end{vmatrix} + x_1 \begin{vmatrix} x_2 & 0 \\ x_1 & 1 \end{vmatrix} \\ &= 2x_1 x_2 > 0 \end{aligned}$$

For all $x_1, x_2 > 0$, this satisfies $(-1)^i |M_i| < 0$ for $2 \le i \le 3$. We conclude that $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbb{R}^2_+$.

4.5.7 Strict Quasiconcavity

Like with strict concavity, strict quasiconcavity is defined by taking the inequality condition of quasiconcavity and making the inequalities strict.

Definition 4.28

A function $f(\vec{x})$ is **strictly quasiconcave** if for any points $\vec{a}$ and $\vec{b}$ in the domain of $f(\vec{x})$,

$$f((1-t)\vec{a}+t\vec{b}) > \min\{f(\vec{a}), f(\vec{b})\} \text{ for all } 0 < t < 1$$

It is **strictly quasiconvex** if

$$f((1-t)\vec{a}+t\vec{b}) < \max\{f(\vec{a}), f(\vec{b})\} \text{ for all } 0 < t < 1$$

We can see from this definition that it is stronger than quasiconcavity. Every strictly quasiconcave function is quasiconcave.

The condition on the upper level sets is more complicated.

Variant of Lemma 4.21

$f(\vec{x})$ is a strictly quasiconcave function, if and only if

1. every upper level set $f(\vec{x}) \geq c$ is a strictly convex set, and
2. there are no neighborhoods on which $f(\vec{x})$ is constant

A function with a thick level set will fail condition 2.

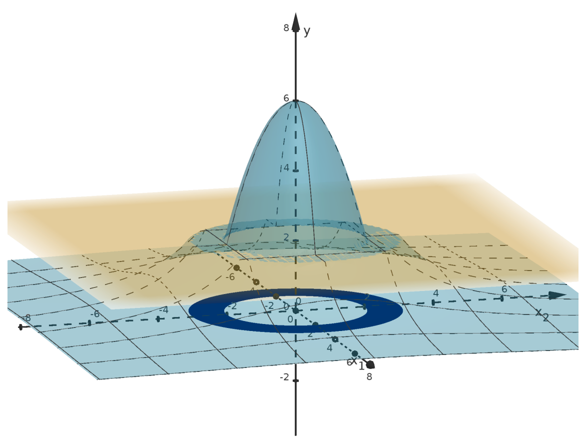


Figure 4.42: A thick level set of $f(\vec{x})$

As we might expect, strict quasiconcavity is a weaker condition than strict concavity.

Variant of Theorem 4.19

If a function $f(\vec{x})$ is strictly concave, then it is also strictly quasiconcave.

We can summarize the relationships different forms of (quasi)-concavity in the following diagram.

$$\begin{tikzcd}
f(\vec{x}) \text{ is strictly concave} \arrow[r] \arrow[d] & f(\vec{x}) \text{ is strictly quasiconcave} \arrow[d] \\
f(\vec{x}) \text{ is concave} \arrow[r] & f(\vec{x}) \text{ is quasiconcave}
\end{tikzcd}$$

Like with strict concavity, strict quasiconcavity can ensure that the Kuhn-Tucker conditions generate a unique maximizer.

Variant of Theorem 4.23

Given an objective function $f(\vec{x})$ and constraints $g_j(\vec{x}) \geq 0$, suppose $\nabla f(\vec{x}^*) \neq \vec{0}$ and $(\vec{x}^*, \vec{\lambda}^*)$ satisfies the Kuhn-Tucker conditions. If $f(\vec{x})$ and the binding $g_j(\vec{x})$ are quasiconcave, and additionally either

1. $f(\vec{x})$ is strictly quasiconcave or
2. the binding $g_j(\vec{x})$ are strictly quasiconcave,

then $\vec{x}^*$ is the unique maximizer of $f(\vec{x})$, subject to the constraints.

Like regular quasiconcavity, strict quasiconcavity is preserved under positive transformation.

Variant of Theorem 4.25

Let $f(\vec{x})$ be a function and $g(\vec{x})$ be a positive transformation of $f(\vec{x})$. $f(\vec{x})$ is strictly quasiconcave if and only if $g(\vec{x})$ is strictly quasiconcave.

The proof of Lemma 4.26 could instead produce a strict inequality with no additional reasoning. As a result we can strengthen our Hessian/bordered Hessian test to guarantee strict quasiconcavity with no modifications.

Improvement on Theorem 4.27

Suppose $f(\vec{x})$ is a continuously differentiable function on a convex domain. If for each $\vec{a}$ in the domain either

1. $Hf(\vec{x})$ satisfies $(-1)^i|M_i| > 0$ for $1 \leq i \leq n$, or
2. $BHf(\vec{x})$ satisfies $(-1)^i|M_i| < 0$ for $2 \leq i \leq n+1$,

then $f(\vec{x})$ is strictly quasiconcave.

4.5.8 Section Summary

The most important definitions and results from this section were

- The definition of quasiconcavity (Definition 4.18)
- Concave functions are quasiconcave (Theorem 4.19)
- The upper level sets of quasiconcave function are convex (Lemma 4.21)
- Sufficient conditions for a maximizer of quasiconcave functions (Theorems 4.22 and 4.23)
- Verifying quasiconcavity via composition (Theorem 4.25)
- Verifying quasiconcavity with the bordered Hessian (Theorem 4.27)
- Definition and variants for strict quasiconcavity (Definition 4.28 etc.)