Section 15.1

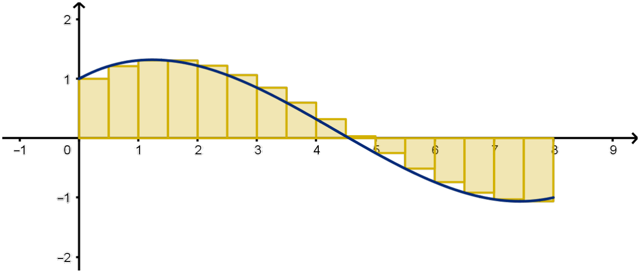
Double Integrals

Goals:

- Approximate the volume under a graph by adding prisms.
- Calculate the volume under a graph using a double integral.

The Single Variable Integral

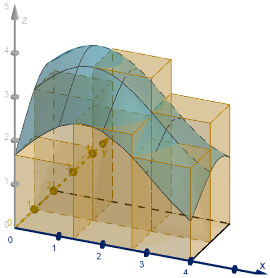
We approximate the area under the graph $y = f(x)$ by rectangles. Smaller rectangles gives a better approximation, and we define the limit of these approximations to be the **definite integral**.



$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The Two Variable Integral

A similar method approximates the signed volume under the graph $z = f(x, y)$ (where volume below the xy-plane counts as negative). We divide the domain



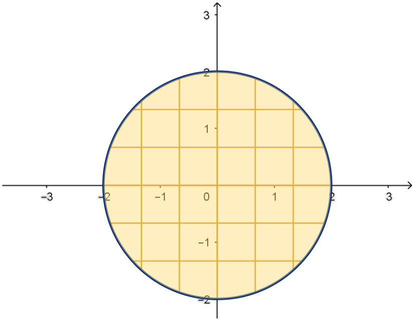
$$0 \le x \le 4$$
$$0 \le y \le 2$$

into rectangles of area A. We draw a prism over each square whose height is the value of the function over some test point (x_i^*, y_i^*).

$$\text{Volume} \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)A.$$

Non-Rectangular Subdivisions

If our domain is not a rectangle, we may not be able to divide it into rectangles. Luckily, the formula for volume of a prism works for any shape base.



$$\text{Volume} \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)A_i.$$

The Double Integral

For a reasonably well-behaved function $f(x, y)$ the volume can be computed by taking a limit of these approximations. We call this limit the double-integral.

Definition

Let D be a domain in $\mathbb{R}^2$. For a given division of D into n regions denote

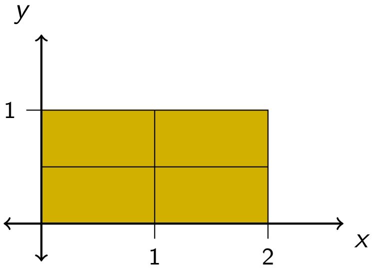
- A_i, the area of the i^{th} region.
- (x_i^*, y_i^*), any point in the i^{th} region
- A is the area of the largest region.

We define the **double integral** of $f(x, y)$ to be a limit over all possible divisions of D.

$$\iint_D f(x, y)dA = \lim_{A \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)A_i$$

Example 1

Consider $\iint_D x^2 y dA$, where D is the region shown here. Approximate the integral using the division of D shown, and evaluating $f(x, y)$ at the midpoint of each rectangle.



Fubini's Theorem

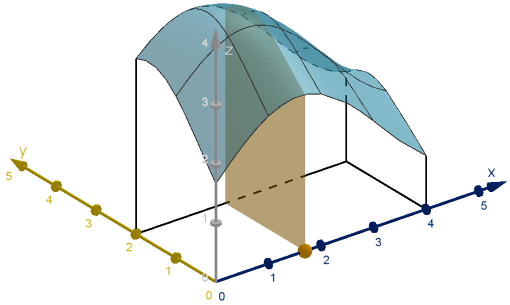
Theorem

For any domain D we have

$$\iint_D f(x,y)dA = \iint_D f(x,y)dxdy = \iint_D f(x,y)dydx.$$

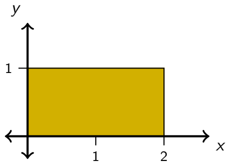
Fubini's Theorem

$\int_0^4 \int_0^2 f(x,y)\,dy\,dx$ is defined as $\int_0^4 \left(\int_0^2 f(x,y)\,dy \right) dx$. The inner integral computes the area of the cross section at each x. As in single variable calculus, integrating these areas $\int_0^4 A(x)\,dx$ gives volume.



Example 2

Compute $\iint_D x^2 y dA$, where D is the region shown here:



Integrals of Products

We can rewrite some double integrals as a product of single integrals.

Theorem

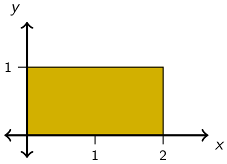
$$\int_a^b \int_c^d f(x)g(y)dydx = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right)$$

We won't be able to use this theorem all the time. It has two important requirements:

1. The bounds of integration (a, b, c, d) are constants. We'll see integrals soon where this is not the case.
2. The function can be factored into a function of x times a function of y. Most two-variable functions cannot.

Example 2 Again

Compute $\iint_D x^2 y dA$, where D is the region shown here:



Applications of Single and Double Integrals

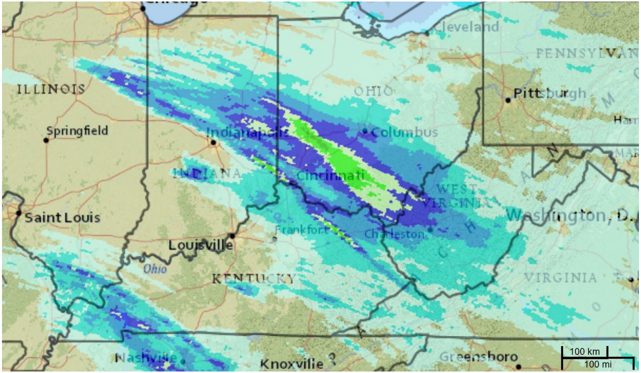
Single integrals are useful for computing total change given a rate of change.

- meters traveled per second $\longrightarrow$ total meters traveled.
- GDP growth per year $\longrightarrow$ total GDP growth.
- mass of a chemical produced per second $\longrightarrow$ total mass produced.

Double integrals are useful when we have a rate per unit of area

Application of Double Integrals - Rainfall

Integrating rainfall per square kilometer gives the total rain that fell in a watershed.

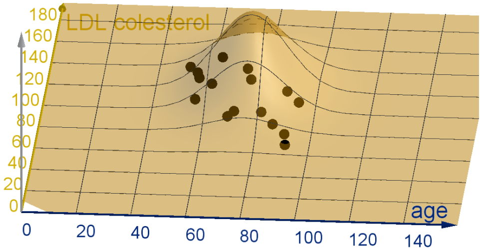


Application of Double Integrals - Energy

Integrating watts per square meter on a solar array gives the total energy generated.

Application of Double Integrals - Probability

If we generate a data set in which we've measured two variables, then the probability that a random data point lies in a given region is the double integral of a probability distribution over that area.



Summary

- What shape do we use to approximate volume under a curve? What is the formula for its volume?
- What does Fubini's Theorem tell us?
- What conditions do you need in order to write a double integral as a product of single integrals?

Section 15.2

Double Integrals over General Regions

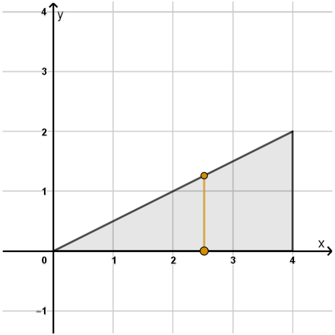
Goals:

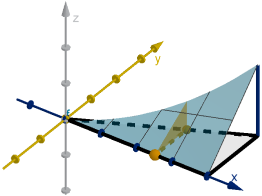
- Set up double integrals over regions that are not rectangles.
- Evaluate integrals where the bounds contain variables.
- Decide when to make $\int dy$ the outer integral, and compute the change of bounds.

Example 1

Let D be the triangle with vertices $(0, 0)$, $(4, 0)$ and $(4, 2)$. Calculate

$$\iint_D 4xy dA$$





Rewriting Integrals of the First Type

To find the bounds of a double integral

1. Find the x value where the domain begins and ends. These numbers are the bound of the outer integral.
2. Find the functions (of the form $y = g(x)$) which define the top and bottom of the domain. These functions are the bounds of the inner integral.

Exercise

Let $f(x, y)$ be a function and D be the trapezoid with vertices $(3, 1), (3, 6), (6, 5)$ and $(6, 4)$. Draw D and set up the bounds of $\iint_D f(x, y)dA$.

Integral Laws

Some single variable integral laws apply to double integrals as well (provided the integrals exist).

1. The sum rule:

$$\iint_D f(x,y) + g(x,y)dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$$

2. The constant multiple rule:

$$\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$$

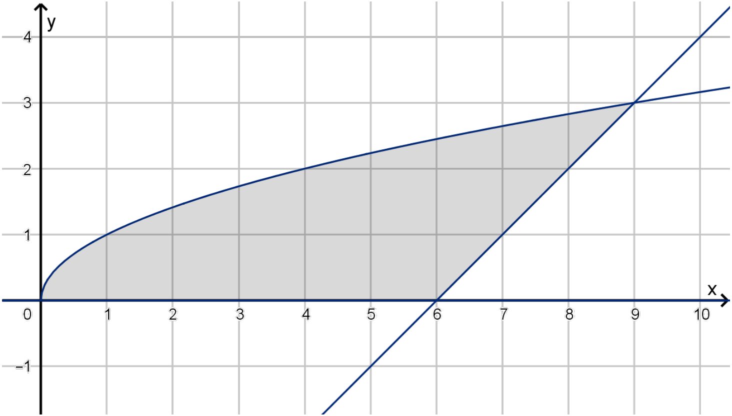
3. If D is the union of two non-overlapping subdomains D_1 and D_2 then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

Example 2

Let D be the region bounded by $y = \sqrt{x}$, $y = 0$ and $y = x - 6$. Calculate

$$\iint_D (x + y) dA.$$

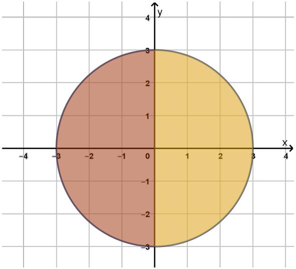


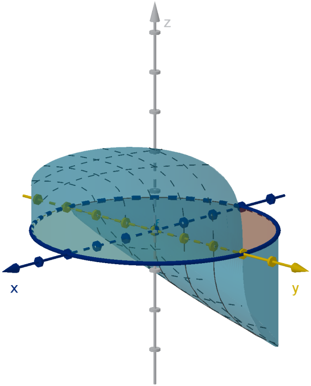
Exercise

Let $R = \{(x, y) : x^2 + y^2 \leq 9, x \geq 0\}$. Draw D and set up $\iint_D f(x, y)dA$ in two different ways.

Example 3

Let D be the region $x^2 + y^2 \leq 9$. Evaluate $\iint_D \sqrt[3]{x}\sqrt{y+3}\,dA$.



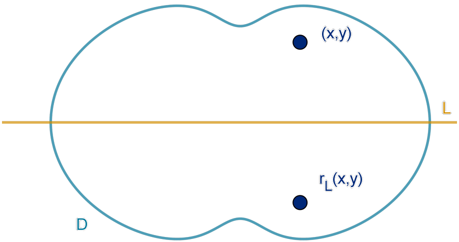


Anti-Symmetry Arguments

We can argue that an integral $\iint_D f(x,y)dA$ is equal to zero when

1. D is **symmetric** about some line L.
2. f is **antisymmetric** about L. For each point (x, y) in D the image of (x, y) across L, denoted $r_L(x, y)$ has the property:

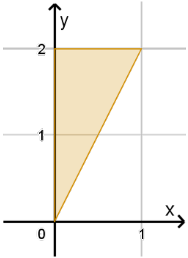
$$f(r_L(x,y)) = -f(x,y).$$



Example 4

Let D be the triangle with vertices $(0,0)$, $(0,2)$ and $(1,2)$. Calculate

$$\iint_D e^{(y^2)}\,dA.$$



Additional Applications

Theorem

The area of a region D can be calculated:

$$\iint_D 1 dA.$$

Definition

The average value of a function f over a region D is defined:

$$f_{ave} = \frac{\iint_D f(x,y)dA}{\text{Area of } D} \quad \text{or} \quad \frac{\iint_D f(x,y)dA}{\iint_D 1dA}$$

Summary

- What are the steps for writing a double integral over a general region?
- How do you decide whether dx or dy is the inner variable?

Section 15.4

Applications of Double Integrals

Goals:

- Integrate a probability distribution to calculate a probability.

Probabilities

Most probabilities that people think about are discreet.

- A flipped coin has a $\frac{1}{2}$ chance to be heads, $\frac{1}{2}$ to be tails.
- A random M&M has a $\frac{1}{6}$ chance to be red, $\frac{1}{6}$ orange, $\frac{1}{6}$ yellow, $\frac{1}{6}$ green, $\frac{1}{6}$ blue and $\frac{1}{6}$ brown.

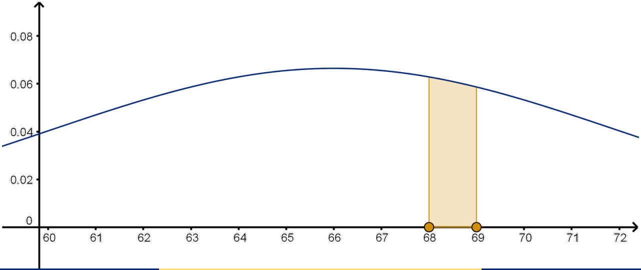
Continuous Probability Distributions

A person's chance of being exactly 68 inches tall is zero. Even people who say they are $5'8''$ are slightly more or slightly less.

Instead we can ask what your chance is of being between 68 and 69 inches tall.

Definition

A function f is a **probability distribution** for an event, if the chance of an outcome between a and b is $\int_a^b f(x)dx$.



Example 1

Darmok and Jalad each travel to the island of Tanagra and arrive between noon and 4PM. Let (x, y) represent their respective arrival times in hours after noon. Suppose the probability that (x, y) falls in a certain domain D which is a subset of $\{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq 4\}$ is $\iint_D \frac{x}{32} dydx$.

Calculate the probability that:

1. Darmok arrives after 3PM.
2. Jalad arrives before 1PM.
3. They both arrive before 2PM.
4. Darmok arrives before Jalad.
5. They arrive within an hour of each other (set it up, don't evaluate).
6. What does the distribution say about when Darmok is likely to arrive? What about Jalad?

Section 15.6

Triple Integrals

Goals:

- Set up triple integrals over three-dimensional domains.
- Evaluate triple integrals.

The Triple Integral

Definition

Given a domain D in three dimension space, and a function $f(x, y, z)$. We can subdivide D into regions

- V_i is the volume of the i^{th} region.
- (x_i^*, y_i^*, z_i^*) is a point in the i^{th} region.
- V is the volume of the largest region.

We define the **triple integral** of f over D to be the following limit over all possible divisions of D:

$$\iiint_D f(x, y, z)dV = \lim_{V \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)V_i$$

Fubini's Theorem Again

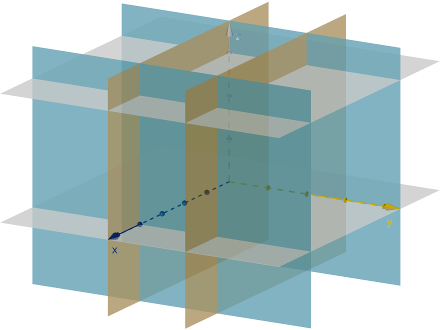
Theorem

$$\iiint_D f(x, y, z)dV = \iiint_D f(x, y, z)dxdydz$$

where the dx, dy and dz can occur in any order.

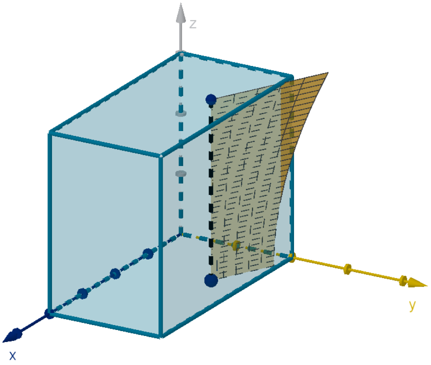
Example 1

Let $R = \{(x, y, z : 0 \leq x \leq 4, 0 \leq y \leq 2, 0 \leq z \leq 3)\}$. Set up $\iiint_R 3zy + x^2 dV$.



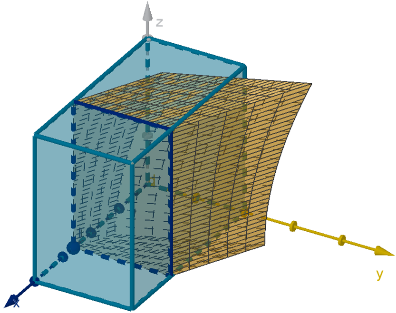
The Geometry of Triple Integrals

$\int_0^3 f(x,y,z)dz$ computes the area under the graph $w = f(x,y,z)$ over each vertical string in the domain.



The Geometry of Triple Integrals

$\int_0^2 \int_0^3 f(x, y, z) dzdy$ computes the volume under the graph $w = f(x, y, z)$ over each $x = c$ cross-section of the domain.



Applications of the Triple Integral

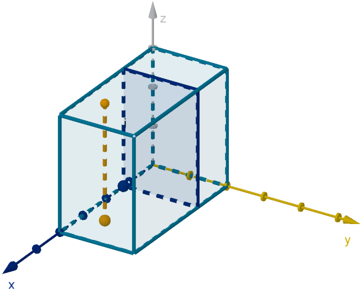
1. Integrating density gives mass.
2. Integrating density $\times$ position gives center of mass.
3. Integrating a three-dimensional probability distribution gives probability.
4. Integrating $1dV$ gives volume.

Visualizing Density

Density is a useful model for visualizing a triple integral without referring to a fourth (geometric) dimension.

$\int_0^3 f(x, y, z)dz$ computes the density of the vertical string at each (x, y).

$\int_0^2 \int_0^3 f(x, y, z)dzdy$ computes the density of the rectangle at each x.



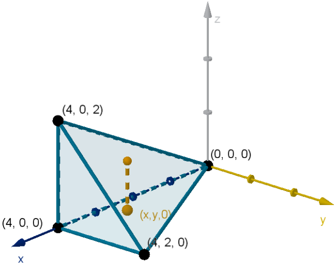
$\int_0^4 \int_0^2 \int_0^3 f(x, y, z)dzdydx$ computes the total mass of the prism.

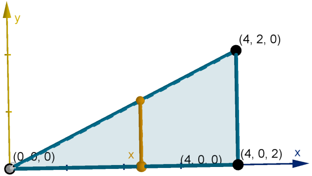
Exercise

Suppose we want to integrate over T, the tetrahedron (pyramid) with vertices $(0,0,0)$, $(4,0,0)$, $(4,2,0)$ and $(4,0,2)$.

1. Draw a careful picture of T.
2. For each (x,y) what face defines the lower bound of z. Which face defines the upper bound?
3. Sketch the set of (x,y) coordinates that belong to T. How would you set up the bounds of integration of this set?
4. Can you write equations for the faces your found in 2? How would you set up the integral?

Exercise



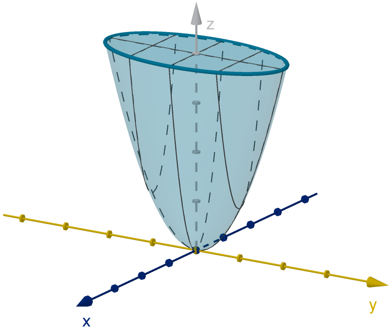


z bounds of T

x, y bounds of T

Example 2

Suppose D is the bounded region between the graph of $z = 4x^2 + y^2$ and the plane $z = 4$. Set up the bounds of the integral $\iiint_D f(x, y, z)dV$.



Exercise

Let R be the region given by $x^2 + y^2 + z^2 \leq 25$.

1. Describe R geometrically.

2. Set up the bounds of integration for $\iiint_R f(x, y, z)dV$.

3. If we plug in the function $f(x, y, z) = 1$ do you happen to know the value of this integral?

Integrals of Products

The product theorem from double integrals also works here:

Theorem

$$\int_a^b \int_c^d \int_e^f f(x)g(y)h(z)dzdydx$$
$$= \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right)\left(\int_e^f h(z)dz\right)$$

This will actually come in handy once we get to spherical integrals. They frequently have constant bounds.

Summary Questions

- What does Fubini's theorem say about dV?
- How do you find the bounds of the inner variable in a triple integral?
- How to you find the bounds of the other two variables?

Section 15.9

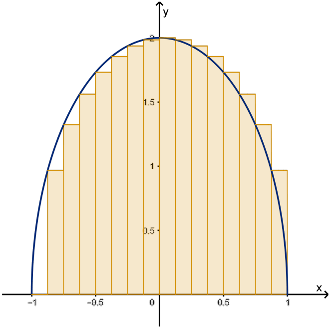
Change of Variables

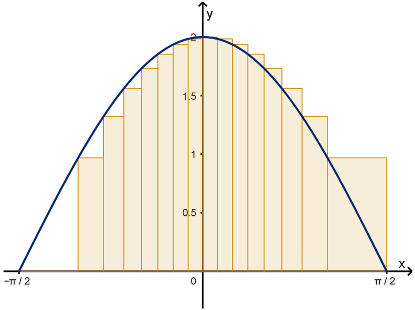
Goals:

- Calculate the Jacobian to convert from one coordinate system to another.

Single Variable Substitution

The substitution rule lets us change the variable of integration. Once we decide what variable to use, there are three things to change.



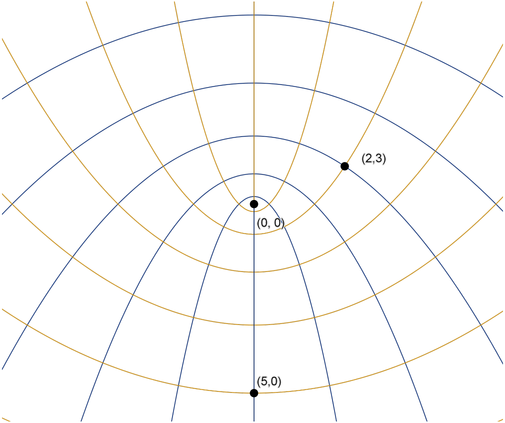


$$\int_{-1}^{1} 2\sqrt{1-x^2}dx$$

$$\underbrace{\int_{-\pi/2}^{\pi/2}}_{\text{bounds}} \underbrace{2\sqrt{1-\sin^2 u}}_{\text{integrand}} \underbrace{\cos u du}_{\text{differential}}$$

Multi-variable Example - Parabolic Coordinates

Instead of the Cartesian coordinates, we can use other coordinate systems for the plane. Here is some parabolic graph paper. Each point has coordinates (σ, τ). The gold curves are $\sigma = 0, 1, 2, 3, \ldots$. The blue curves are $\tau = 0, 1, 2, 3, \ldots$.



Converting Cartesian Coordinates to Parabolic

Formula

For a given point (σ, τ), we can calculate the corresponding (x, y) coordinates:

$$x = \sigma\tau$$
$$y = \frac{1}{2}(\tau^2 - \sigma^2)$$

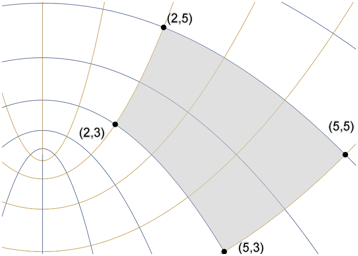
We can express this as a function

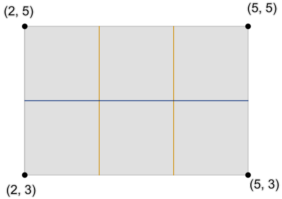
$$\mathbf{r}(\sigma, \tau) = \left\langle \sigma\tau, \frac{1}{2}(\tau^2 - \sigma^2) \right\rangle.$$

Multi-variable Substitution - The Problem

Suppose we want to integrate the function $f(x, y) = x^2$ over the domain below left. It's easier to describe this domain in (σ, τ) coordinates.

1. The bounds of integration are $2 \leq \sigma \leq 5$, and $3 \leq \tau \leq 5$.
2. We can substitute the integrand: $x^2 = \sigma^2\tau^2$.
3. But $\int_2^5 \int_3^5 \sigma^2\tau^2 d\tau d\sigma$ computes the volume over the rectangle (below right), not over our domain, which provides a larger base.

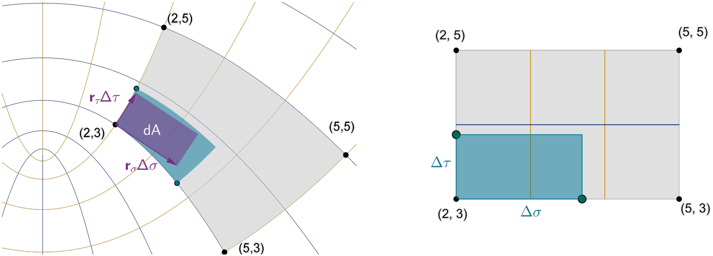




Multi-variable Substitution

When we take a $\Delta\sigma$ by $\Delta\tau$ rectangle in a Cartesian coordinate system, how much bigger does it get when we map it into the parabolic coordinate system? Its too difficult to compute it precisely. Instead, we can approximate the effect of $\Delta\sigma$ and $\Delta\tau$ by linearization.

change in σ	change in τ
$dx = \frac{\partial x}{\partial \sigma} d\sigma$	$dx = \frac{\partial x}{\partial \tau} d\tau$
$dy = \frac{\partial y}{\partial \sigma} d\sigma$	$dy = \frac{\partial y}{\partial \tau} d\tau$



Example 1

Given the formula:

$$\langle x, y \rangle = \mathbf{r}(\sigma, \tau) = \left\langle \sigma\tau, \frac{1}{2}(\tau^2 - \sigma^2) \right\rangle.$$

Find an expression for dA in terms of $d\tau d\sigma$.

The Jacobian

Definition

Given a coordinate system (u, v), the matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$$

is called the **Jacobian matrix**. The **Jacobian** is the absolute value of the determinant and is denoted:

$$\frac{\partial(x, y)}{\partial(u, v)} = |\det \mathbf{J}|$$

In an integral, $dxdy = \frac{\partial(x,y)}{\partial(u,v)} dudv$.

We will define the Jacobian similarly for a three variable coordinate system.

Summary Questions

- What does the Jacobian do?
- What three steps must we follow when rewriting an integral with a new coordinate system?

Section 15.3

Polar Coordinates

Goals:

- Convert integrals from Cartesian to polar coordinates.
- Evaluate integrals in polar coordinates.

Polar Coordinates

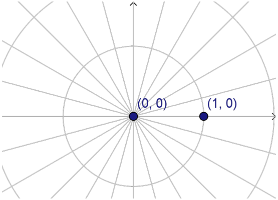
Definition

The **polar coordinates** of a point are denoted (r, θ) where

- θ ("theta") is the **direction** to the point from the origin (measured anticlockwise from the positive x axis).
- r is the distance to the point in that direction (negative r means travel backwards).

Unlike Cartesian coordinates, a point can be represented in several different ways.

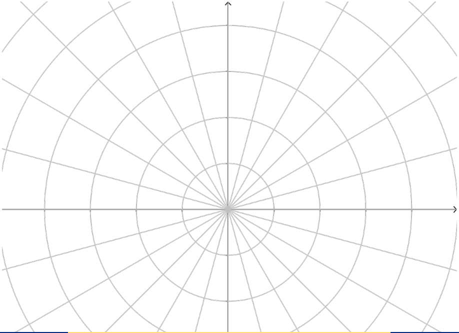
- $(1, 0) = (1, 2\pi) = (1, 4\pi)$.
- $(1, 0) = (-1, \pi)$
- $(0, \alpha) = (0, \beta)$ for all α, β.



Exercise

Plot and label the following points and sets in polar coordinates

- $A = (2, \frac{\pi}{3})$
- $B = (1.5, 3\pi)$
- $C = (-3, -\frac{\pi}{4})$
- $R = \{(r, \theta) : r \leq 2\}$
- $S = \{(r, \theta) : \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}, r \geq 1\}$



Converting to Polar Coordinates

Cartesian to Polar

$$\mathbf{p}(r,\theta) = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j}$$

$$x = r\cos\theta$$

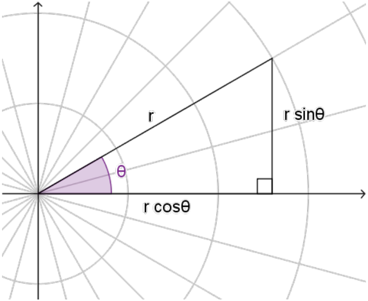
$$y = r\sin\theta$$

Notice: $x^2 + y^2 = r^2$

$$r = \sqrt{x^2+y^2}$$

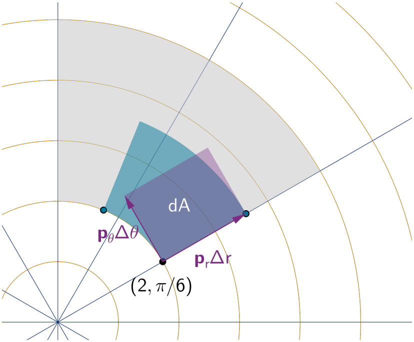
$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & x < 0 \end{cases}$$

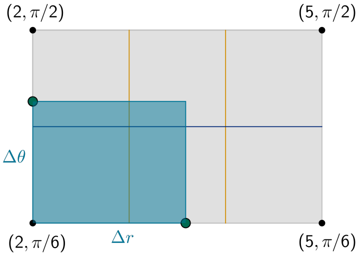
A full circle is $0 \leq \theta \leq 2\pi$.



Example 1

Calculate the Jacobian $\dfrac{\partial(x,y)}{\partial(r,\theta)}$ such that $dxdy = \dfrac{\partial(x,y)}{\partial(r,\theta)} dr d\theta$.

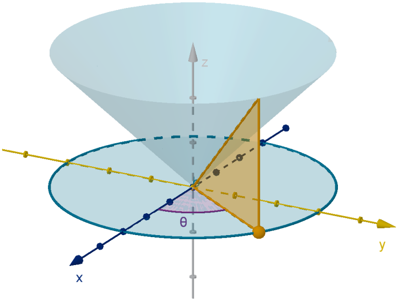




Example 2

Let D be the disk: $x^2 + y^2 \leq 9$. Calculate

$$\iint_D \sqrt{x^2 + y^2}\, dA.$$



Example 3

Let $D = \{(x, y) : x \geq 0, x \leq y, x^2 + y^2 \leq 2\}$. Sketch D and calculate

$$\iint_D x^2 dA.$$

Exercise

For each of the integrals below, sketch the domain of integration then convert to polar. You need not evaluate.

1. $\displaystyle\iint_D 2x - 3y^2 dydx$
   where $D = \{(x, y) : x^2 + y^2 \leq 16, -y \leq x \leq y\}$

2. $\displaystyle\iint_D x^2 ydydx$
   where $D = \{(x, y) : 4 \leq x^2 + y^2 \leq 9, y \leq 0\}$

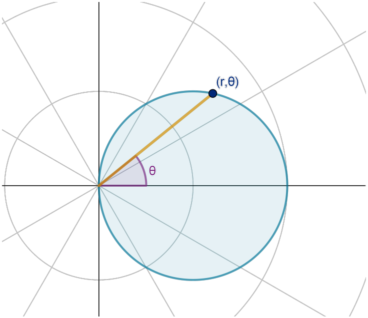
3. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} x^2 + y^2 dxdy$

Which of your integrals can be solved using the product formula?

Example 4

Let D be the domain $(x-1)^2 + y^2 \leq 1$. Evaluate

$$\iint_D x^2 + y^2 dA.$$



Trig Formulas

Higher powers of sine and cosine arise naturally in polar integrals. You'll be responsible for applying the following formulas.

Formulas

$$\sin^2\theta = \frac{1}{2} - \frac{\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1}{2} + \frac{\cos(2\theta)}{2}$$
$$\sin^3\theta = \sin\theta - \cos^2\theta\sin\theta$$
$$\cos^3\theta = \cos\theta - \sin^2\theta\cos\theta$$

Summary Questions

- How do you recognize when an integral is better evaluated in polar coordinates?
- What is the dA in polar coordinates?

Section 15.7

Cylindrical Coordinates

Goal:

- Convert an integral to cylindrical coordinates.

The Jacobian in Three Dimensions

The following generalizes the Jacobian to three dimensional coordinates:

Definition

Given a coordinate system (u, v, w), The Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix}$$

The Jacobian is the absolute value of the determinant.

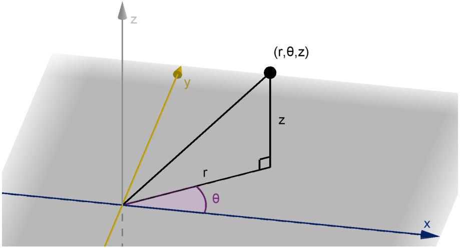
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = |\det \mathbf{J}|$$

Cylindrical Coordinates

Cylindrical coordinates are a three dimensional coordinate system, where the *xy* coordinates are replaced by polar coordinates. The conversions are

Cartesian to Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$



Exercises

1. Describe (or draw?) the following regions in cylindrical coordinates.
   a. $R = \{(r, \theta, z) : r = 2\}$
   b. $R = \{(r, \theta, z) : r \leq 5\}$
   c. $R = \{(r, \theta, z) : 0 \leq \theta \leq \frac{\pi}{4}\}$
   d. $R = \{(r, \theta, z) : z = 3\}$
2. Compute the Jacobian $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$.

Setting up Bounds in Cylindrical Integrals

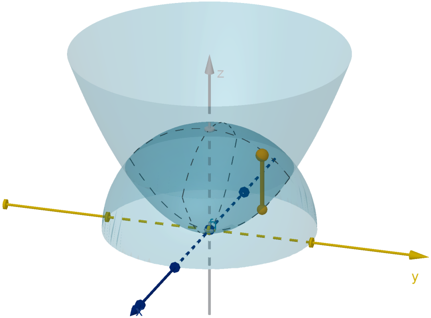
1. Most frequently, the region is bounded below by $z = f(r, \theta)$ and bounded above by $z = g(r, \theta)$. In this case, z is your inner variable.

2. If R is the region between two graphs, you'll need to find their intersection in order to determine the (r, θ) values of the domain.

3. The set of (r, θ) values is two-dimensional. Sketch this set in the plane and set up the bounds as in polar coordinates.

Example 1

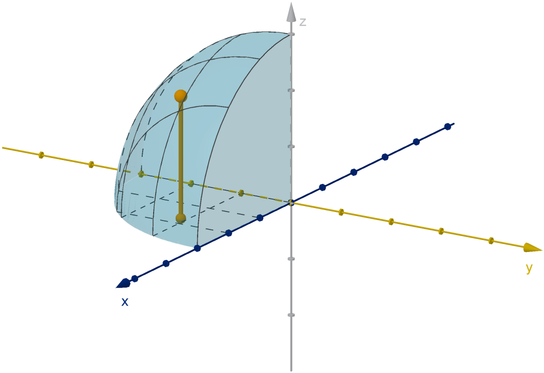
Set up the integral for $f(x, y, z)$ over the region R enclosed between the graphs $z = x^2 + y^2$ and $z = \sqrt{1 - x^2 - y^2}$.



Example 2

Convert the following triple integral to cylindrical coordinates:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{0} \int_0^{\sqrt{9-x^2-y^2}} yz^2 \, dz\,dy\,dx$$



Exercise

Set up the integrals of $f(x, y, z)$ over the following regions using cylindrical coordinates.

1. The cylinder of radius 4 about the z-axis between $z = -2$ and $z = 2$.
2. The intersection of the sphere $x^2 + y^2 + z^2 \leq 1$ and the half-spaces $x \geq 0$ and $y \leq x$.

Example 3

The (r, θ) domain might be described by some function $r \leq h(\theta)$. In this case, r goes inside θ in the order of integration.

Set up the integral for $f(x, y, z)$ over the region R which lies between the graphs $z = x^2 + y^2$ and $z = 4x$.

Exercise

Let R be the region between the graphs of $z = 1 - x^2 - (y-1)^2$ and $z = 0$. Evaluate $\iiint_R y dV$.

Summary Questions

- How do you recognize when an integral is better evaluated in cylindrical coordinates?
- What is the dV in cylindrical coordinates?

Section 15.8

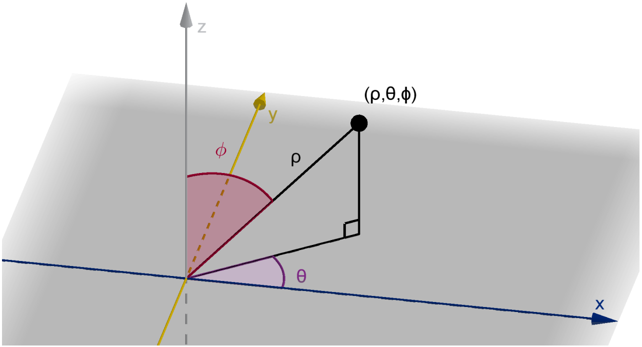
Spherical Coordinates

Goal:

- Convert an integral to spherical coordinates.

Spherical Coordinates

Spherical coordinates are a three dimensional coordinate system. Here ρ ("rho") is the (three dimensional) distance from the origin. ϕ ("phi") is the angle the radius makes with the positive z axis. θ is the angle that the projection to the xy-plane makes with the positive x-axis.



Converting to Spherical

The following formulas follow from trigonometry.

Cartesian to Spherical

$$x = \rho \cos\theta \sin\phi$$

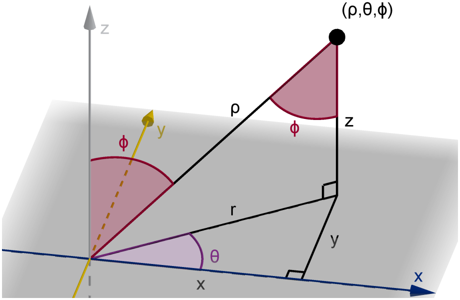
$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\phi$$

Notice: $x^2 + y^2 + z^2 = \rho^2$

A full sphere is $0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$



Exercise

Describe (or draw?) the following regions in spherical coordinates.

1. $R = \{(\rho, \theta, \phi) : \phi = \frac{\pi}{2}\}$
2. $R = \{(\rho, \theta, \phi) : \rho \leq 5\}$
3. $R = \{(\rho, \theta, \phi) : 0 \leq \theta \leq \frac{\pi}{4}\}$
4. $R = \{(\rho, \theta, \phi) : \phi \geq \frac{2\pi}{3}\}$

The Jacobian

Theorem

The Jacobian for spherical coordinates is

$$\rho^2 \sin\phi.$$

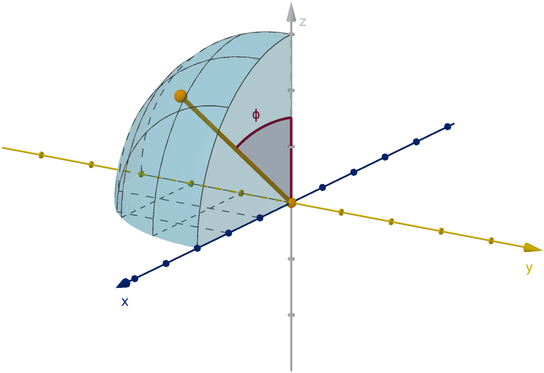
Example 1

Calculate the volume of a sphere of radius R.

Example 2

Convert the following triple integral to spherical coordinates:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{0} \int_0^{\sqrt{9-x^2-y^2}} yz^2 \, dz\,dy\,dx$$



Setting up Bounds in Spherical Integrals

Spherical coordinates are only worth using if the domain is reasonably well behaved.

1. In many cases, all the bounds of integration are constants.
2. The bounds of ρ involve the expression $x^2 + y^2 + z^2$.
3. The bounds of θ are given by inequalities containing only x and y. Draw these in the plane.
4. The bounds of ϕ are given by inequalities concerning z.
5. In some more advanced applications, the ρ bounds may be a function of ϕ or θ, meaning ρ should be the inner variable.

Exercise

Set up the integrals of $g(x, y, z)$ over the following regions using spherical coordinates.

1. The intersection of $x^2 + y^2 + z^2 \leq 4$ and $z \leq 0$.
2. The intersection of the sphere $x^2 + y^2 + z^2 \leq 1$ and the half-spaces $x \geq 0$ and $y \leq x$.
3. The intersection of the cone $z \geq \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 \leq 9$.

Summary Questions

- How do you recognize when an integral is better calculated in spherical coordinates?
- What is dV in spherical coordinates?