Section 14.1

Functions of Several Variables

Goals:
For functions of several variables be able to:

- Convert an implicit function to an explicit function.
- Calculate the domain.
- Calculate **level curves** and **cross sections**.

Functions of Two Variables

Definition

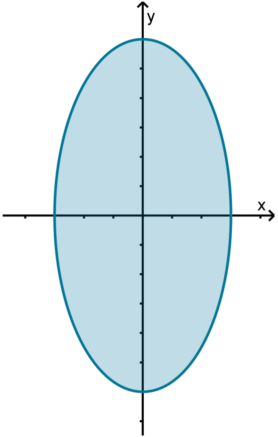
A function of two variables is a rule that assigns a number (the **output**) to each ordered pair of real numbers (x, y) in its **domain**. The output is denoted $f(x, y)$.

Some functions can be defined algebraically. If $f(x, y) = \sqrt{36 - 4x^2 - y^2}$ then

$$f(1, 4) = \sqrt{36 - 4 \cdot 1^2 - 4^2} = 4.$$

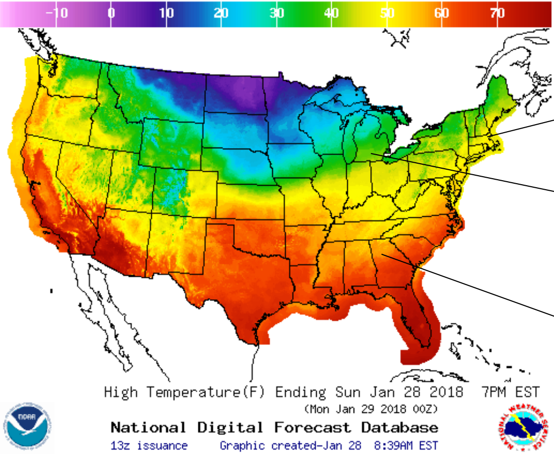
Example 1

Identify the domain of $f(x, y) = \sqrt{36 - 4x^2 - y^2}$.



Example - Temperature Function

Many useful functions cannot be defined algebraically. If (x, y) are the longitude and latitude of a position on the earth's surface. We can define $T(x, y)$ which gives the temperature at that position.



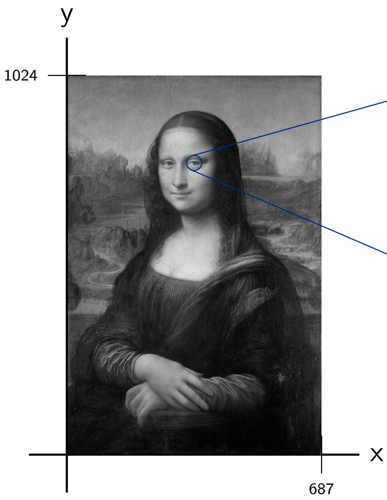
$T(-71.06, 42.36) = 50$

$T(-83.74, 42.28) = 41$

$T(-84.38, 33.75) = 59$

Example - Digital Images

A digital image can be defined by a brightness function $B(x, y)$.

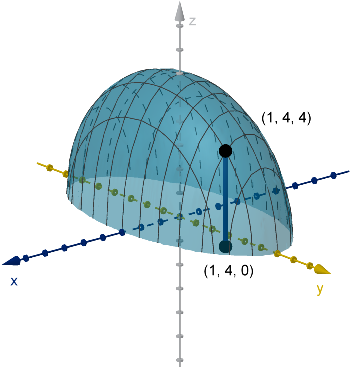


The Graph of a Function of Two Variables

Definition

The graph of a function $f(x, y)$ is the set of all points (x, y, z) that satisfy

$$z = f(x, y).$$



Here is the graph

$$z = \sqrt{36 - 4x^2 - y^2}$$

We see that $f(1, 4)$ is realized geometrically by the height of the graph above $(1, 4, 0)$.

Level Curves

Definition

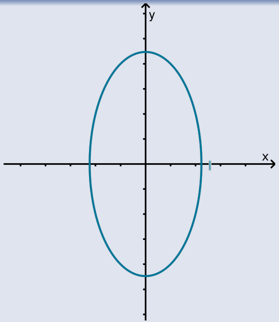
A **level set** of a function $f(x, y)$ is the graph of the equation $f(x, y) = c$ for some constant c. For a function of two variables this graph lies in the xy-plane and is called a **level curve**.

Example

Consider the function

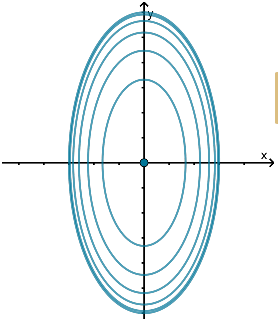
$$f(x, y) = \sqrt{36 - 4x^2 - y^2}.$$

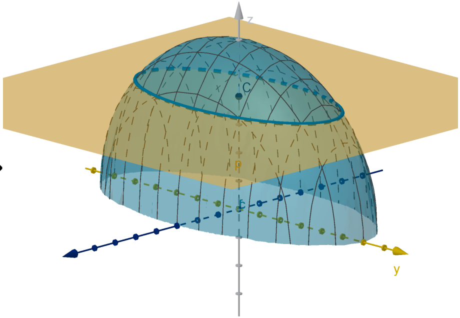
The level curve $\sqrt{36 - 4x^2 - y^2} = 4$ simplifies to $4x^2 + y^2 = 20$. This is an ellipse.



The Geometry of Level Curves

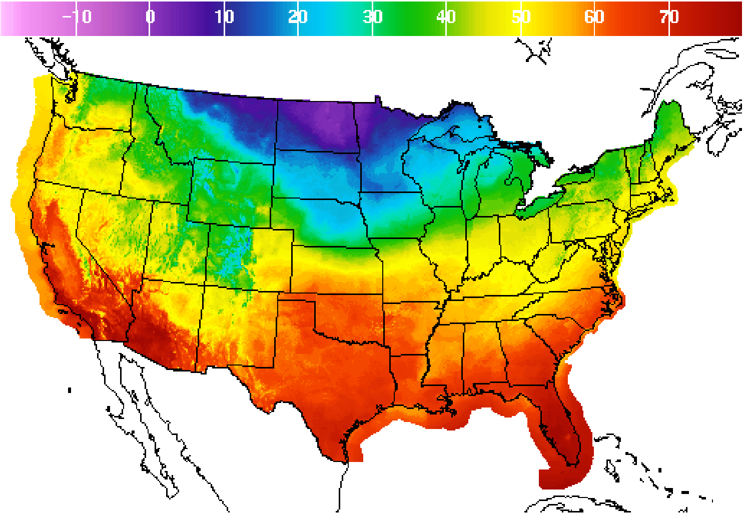
Level curves take their shape from the intersection of $z = f(x, y)$ and $z = c$. Seeing many level curves at once can help us visualize the shape of the graph.





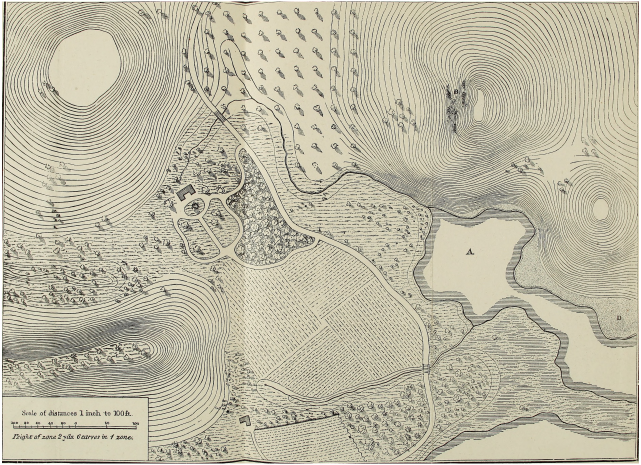
Example 3

Where are the level curves on this temperature map?



Example 4

What features can we discern from the level curves of this topographical map?

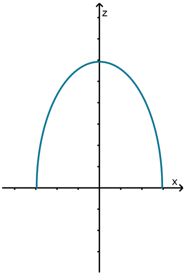


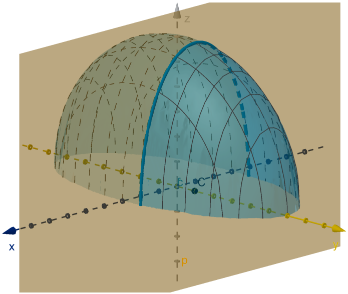
Example 5 - Cross Sections

Definition

The intersection of a plane with a graph is a **cross section**. A level curve is a type of cross section, but not all cross sections are level curves.

Find the cross section of $z = \sqrt{36 - 4x^2 - y^2}$ at the plane $y = 1$.



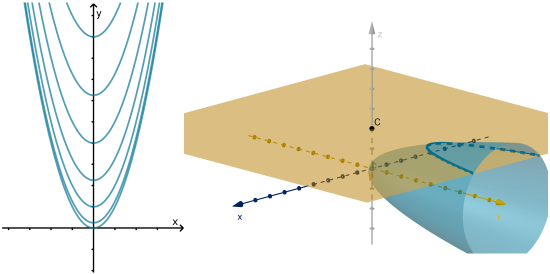


Example 6 - Converting to Explicit Functions

Definition

We sometimes call an equation in x, y and z an **implicit function**. Often in order to graph these, we convert them to **explicit functions** of the form $z = f(x, y)$

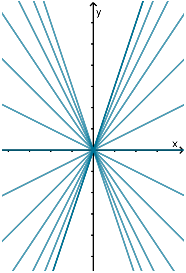
Write the equation of a paraboloid $x^2 - y + z^2 = 0$ as one or more explicit functions so it can be graphed. Then find the level curves.

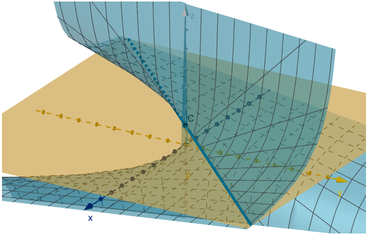


Exercise

Consider the implicit equation $zx = y$

1. Rewrite this equation as an explicit function $z = f(x, y)$.
2. What is the domain of f?
3. Solve for and sketch a few level sets of f.
4. What do the level sets tell you about the graph $z = f(x, y)$?





Functions of More Variables

We can define functions of three variables as well. Denoting them

$$f(x, y, z).$$

The definitions of this section can be extrapolated as follows.

Variables	2	3
Function	$f(x, y)$	$f(x, y, z)$
Domain	subset of $\mathbb{R}^2$	subset of $\mathbb{R}^3$
Graph	$z = f(x, y)$ in $\mathbb{R}^3$	$w = f(x, y, z)$ in $\mathbb{R}^4$
Level Sets	level curve in $\mathbb{R}^2$	**level surface** in $\mathbb{R}^3$

Functions and Level Sets

Observation

We might hope to solve an implicit equation of n variables to obtain an explicit function of $n-1$ variables. However, we can also treat it as a level set of an explicit function of n variables (whose graph lives in $n+1$ dimensional space).

$$\begin{tikzcd} x^2+y^2+z^2=25 \arrow[rr] \arrow[dr] & & z=\pm\sqrt{25-x^2-y^2} \\ & \begin{array}{c} w=x^2+y^2+z^2 \\ w=25 \end{array} & \end{tikzcd}$$

Both viewpoints will be useful in the future.

Summary

- What does the height of the graph $z = f(x, y)$ represent?
- What is the distinction between a level set and a cross section?
- What is the difference between an implicit equation and explicit function?

Section 14.2

Limits and Continuity

Goals:

- Understand the definition of a **limit** of a multivariable function.
- Use the Squeeze Theorem
- Apply the definition of **continuity**.

The Definition of a Limit

Definition

We write

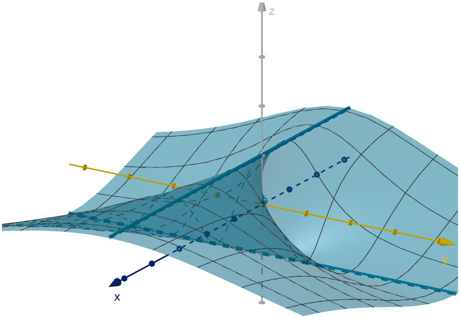
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if we can make the values of f stay arbitrarily close to L by restricting to a sufficiently small neighborhood of (a, b).

Proving a limit exists requires a formula or rule. For any amount of closeness required (ϵ), you must be able to produce a radius δ around (a, b) sufficiently small to keep $|f(x, y) - L| < \epsilon$.

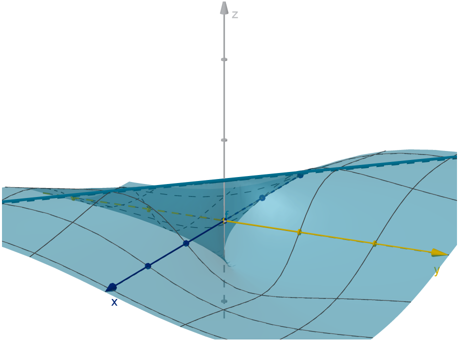
Non-Example 1

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.



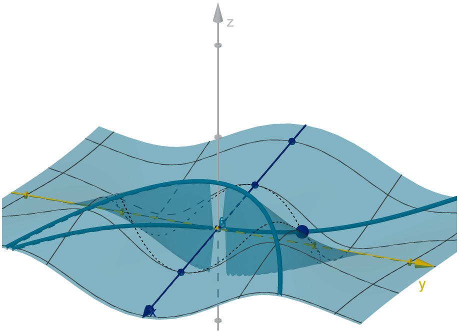
Non-Example 2

Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.



Non-Example 3

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.



Limit Laws and the Squeeze Theorem

The limit laws from single variable limits transfer comfortably to multi variable functions.

1. Sum/Difference Rule
2. Constant Multiple Rule
3. Product/Quotient Rule

The Squeeze Theorem

If $g < f < h$ in some neighborhood of (a, b) and

$$\lim_{(x,y)\to(a,b)} g(x, y) = \lim_{(x,y)\to(a,b)} h(x, y) = L,$$

then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

Continuity

Definition

We say $f(x, y)$ is **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Three of More Variables

Everything we've done has a three or n-dimensional analogue.

Summary

- Why is it harder to verify a limit of a multivariable function?
- What do you need to check in order to determine whether a function is continuous?

Section 14.3

Partial Derivatives

Goals:

- Calculate **partial derivatives**.
- Realize when not to calculate partial derivatives.

Motivational Example

The force due to gravity between two objects depends on their masses and on the distance between them. Suppose at a distance of $8,000$km the force between two particular objects is 100 newtons and at a distance of $10,000$km, the force is 64 newtons.

How much do we expect the force between these objects to increase or decrease per kilometer of distance?

Derivatives of Single-Variable Functions

Derivatives of a single-variable function were a way of measuring the change in a function. Recall the following facts about $f'(x)$.

1. Average rate of change is realized as the slope of a secant line:

$$\frac{f(x) - f(x_0)}{x - x_0}$$

2. The derivative $f'(x)$ is defined as a limit of slopes:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3. The derivative is the instantaneous rate of change of f at x.

4. The derivative $f'(x_0)$ is realized geometrically as the slope of the tangent line to $y = f(x)$ at x_0.

5. The equation of that tangent line can be written in point-slope form:

$$y - y_0 = f'(x_0)(x - x_0)$$

Limit Definition of Partial Derivatives

A partial derivative measures the rate of change of a multivariable function as one variable changes, but the others remain constant.

Definition

The **partial derivatives** of a two-variable function $f(x, y)$ are the functions

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Notation

The partial derivative of a function can be denoted a variety of ways. Here are some equivalent notations

- f_x
- $\frac{\partial f}{\partial x}$
- $\frac{\partial z}{\partial x}$
- $\frac{\partial}{\partial x} f$
- $D_x f$

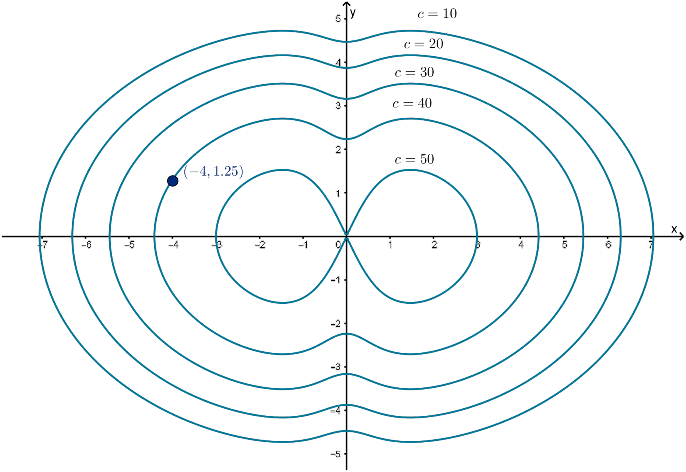
Example 1

To find f_y, we perform regular differentiation, treating y as the independent variable and x as a constant.

1. Find $\frac{\partial}{\partial y}(y^2 - x^2 + 3x \sin y)$.

Example 2

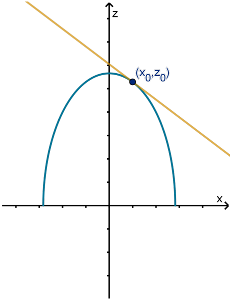
Below are the level curves $f(x, y) = c$ for some values of c. Can we tell whether $f_x(-4, 1.25)$ and $f_y(-4, 1.25)$ are positive or negative?

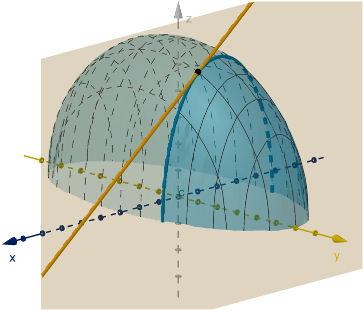


Geometry of the Partial Derivative

The partial derivative $f_x(x_0, y_0)$ is realized geometrically as the slope of the line tangent to $z = f(x, y)$ at (x_0, y_0, z_0) and traveling in the x direction.

Since y is held constant, this tangent line lives in the cross section $y = y_0$. For instance, here is $f_x(x_0, y_0)$ for $f(x, y) = \sqrt{36 - 4x^2 - y^2}$.





Example 3

Find f_x for the following functions $f(x, y)$:

1. $f = \sqrt{xy} \qquad x > 0, y > 0$
2. $f = \frac{y}{x}$
3. $f = \sqrt{x + y}$
4. $f = \sin(xy)$

Exercises

Find f_x and f_y for the following functions $f(x, y)$

1. $f(x, y) = x^2 - y^2$

2. $f(x, y) = \sqrt{\frac{y}{x}} \qquad x > 0, y > 0$

3. $f(x, y) = ye^{xy}$

Limitations of the Partial Derivative

Suppose Jinteki Corporation makes widgets which is sells for \$100 each. If W is the number of widgets produced and C is their operating cost, Jinteki's profit is modeled by

$$P = 100W - C.$$

Since $\frac{\partial P}{\partial W} = 100$ does this mean that increasing production can be expected to increase profit at a rate of \$100 per widget?

Functions of More Variables

We can also calculate partial derivatives of functions of more variables. All variables but one are held to be constants. For example if

$$f(x, y, z) = x^2 - xy + \cos(yz) - 5z^3$$

then we can calculate $\frac{\partial f}{\partial y}$:

Exercise

For an ideal gas, we have the law $P = \frac{nRT}{V}$, where P is pressure, n is the number of moles of gas molecules, T is the temperature, and V is the volume.

1. Calculate $\frac{\partial P}{\partial V}$.
2. Calculate $\frac{\partial P}{\partial T}$.
3. (Science Question) Suppose we're heating a sealed gas contained in a glass container. Does $\frac{\partial P}{\partial T}$ tell us how quickly the pressure is increasing per degree of temperature increase?

Higher Order Derivatives

Taking a partial derivative of a partial derivative gives us a higher order derivative. We use the following notation.

Notation

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

We need not use the same variable each time

Notation

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial^2 f}{\partial y \partial x}$$

Example 4

If $f(x, y) = \sin(3x + x^2 y)$ calculate f_{xy}.

Exercise

If $f(x, y) = \sin(3x + x^2y)$ calculate f_{yx}.

Does Order Matter?

No. Specifically, the following is due to Clairaut:

Theorem

If f is defined on a neighborhood of (a, b) and the functions f_{xy} and f_{yx} are both continuous on that neighborhood, then $f_{xy}(a, b) = f_{yx}(a, b)$.

This readily generalizes to larger numbers of variables, and higher order derivatives. For example $f_{xyyz} = f_{zyxy}$.

Summary Questions

- What does the partial derivative $f_y(a, b)$ mean geometrically?
- Can you think of an example where the partial derivative does not accurately model the change in a function?
- What is Clairaut's Theorem?

Section 14.4

Tangent Planes and Linear Approximations

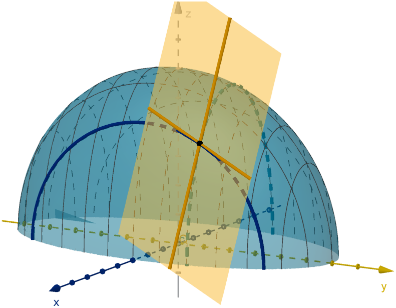
Goals:

- Calculate the equation of a **tangent plane**.
- Rewrite the tangent plane formula as a **linearization** or **differential**.
- Use linearizations to estimate values of a function.
- Use a differential to estimate the error in a calculation.

Tangent Planes

Definition

A **tangent plane** at a point $P = (x_0, y_0, z_0)$ on a surface is a plane containing the tangent lines to the surface through P.



The Equation of a Tangent Plane

Equation

If the graph $z = f(x, y)$ has a tangent plane at (x_0, y_0), then it has the equation:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Remarks:

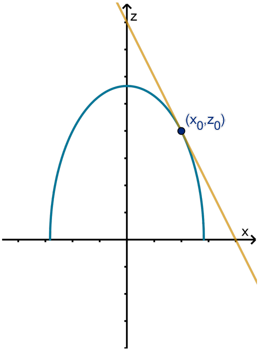
1. This is the normal equation of a plane if we move the $z - z_0$ terms to the other side.
2. x_0 and y_0 are numbers, so $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are numbers. The variables in this equation are x, y and z.

Understanding the Tangent Plane Equation

The cross sections of the tangent plane give the equation of the tangent lines we learned in single variable calculus.

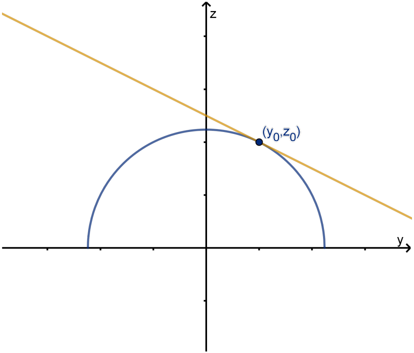
$$y = y_0$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + 0$$



$$x = x_0$$

$$z - z_0 = 0 + f_y(x_0, y_0)(y - y_0)$$



Example 1

Give an equation of the tangent plane to $f(x, y) = \sqrt{xe^y}$ at $(4, 0)$

Exercise

Compute the equation of the tangent plane to $z = \sqrt{36 - 4x^2 - y^2}$ at $(2, 2, 4)$.

Linearization

Definition

If we write z as a function $L(x, y)$, we obtain the **linearization** of f at (x_0, y_0).

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If the graph $z = f(x, y)$ has a tangent plane, then $L(x, y)$ approximates the values of f near (x_0, y_0).

Notice $f(x_0, y_0)$ just calculates the value of z_0.

Example 2

Use a linearization to approximate the value of $\sqrt{4.02e^{0.05}}$.

The Two-Variable Differential

The differential dz measures the change in the linearization given particular changes in the inputs: dx and dy. It is a useful shorthand when one is estimating the error in an initial computation.

Definition

For $z = f(x, y)$, the **differential** or **total differential** dz is a function of a point (x, y) and two independent variables dx and dy.

$$\begin{aligned} dz &= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \end{aligned}$$

Exercise

Suppose I decide to invest \$10,000 expecting a 6% annual rate of return for 12 years, after which I'll use it to purchase a house. The formula for compound interest

$$P = P_0 e^{rt}$$

indicates that when I want to buy a house, I will have $P = 10,000e^{0.72}$.

I accept that my expected rate of return might have an error of up to $dr = 2\%$. Also, I may decide to buy a house up to $dt = 3$ years before or after I expected.

1. Write the formula for the differential dP at $(r, t) = (0.06, 12)$.

2. Given my assumptions, what is the maximum estimated error dP in my initial calculation?

3. What is the actual maximum error in P?

Summary Questions

- What do you need to compute in order to give the equation of a tangent plane?
- When is it preferable to approximate using a linearization?
- How is the differential defined for a two variable function? What does each variable in the formula mean?

Section 14.5

The Chain Rule

Goals:

- Use the **chain rule** to compute derivatives of compositions of functions.
- Perform implicit differentiation using the chain rule.

Motivational Example

Suppose a Jinteki Corporation makes widgets which is sells for \$100 each. It commands a small enough portion of the market that its production level does not affect the price of its products. If W is the number of widgets produced and C is their operating cost, Jinteki's profit is modeled by

$$P = 100W - C$$

The partial derivative $\frac{\partial P}{\partial W} = 100$ does not correctly calculate the effect of increasing production on profit. How can we calculate this correctly?

The Chain Rule

Recall that for a differentiable function $z = f(x, y)$, we computed $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$.

Theorem

Consider a differentiable function $z = f(x, y)$. If we define $x = g(t)$ and $y = h(t)$, both differential functions, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1

If $P = R - C$ and we have $R = 100W$ and $C = 3000 + 70W - 0.1w^2$, calculate $\frac{dP}{dW}$.

The Chain Rule, Generalizations

The chain rule works just as well if x and y are functions of more than one variable. In this case it computes partial derivatives.

Theorem

If $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, are all differentiable, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

We can also modify it for functions of more than two variables.

Theorem

If $w = f(x, y, z)$, $x = g(s, t)$, $y = h(s, t)$ and $z = j(s, t)$ are all differentiable, then

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Application to Implicit Differentiation

Recall that an implicit function on n variables is a level curve of a n-variable function. How can we use this to calculate $\frac{dy}{dx}$ for the graph $x^3 + y^3 - 6xy = 0$?

Exercise

Recall that for an ideal gas $P(n, T, V) = \frac{nRT}{V}$. R is a constant. n is the number of molecules of gas. T is the temperature in Celsius. V is the volume in meters. Suppose we want to understand the rate at which the pressure changes as an air-tight glass container of gas is heated.

1. Apply the chain rule to get an expression for $\frac{dP}{dT}$.
2. What is $\frac{dn}{dT}$?
3. What is $\frac{dT}{dT}$?
4. Suppose that $\frac{dV}{dT} = (5.9 \times 10^{-6})V$. Calculate and simplify the expression you got for $\frac{dP}{dT}$.

Summary Questions

- What is the difference between $\frac{dz}{dx}$ and $\frac{\partial z}{\partial x}$? How is the first one computed?
- How do you use the chain rule to differentiate implicit functions?

Section 14.6

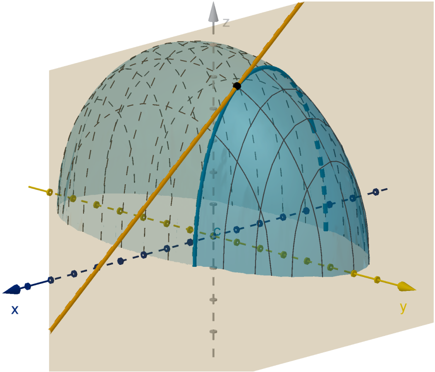
Directional Derivatives and the Gradient Vector

Goals:

- Calculate the **gradient vector** of a function.
- Relate the gradient vector to the shape of a graph and its level curves.
- Compute **directional derivatives**.

Generalization of the Partial Derivative

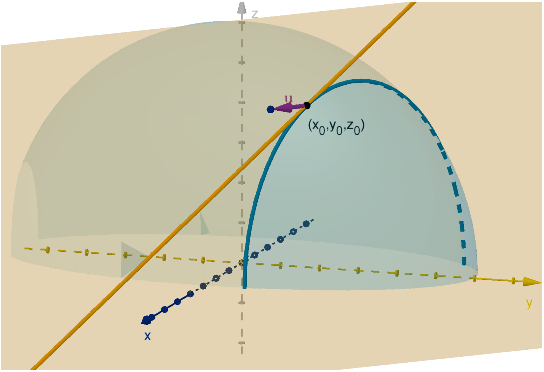
The partial derivatives of $f(x, y)$ give the instantaneous rate of change in the x and y directions. This is realized geometrically as the slope of the tangent line. What if we want to travel in a different direction?



The Directional Derivative

Definition

For a function $f(x,y)$ and a unit vector $\mathbf{u} \in \mathbb{R}^2$, we define $D_{\mathbf{u}}f$ to be the instantaneous rate of change of f as we move in the $\mathbf{u}$ direction. This is also the slope of the tangent line to f in the direction of $\mathbf{u}$.



Limit Definition of the Directional Derivative

Recall that we compute the $D_x f$ by comparing the values of f at (x, y) to the value at $(x + h, y)$, a displacement of h in the x-direction.

$$D_x f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

To compute $D_{\mathbf{u}} f$ for $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, we compare the value of f at (x, y) to the value at $(x + ta, y + tb)$, a displacement of t in the $\mathbf{u}$-direction.

Limit Formula

$$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + ta, y + tb) - f(x, y)}{t}$$

Other Cross Sections Worksheet

1. What direction produces the greatest directional derivative? The smallest?
2. How are these directions related to the geometry (specifically the level curves) of the graph?
3. How these directions related to the partial derivatives?

Conclusions from the Other Cross Sections Worksheet

The Gradient Vector

Definition

The **gradient vector** of f at (x, y) is

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

Remarks:

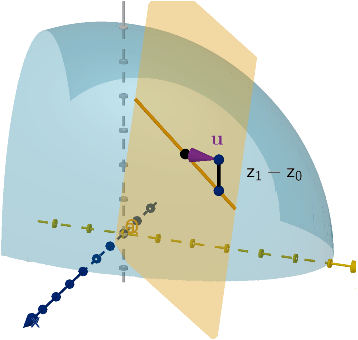
1. The gradient vector is a function of (x, y). Different points have different gradients.
2. $\mathbf{u}_{\max}$, which maximizes $D_{\mathbf{u}} f$, points in the same direction as ∇f.
3. $\mathbf{u}_0$, which is tangent to the level curves, is orthogonal to ∇f.

The Gradient Vector and Directional Derivatives

The tangent lines live in the tangent plane. We can compute their slope by rise over run.

Let $\mathbf{u}$ be a unit vector from (x_0, y_0) to (x_1, y_1). Let the associated z values in the tangent plane be z_0 and z_1 respectively.

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{\text{rise}}{\text{run}} = \frac{z_1 - z_0}{|\mathbf{u}|}$$
$$= f_x(x_0, y_0)(x_1 - x_0) + f_y(x_0, y_0)(y_1 - y_0)$$
$$= \nabla f(x_0, y_0) \cdot \mathbf{u}.$$



This formula generalizes to functions of three or more variables.

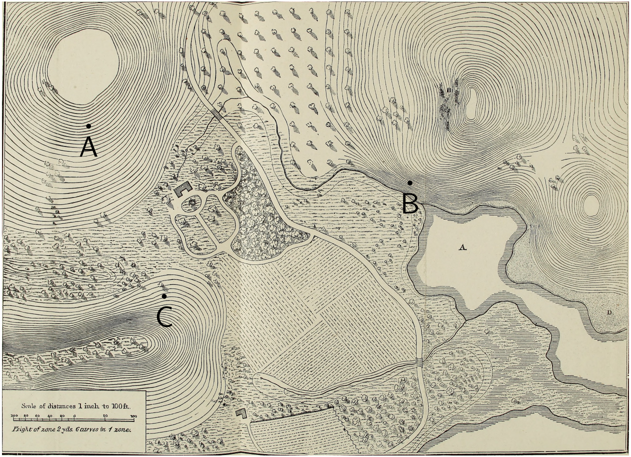
Example 1

Let $f(x, y) = \sqrt{9 - x^2 - y^2}$ and let $\mathbf{u} = \langle 0.6, -0.8 \rangle$.

1. What are the level curves of f?
2. What direction does $\nabla f(1, 2)$ point?
3. Without calculating, is $D_{\mathbf{u}} f(1, 2)$ positive or negative?
4. Calculate $\nabla f(1, 2)$ and $D_{\mathbf{u}} f(1, 2)$.

Example 2

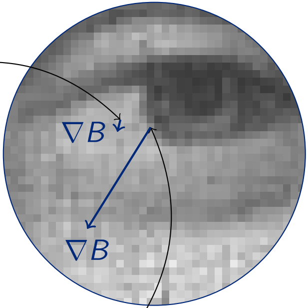
Let $h(x, y)$ give the altitude at longitude x and latitude y. Assuming h is differentiable, draw the direction of $\nabla h(x, y)$ at each of the points labeled below. Which gradient is the longest?



Application - Edge Detection

The length of the gradient of a brightness function detects the edges in a picture, where the brightness is changing quickly.

$$\frac{\partial B}{\partial x}(336,785) \approx \frac{185-187}{1}$$
$$\frac{\partial B}{\partial y}(336,785) \approx \frac{179-187}{1}$$
$$\nabla B(336,785) \approx (-2,-8)$$



$$\frac{\partial B}{\partial x}(340,784) \approx \frac{97-139}{1}$$
$$\frac{\partial B}{\partial y}(340,784) \approx \frac{72-139}{1}$$
$$\nabla B(340,784) \approx (-42,-67)$$

Application - Tangent Planes

Use a gradient vector to find the equation of the tangent plane to the graph $x^2 + y^2 + z^2 = 14$ at the point $(2, 1, -3)$.

Summary Questions

- What does the direction of the gradient vector tell you?
- What does the directional derivative mean geometrically? How do you compute it?
- How is the gradient vector related to a level set?

Section 14.7

Maximum and Minimum Values

Goals:

- Find **critical points** of a function.
- Test critical points to find local maximums and minimums.
- Use the **Extreme Value Theorem** to find the global maximum and global minimum of a function over a closed set.

Local Maximum and Minimum

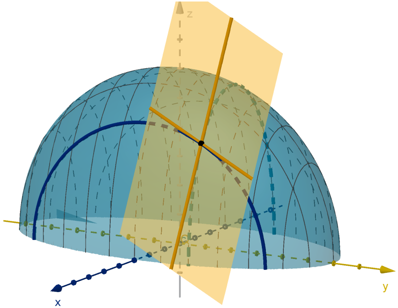
Definition

Given a function $f(x, y)$ we say

1. (a, b) is a **local maximum** if $f(a, b) \geq f(x, y)$ for all (x, y) in some neighborhood around (a, b).
2. (a, b) is a **local minimum** if $f(a, b) \leq f(x, y)$ for all (x, y) in some neighborhood around (a, b).

Relation to the Gradient

If $f_x(a, b) \neq 0$ or $f_y(a, b) \neq 0$, then (a, b) is not a local extreme. The nonzero partial derivative guarantees we can find higher and lower values in that direction.



If both these partial derivatives are 0, then $\nabla f(a, b) = \langle 0, 0 \rangle$.

Critical Points

Definition

We say (a, b) is a **critical point** of f if either

1. $\nabla f(a, b) = \langle 0, 0 \rangle$ or
2. $\nabla f(a, b)$ does not exist (because one of the partial derivatives does not exist).

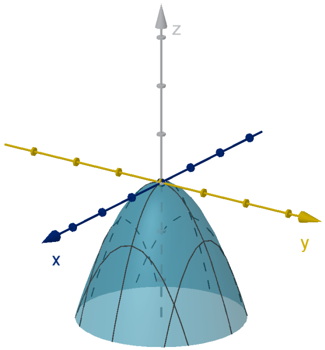
From the observation on the previous slide, it follows that local extremes must reside at critical points.

Example 1

The function $z = 2x^2 + 4x + y^2 - 6y + 13$ has a minimum value. Find it.

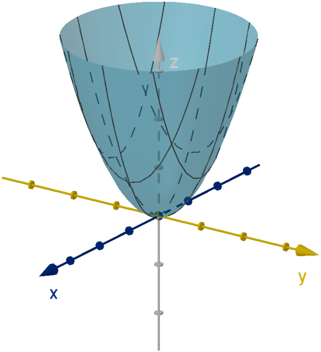
Identifying Local Extremes

A critical point could be a local maximum. f decreases in every direction from $(0, 0)$.



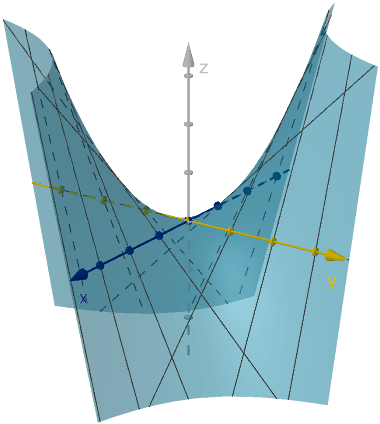
Identifying Local Extremes

A critical point could be a local minimum. f increases in every direction from $(0, 0)$.



Identifying Local Extremes

A critical point could be neither. f increases in some directions from $(0,0)$ but decreases in others. This configuration is called a **saddle point**.



The Second Derivatives Test

Theorem

Suppose f is differentiable at (a, b) and $f_x(a, b) = f_y(a, b) = 0$. Then we can compute

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum.*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum.*
3. *If $D < 0$ then (a, b) is a saddle point.*

Unfortunately, if $D = 0$, this test gives no information.

The Hessian Matrix

Definition

The quantity D in the second derivatives test is actually the determinant of a matrix called the **Hessian** of f.

$$f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 = \det \underbrace{\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}}_{Hf(a,b)}$$

This can be a useful mnemonic for the second derivatives test.

Exercise

Let $f(x, y) = \cos(2x + y) + xy$

1. Verify that $\nabla f(0, 0) = \langle 0, 0 \rangle$.
2. Is $(0, 0)$ a local minimum, a local maximum, or neither?

The Extreme Value Theorem

Theorem

A continuous function f on a closed and bounded domain D has an absolute maximum and an absolute minimum somewhere in D.

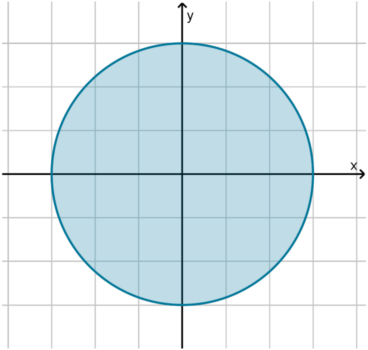
Definition

Let D be a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$.

- D is **closed** if it contains all of the points on its boundary.
- D is **bounded** if there is some upper limit to how far its points get from the origin (or any other fixed point). If there are points of D arbitrarily far from the origin, then D is **unbounded**.

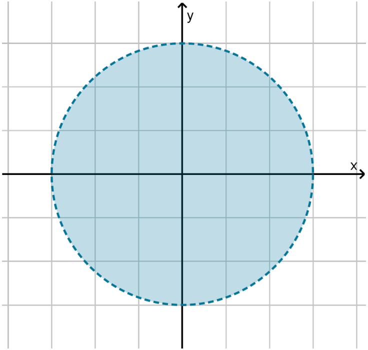
Closed Domains

Closed



$x^2 + y^2 \leq 9$

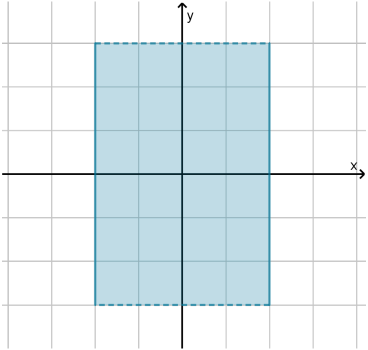
Not Closed



$x^2 + y^2 < 9$

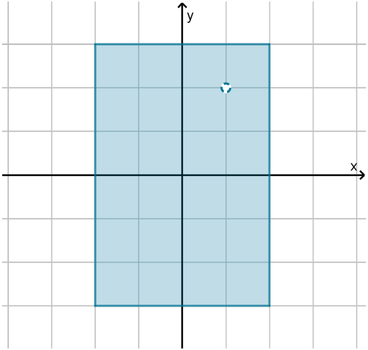
Closed Domains

Not Closed



$-2 \leq x \leq 2$ and $-3 < y < 3$

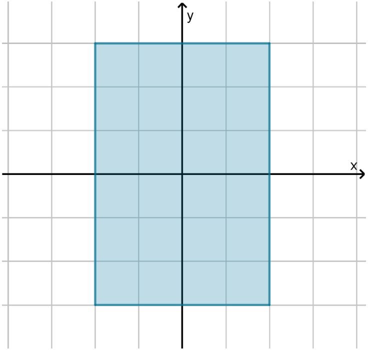
Not Closed



$-2 \leq x \leq 2$ and $-3 \leq y \leq 3$
and $(x, y) \neq (1, 2)$

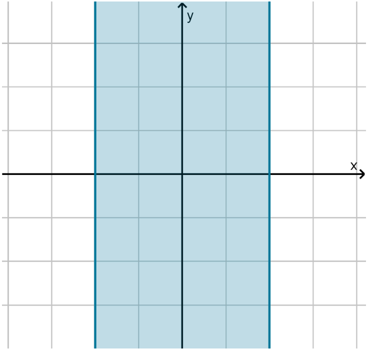
Bounded Domains

Bounded



$-2 \leq x \leq 2$ and $-3 \leq y \leq 3$

Unbounded

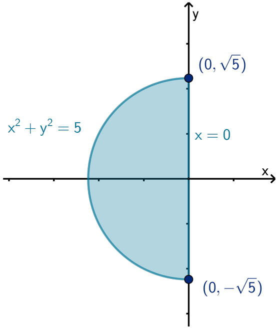


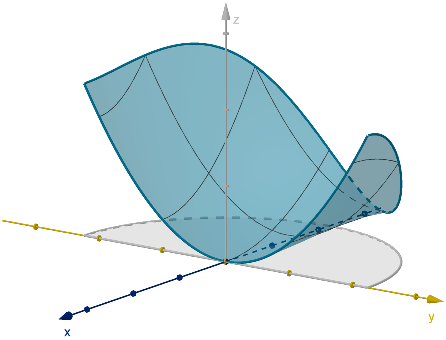
$-2 \leq x \leq 2$

Example 2

Find the maximum value of $f(x, y) = x^2 + 2y^2 - x^2y$ on the domain

$$D = \{ \underbrace{(x, y)}_{\text{points in } \mathbb{R}^2} : \underbrace{x^2 + y^2 \leq 5, x \leq 0}_{\text{conditions}} \}$$





Exercise

Let $f(x, y)$ be a differentiable function and let

$$D = \{(x, y) : y \geq x^2 - 4, x \geq 0, y \leq 5\}.$$

1. Sketch the domain D.
2. Does the Extreme Value Theorem guarantee that f has an absolute minimum on D? Explain.
3. List all the places you would need to check in order to locate the minimum.

Summary

- Where must the local maximums and minimums of a function occur? Why does this make sense?
- What does the second derivatives test tell us?
- What hypotheses does the Extreme Value Theorem require? What does it tell us?

Section 14.8

Lagrange Multipliers

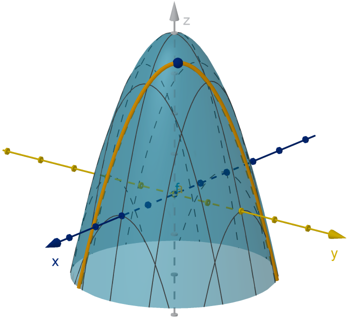
Goal:

- Find minimum and maximum values of a function subject to a **constraint**.
- If necessary, use **Lagrange multipliers**.

Maximums on a Constraint Worksheet

Sometimes we aren't interested in the maximum value of $f(x, y)$ over the whole domain, we want to restrict to only those points that satisfy a certain **constraint** equation.

The maximum on the constraint is unlikely to be the same as the unconstrained maximum (where $\nabla f = 0$). Can we still use ∇f to find the maximum on the constraint?



$\max f$ such that $x + y = 1$

Conclusions from Maximums on a Constraint Worksheet

The Method of Lagrange Multipliers

Theorem

Suppose an **objective function** $f(x, y)$ *and a* **constraint function** $g(x, y)$ *are differentiable. The local extremes of* $f(x, y)$ *given the constraint* $g(x, y) = k$ *occur where*

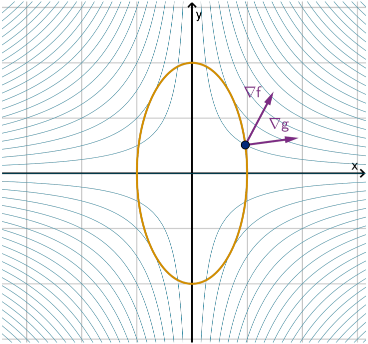
$$\nabla f = \lambda \nabla g$$

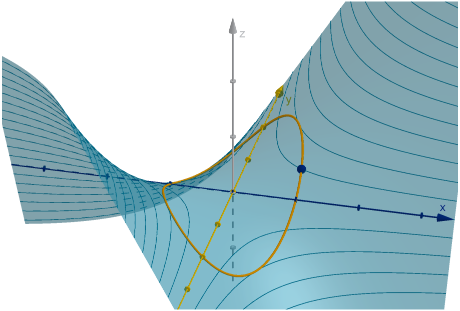
for some number λ*, or else where* $\nabla g = 0$*. The number* λ *is called a* **Lagrange Multiplier***.*

This theorem generalizes to functions of more variables.

Lagrange and the Directional Derivatives and Level Curves

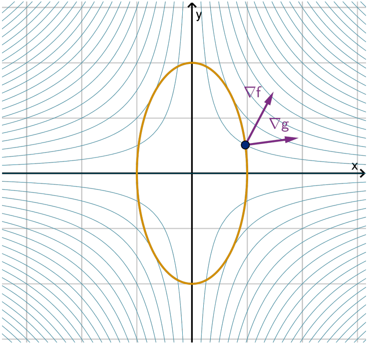
When ∇f is not parallel to ∇g, we can see that we can travel along $g(x, y) = k$ and increase the value of f. This is because $D_{\mathbf{u}} f > 0$ for some $\mathbf{u}$ tangent to the constraint. When ∇f is parallel to ∇g, we see that the level curves of f are tangent to the level curve $g(x, y) = k$.





Example 1

Find the point(s) on the ellipse $4x^2 + y^2 = 4$ on which the function $f(x, y) = xy$ is maximized.



Exercise

Refer to your "Maximums on a Constraint" worksheet.

1. What system of equations would you set up to find the critical points of f on the constraint $p(x, y, z) = 7$?

2. Can you solve it?

3. Which was easier, using Lagrange or using substitution?

Using the Extreme Value Theorem (Two Variable)

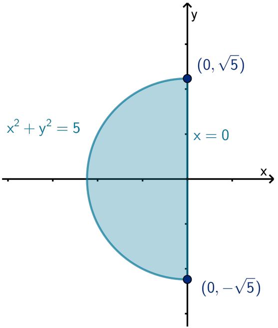
To find the absolute minimum and maximum of $f(x, y)$ over a closed domain D with boundaries $g(x, y) = c$.

1. Compute ∇f and find the critical points inside D.
2. Identify the boundary components. Find the critical points on each using substitution **or Lagrange multipliers**.
3. Identify the intersection points between boundary components.
4. Evaluate $f(x, y)$ at all of the above. The minimum is the lowest number, the maximum is the highest.

Example 2

Use Lagrange multipliers to find the maximum value of $f(x, y) = x^2 + 2y^2 - x^2 y$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 5, x \leq 0\}$$



More than One Constraint?

If we have two constraints, $g(x, y, z) = k$ and $h(x, y, z) = l$, then their intersection is a curve. The gradient vectors to g and h are both normal to the curve. For the curve to be tangent to the level curves of f, we need that ∇f lies in the normal plane to the curve. In other words:

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

This system of equations is usually difficult to solve.

Summary

- What is a constraint?
- What equations do you write when you apply the method of Lagrange multipliers?
- How does a constraint arise when finding the maximum over a closed and bounded domain?